QUANTUM FLUCTUATION RELATIONS

Michele Campisi, Peter Hänggi, and Peter Talkner

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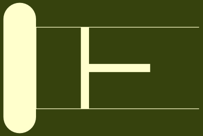
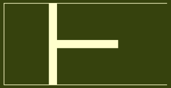
arXiv:1012.2268

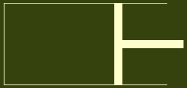
T 

$$H = H(\mathbf{z}, \lambda_t)$$



λ_0

eq.



λ_τ

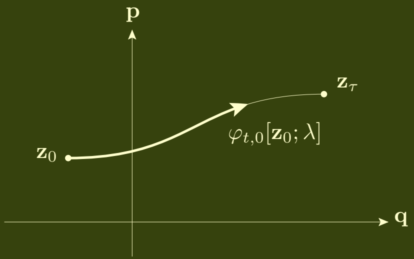
non-eq

$$\rho_0(\mathbf{z}) = \frac{e^{-\beta H(\mathbf{z},\lambda_0)}}{Z(\lambda_0)} \qquad \rho_\tau(\mathbf{z}) \neq \frac{e^{-\beta H(\mathbf{z},\lambda_\tau)}}{Z(\lambda_\tau)}$$

$$\lambda : [0, \tau] \rightarrow R$$
$$t \rightarrow \lambda_t$$

protocol

CLASSICAL WORK



$$W[\mathbf{z}_0;\lambda] = H(\mathbf{z}_\tau,\lambda_\tau) - H(\mathbf{z}_0,\lambda_0) = \int \mathrm{d}t\,\dot{\lambda}_t \frac{\partial H(\varphi_{t,0}[\mathbf{z}_0;\lambda],\lambda_t)}{\partial \lambda_t}$$

$$p[W;\lambda] = \int \mathrm{d}\mathbf{z}_0 \rho_0(\mathbf{z}_0)\delta[W - H(\mathbf{z}_\tau,\lambda_\tau) + H(\mathbf{z}_0,\lambda_0)]$$

Work is a RANDOM quantity

JARZYNSKI EQUALITY

C. Jarzynski, PRL **78** 2690 (1997)

$$\left\langle e^{-\beta W} \right\rangle = e^{-\beta \Delta F}$$

$$\Delta F = F(\lambda_\tau) - F(\lambda_0) = -\beta^{-1} \ln \frac{Z(\lambda_\tau)}{Z(\lambda_0)}$$

$$Z(\lambda_t) = \int \mathrm{d}\mathbf{z}\, e^{-\beta H(\mathbf{z}, \lambda_t)}$$

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Since exp is convex, then

$$\langle W \rangle \geq \Delta F \qquad \text{Second Law}$$

INCLUSIVE VS. EXCLUSIVE VIEWPOINT

C. Jarzynski, C.R. Phys. **8** 495 (2007)

$$H(\mathbf{z}, \lambda_t) = H_0(\mathbf{z}) - \lambda_t Q(\mathbf{z})$$

$$\begin{aligned} W &= -\int Q \mathrm{d}\lambda \\ &= H(\mathbf{z}_\tau, \lambda_\tau) - H(\mathbf{z}_0, \lambda_0) \end{aligned}$$

Not necessarily of the textbook form $\int \mathrm{d}$(displ.)$\times$ (force)

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$$\begin{aligned} W_0 &= \int \lambda \mathrm{d}Q \\ &= H_0(\mathbf{z}_\tau) - H_0(\mathbf{z}_0) \quad \Rightarrow \quad \langle e^{-\beta W_0} \rangle = 1 \end{aligned}$$

G.N. Bochkov and Yu. E. Kuzovlev JETP **45**, 125 (1977)

GAUGE FREEDOM

$$\begin{aligned} H'(\mathbf{z}, \lambda_t) &= H(\mathbf{z}, \lambda_t) + g(\lambda_t) \\ W' &= W + g(\lambda_\tau) - g(\lambda_0) \\ \Delta F' &= \Delta F + g(\lambda_\tau) - g(\lambda_0) \end{aligned}$$

W and ΔF are not gauge invariant

GAUGE FREEDOM

$$
\begin{aligned}
H'(\mathbf{z}, \lambda_t) &= H(\mathbf{z}, \lambda_t) + g(\lambda_t) \\
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\Delta F' &= \Delta F + g(\lambda_\tau) - g(\lambda_0)
\end{aligned}
$$

