COLORED NOISE IN DYNAMICAL SYSTEMS

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CONTENTS

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I. INTRODUCTION

The general subject of colored noise driven dynamical flows is rooted in the study of the motion of small particles suspended in a fluid and moving under the influence of random forces that result from collisions with molecules of the fluid induced by thermal fluctuations. In short, the phenomenon of *Brownian motion* [1]. In the earliest studies of Brownian motion, the damping of the motion of the suspended particles was very large compared to that of the fluid molecules, so that inertial effects could be neglected. Moreover, the thermal fluctuations occur on a time scale that is very much shorter than that of the Brownian particle. It is then a good approximation to assume that the random forces are uncorrelated delta functions as perceived by the particle on its own, much slower time scale. This assumption considerably simplifies the problem, because it allows one to treat the stochastic dynamical motions as a Markovian process for which many methods and approximation schemes are available. The fluctuations that can be treated under this assumption have often been termed "white noise". Thus, white noise fluctuations $\xi(t)$, are those for which the autocorrelation function is given by

$$\langle \xi(t)\xi(s) \rangle = 2D\delta(t-s) \tag{1.1}$$

where we designate the noise intensity as D. This noise has no time scale and exists independently of any other physical system. A large body of literature on white noise applications exists, and appropriate starting points are the now classic reviews of Chandrasekhar, Uhlenbeck and Ornstein, and others in the collection of Wax [2]; the texts by Stratonovich [3], van Kampen [4], Risken [5], and Horsthemke and Lefever [6]; or the reports by Hänggi and Thomas [7], and Fox [8].

In the physical world this idealization, however, is never exactly realized. What must be done is to consider the noise and the physical system within which, or upon which, it is operating together. Specifically, the time scales of the two systems must be taken into account. Therefore, we seek, in the first instance, a noise with a well-defined characteristic time. One of the simplest examples for an introductory discussion of time-correlated noise is the Ornstein–Uhlenbeck process, which exhibits an exponential correlation function,

$$\langle \xi(t)\xi(s) \rangle = (D/\tau_n) \exp(-|t-s|/\tau_n) \tag{1.2}$$

with noise correlation time τ_n. This fluctuation process is called "colored noise" in analogy with the effects of filtering on white light. The terms "white" and "colored noise", are, of course, jargon words. Nevertheless, because they are widely recognized and understood, even though somewhat imprecise, we shall use them throughout. Now it is important to understand what clock is being used to measure the noise correlation time. It is the physical system itself, which is either generating the noise internally or is subject to the noise as an external forcing which, according to its own characteristic response time, perceives the time scale of the noise. The physical system is frequently and simply modeled with a stochastic differential equation termed a Langevin equation. A damped oscillator subject to a one-dimensional (1-D) deterministic potential $U(x)$, and additive noise serves as a simple example:

$$m\ddot{x} + m\gamma\dot{x} = -\frac{dU(x)}{dx} + \xi(t) \tag{1.3}$$

where m is the mass and γ is the damping factor. By dividing through by $m\gamma$, neglecting the inertial term in the limit of large γ, and properly scaling the system coordinate x to be dimensionless, that is, $x \rightarrow \alpha x$ and $D \rightarrow D(\alpha)$, we have

$$\dot{x} = \left[\frac{1}{\tau_s}\right]\left\{-\left[\frac{dU(x)}{dx}\right] + \xi(t)\right\} \tag{1.4}$$

where τ_s is the system characteristic time. Now for simplicity, we can scale time in the Langevin equation so that τ_s is removed by letting $t' = t/\tau_s$. But we must measure time in Eq. (1.2) on the same scale, where with $\tilde{D} = D(\alpha)/\tau_s$

$$\langle \xi(t')\xi(s') \rangle = \left[\frac{\tau_s}{\tau_n}\right]\tilde{D} \exp\left[-|t'-s'|\left(\frac{\tau_s}{\tau_n}\right)\right] \tag{1.5}$$