W and ΔF are not gauge invariant

$$
\langle e^{-\beta W'} \rangle = e^{-\beta \Delta F'} \iff \langle e^{-\beta W} \rangle = e^{-\beta \Delta F}
$$

The fluctuation theorem is gauge invariant !
Gauge:
irrelevant for dynamics
crucial for ENERGY-HAMILTONIAN connection
[see also D.H. Kobe AJP **49**, 581 (1981)]

QUANTUM WORK FLUCTUATION RELATIONS

$$H(\mathbf{z}, \lambda_t) \longrightarrow \mathcal{H}(\lambda_t)$$

$$\rho(\mathbf{z}, \lambda_t) \longrightarrow \varrho(\lambda_t) = \frac{e^{-\beta \mathcal{H}(\lambda_t)}}{\mathcal{Z}(\lambda_t)}$$

$$Z(\lambda_t) \longrightarrow \mathcal{Z}(\lambda_t) = \mathrm{Tr} e^{-\beta \mathcal{H}(\lambda_t)}$$

$$\varphi_{t,0}[\mathbf{z}_0; \lambda] \longrightarrow U_{t,0}[\lambda] = \mathcal{T} \exp \left[-\frac{i}{\hbar} \int_0^t ds \mathcal{H}(\lambda_s) \right]$$

$$W[\mathbf{z}_0; \lambda] \longrightarrow ?$$

$$
\begin{aligned}
\mathcal{W}[\lambda] &= U^{\dagger}_{t,0}[\lambda]\mathcal{H}(\lambda_t)U_{t,0}[\lambda] - \mathcal{H}(\lambda_0) \\
&= \mathcal{H}^{H}_{t}(\lambda_t) - \mathcal{H}(\lambda_0) \\
&= \int_0^t \mathrm{d}t\dot{\lambda}_t \frac{\partial \mathcal{H}^{H}_{t}(\lambda_t)}{\partial \lambda_t}
\end{aligned}
$$

WORK IS NOT AN OBSERVABLE

P. Talkner, E. Lutz, and P. Hänggi, PRE **75** 050102 (2007)

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Work characterizes PROCESSES, not states! (δW is not exact)

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Right definition of quantum work:

$$W[\mathbf{z}_0;\lambda] \longrightarrow w = E_m^{\lambda_\tau} - E_n^{\lambda_0} \qquad \text{two-measurements}$$

$E_n^{\lambda_t}$ =instantaneous eigenvalue: $\mathcal{H}(\lambda_t)|\psi_{n,\gamma}^{\lambda_t}\rangle = E_n^{\lambda_t}|\psi_{n,\gamma}^{\lambda_t}\rangle$

PROBABILITY OF WORK

$$p[w; \lambda] = \sum_{n,m} \delta(w - [E_m^{\lambda_\tau} - E_n^{\lambda_0}]) p_{m|n}[\lambda] p_n$$

UNITARITY

$$\text{Unitarity} \Rightarrow \sum_n p_{m|n}[\lambda] = 1 \qquad \text{(double stochasticity)}$$

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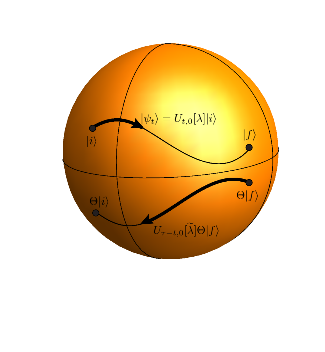
Double Stochasticity + Gibbs distribution

$$\Rightarrow \langle e^{-\beta w} \rangle = e^{-\beta \Delta F}$$

MICROREVERSIBILITY OF QUANTUM NON-AUTONOMOUS SYSTEMS



$$\Theta\mathcal{H}(\lambda_t) = \mathcal{H}(\lambda_t)\Theta$$

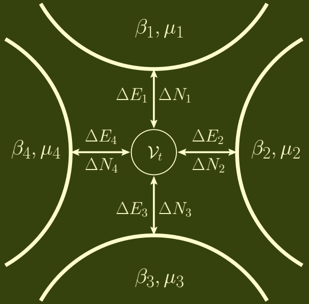
$$\Rightarrow U_{t,\tau}[\lambda] = \Theta^\dagger U_{\tau-t,0}[\widetilde{\lambda}]\Theta$$

$$\widetilde{\lambda} : [0,\tau] \to R$$

$$t \to \widetilde{\lambda}_t = \lambda_{\tau-t}$$

$$\text{Microreversiility} + \text{Gibbs distribution} \Rightarrow \frac{p[w;\lambda]}{p[-w;\widetilde{\lambda}]} = e^{\beta(w-\Delta F)}$$

QUANTUM EXCHANGE FLUCTUATION RELATIONS



$$\mathcal{H}(\mathcal{V}_t) = \sum_{i=1}^{s} \mathcal{H}_i + \mathcal{V}_t$$

$$\varrho_0 = \Pi_i \varrho_i = \Pi_i e^{-\beta_i[\mathcal{H}_i - \mu_i \mathcal{N}_i]} / \Xi_i$$

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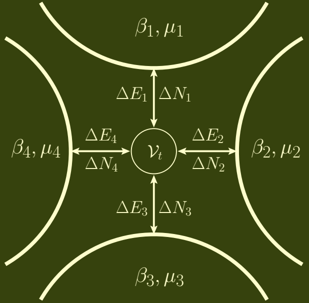

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$$p[\Delta \mathbf{E}, \Delta \mathbf{N}; \mathcal{V}] = \sum_{m,n} \prod_i \delta(\Delta E_i - E_m^i + E_n^i)\delta(\Delta N_i - N_m^i + N_n^i) p_{m|n}[\mathcal{V}] p_n^0$$

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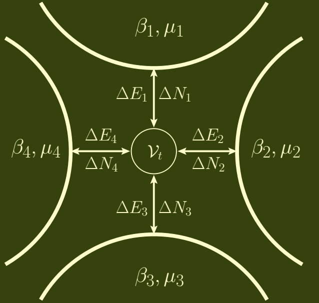

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$$\frac{p[\Delta \mathbf{E}, \Delta \mathbf{N}; \mathcal{V}]}{p[-\Delta \mathbf{E}, -\Delta \mathbf{N}; \widetilde{\mathcal{V}}]} = \prod_i e^{\beta_i [\Delta E_i - \mu_i \Delta N_i]}$$

C. Jarzynski and D.K. Wójcik, PRL **92** 230602 (2004)
K. Saito and Y. Utsumi PRB **78** 115429 (2008)

D. Andrieux, P. Gaspard, T. Monnai, and S. Tasaki ,NJP **11** 043014 (2009)

PROBLEM

Projective quantum measurement of:

$\Delta E =$TOTAL energy
$\Delta N =$ TOTAL number of particles

in MACROSCOPIC systems !!!

ROBUSTNESS OF FLUCTUATION THEOREMS TO QUANTUM MEASUREMENTS

Measure $\mathcal{A}$ at time $t_1 \in (0, \tau)$

$$E_n^{\lambda_0} \to \alpha_r \to E_m^{\lambda_\tau}, \qquad p(m|r|n) = p(m|r)p(r|n)$$

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$$w = E_m^{\lambda_\tau} - E_n^{\lambda_0}$$

$$p(w) = \sum_{m,r,n} \delta(w - [E_m^{\lambda_\tau} - E_n^{\lambda_0}])p(m|r|n)p_n$$

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$$p(w) = \sum_{m,r,n} \delta(w - [E_m^{\lambda_\tau} - E_n^{\lambda_0}])p(m|r|n)p_n$$

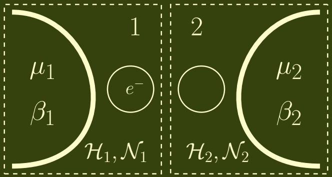
$$p_1(m|n) = \sum_r p(m|r|n) \qquad \text{doubly stochastic}$$

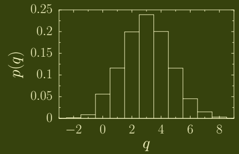
M. Campisi, P. Talkner, and P. Hänggi PRL **105** 210401 (2010)

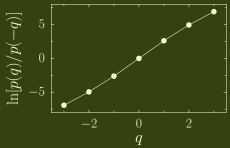
M. Campisi, P. Talkner, and P. Hänggi PRE *in press* (2011)

EXPERIMENTS:
EXCHANGE FLUCTUATION RELATION

Y. Utsumi, D.S. Golubev, M. Marthaler, K. Saito, T. Fujisawa, and G. Schön
PRB **81** 125331 (2010)







$$\frac{p(q)}{p(-q)} = e^{\beta(\mu_1 - \mu_2)q}$$

Continuous monitoring of FLUX