It has become customary to write the Langevin equation in this scale, and the noise correlation function in terms of a dimensionless time $\tau \equiv \tau_n/\tau_s$, so that

$$\dot{x} = -\frac{dU(x)}{dx} + \xi(t) \tag{1.6}$$

and

$$\langle \xi(t)\xi(s) \rangle = (D/\tau)\exp(-|t-s|/\tau) \tag{1.7}$$

where $D \equiv \tilde{D}$ is now a dimensionless noise intensity. Though this is a convenient scale, which we shall adopt throughout this paper, it has sometimes obscured the role of colored noise in real physical systems by hiding the "clock" with which the system measures time. Next, we shall discuss various approximations, some valid only for small values of τ. In order to successfully apply these theories to any real physical system, it is essential to understand what "small" means. As the above discussion indicates, this means that $\tau_n \ll \tau_s$ (or $\tau \ll 1$), but how much smaller depends not only on the particular approximations used but also on the problem or system. A related question is one of measurability and distinguishability. The analogue simulators and associated measurements, which mimic real physical systems, are not accurate enough to convincingly distinguish any colored noise approximate theory from the white noise predictions for the same system for $\tau \cong 0.1$. Digital simulations can, of course, achieve much greater accuracy and further improvements are currently forthcoming. Even so, it is probably not practical (with finite computing time) to expect distinguishability for $\tau \cong 0.001$.

Moving toward larger τ, approximations that are based on perturbations of the white noise theory become progressively less accurate once again for values of τ that depend both on the system and on the particular approximation used. These so-called "small τ" approximate theories have roots that date to the original work of Stratonovich [3] cited above. A different approach is expounded by Risken [5] who has pioneered the use of matrix continued fraction expansions, which offer solutions to the colored noise problem. These expansions are in principle exact, but are in practice rendered approximate by the necessity to truncate and numerically invert a final matrix of infinite dimension. Moreover, in the absence of supercomputers, the matrix continued fraction method is practically limited to systems with a low number of state variables.

One of the earliest definitive results, which indicated that colored noise plays an important role in nature was Kubo's explanation of motional narrowing of the observed magnetic resonance line shapes induced by thermal fluctuations [9]. Kubo's model is exactly solvable and applicable to all ranges of τ, since it treats a linear system: an oscillator with a noisy frequency. The observed statistical properties of the fluctuations of dye laser light [10] offered the next solid evidence that the noise found in some physical systems is colored. Initially, the evidence was provided by numerical simulations of the nonlinear laser Langevin equations using

Ornstein–Uhlenbeck noise, compared to measurements of the correlation function of the actual laser light fluctuations [11]. Soon after, an early success of the so-called "small τ" theory resulted from its application to the same experimental dye laser data [12]. Recently, it has been demonstrated that the pump parameter in dye lasers can be adjusted close enough to the laser transition that the laser fluctuations are driven by (pump) noise of moderate values of τ [13]. The intensity fluctuations in all pumped lasers originate from two sources: the spontaneous emission, or quantum, noise that derives from the statistics of photon emission from the inverted population within the laser cavity, and the pump noise that derives from fluctuations in the intensity of the pump. The pump noise is governed by a much slower time scale than the emission noise, and so has been treated as colored noise, while the emission noise has until recently been assumed to be white. Colored spontaneous emission noise has been shown to have a strong influence on the properties of the proposed correlated spontaneous emission laser [14]. Noise color also has a strong effect on the systematics of noise induced bifurcations among ordered and turbulent states in nematic liquid crystals [15, 16].

II. USE AND ABUSE OF COLORED NOISE

This section first reviews the development of the field of systems driven by noise starting from the early work on Brownian motion around the turn of the century, then continues with the pioneering studies and applications to physical systems during the decade of the 1950s, and concludes with the more recent theoretical developments through the early 1980s.

A. The Role of White Noise

As already mentioned in Section I, the most well-known application of a noisy differential equation for a state variable $x(t)$ is the theory of Brownian motion, described first in 1828 [17], with the first precise experiments carried out in 1888 by Gouy [18]. The description in terms of a noisy differential equation wherein one splits the motion into two parts, a slowly varying systematic part and a rapidly varying random part, is due to Langevin [19], who first wrote the familiar expression for the damped motion of a randomly forced particle, with $\dot{x} = v$,

$$m\dot{v} = -m\gamma v + \xi(t) \tag{2.1}$$

with

$$\langle \xi(t)\xi(t') \rangle = \frac{2\gamma kT}{m} \delta(t-t') \tag{2.2}$$

A study of the solution of this equation, however, had to await Ornstein's early work on Brownian motion (see the historical discussion on colored noise given below). It is important to note that a general study of Eq. (2.1) is nontrivial. In order to make progress, one necessarily must specify the properties of the random force. It is often justifiable to assume that the random forces, which sometimes derive from the environment, are correlated on a very small time scale τ_n compared to the characteristic relaxation time for the system τ_s, around a locally stable state. An idealized treatment then assumes the random force to have zero correlation time, that is $\xi(t)$ is approximated by a (generalized) δ-correlated process:

$$\langle \xi(t)\xi(s) \rangle = 2D\delta(t-s) \tag{2.3}$$

where all the frequencies of its power spectrum $S_\xi(\omega) = \int_{-\infty}^{\infty} \langle \xi(t)\xi(s) \rangle e^{-i\omega t}\, dt \equiv 2D$, are present with equal weight. Obviously, there exist several classes of such white noise processes, all of which are completely understood [20]. The classes are defined in terms of the derivative $\xi(t) = dz(t)/dt$ of processes with stationary, independent increments. For example, the derivative of the Wiener process [21, 22] defines *Gaussian white noise*, whereas the derivative of the Poisson process yields *white shot noise*. These two elementary noise processes form the building blocks for the theory of Markov processes [20, 23–26]. Stochastic differential equations composed of nonlinear drift flows $f_\alpha(x)$ and multiplicative noise forces $g_{\alpha i}(x)\xi_i(t)$, that is,

$$\dot{x}_\alpha = f_\alpha(x) + \sum_{i=1}^{n} g_{\alpha i}(x)\xi_i(t) \qquad \alpha = 1, \ldots, n \tag{2.4}$$

where $\xi_i(t)$ denotes a white (generally) non-Gaussian random force, thus describe a multidimensional Markov process $\mathbf{x}(t)$. The corresponding master equation, which describes the rate of change of the probability, as well as the statistical properties of the nonlinear noise forces, has been discussed in the literature [27–29].

From an historical point of view, the statistical consequences of Eq. (2.1) have first been studied by Ornstein [30, 31], implicitly assuming *Gaussian* white noise (see also [32, 33] and the bibliographical notes given in [34]). For the mean-square displacement in thermal equilibrium, he obtained from Eq. (2.1) the central result [30, 31],

$$\langle (x-x_0)^2\rangle_{x_0} = \left(\frac{2kT}{m\gamma^2}\right)[\gamma t - 1 + e^{-\gamma t}] \tag{2.5}$$

where an average over $x(t_0)=x_0$ is implied by the subscript on the left-hand side. In Eq. (2.5) T denotes the temperature and k is the Boltzmann constant. This result, which accounts also for the inertia effects $m\dot{v}$, generalizes the celebrated result by Einstein [35].

$$\langle x^2\rangle = \left(\frac{2kT}{m\gamma}\right)t \tag{2.6}$$

The inertia induced shift obtained by Ornstein in Eq. (2.5), given by the terms $[-1+\exp(-\gamma t)]/\gamma$ inside the bracket is, of course, the result of the two-dimensional 2-D stochastic motion in phase space, which is equivalent to a colored noise driven dynamics in configuration space. The passage from Eq. (2.1) to a partial differential equation for the probability[1]

$$\frac{\partial}{\partial t} p_t(v) = \gamma \frac{\partial}{\partial v}[v p_t(v)] + \frac{kT\gamma}{m}\frac{\partial^2}{\partial v^2} p_t(v) \tag{2.7}$$

has been achieved by Fokker [36, 37], Smoluchowski [38], and Planck [39]. Actually, Eq. (2.7) was obtained earlier by Lord Rayleigh [40, 41] who employed a limiting procedure from a discrete state Brownian motion model for a heavy particle (the Rayleigh model). This connection between the Langevin equation driven by Gaussian white noise and the parabolic partial differential equation, Eq. (2.7), commonly known as the "Fokker–Planck equation", was subsequently generalized to account for the Brownian motion of the configuration coordinate of a particle moving in an external potential field by Smoluchowski [42, 43] and Fürth [44]. The generalization to the full phase space, that is, $\partial p_t(x,v)/\partial t$, has been obtained first by Klein [45] (see also [46]).

Useful applications of the theory of Brownian motion to the calculation of other statistical quantities, such as the probability density of first passage times or absorption and escape probabilities, had been considered as early as 1915 by Schrödinger [47] and others [42–44]. For nonlinear flows, interesting applications, such as calculations of the stationary probability density of a 2-D noisy limit cycle, and the exact

[1] We will refer throughout this article to the "probability density" as simply the "probability".

quadrature formulas for the mean first-passage time of 1-D Fokker–Planck processes, have been obtained as early as 1933 by Pontryagin et al. [48] (translated by Barber in [49]).

Interesting as these applications are, however, the fact is that noise with zero correlation time leads to stochastic realizations, generated by the noncontinuous noise-sample paths, which are in reality nonphysical. For example, for the state variable $x_\alpha(t)$ driven by a white Gaussian noise source, as given by Eq. (2.4), the sample paths are not of bounded variation, nor are they continuous (as it is the case with white shot noise) or differentiable [21, 22, 50]. Thus, any results obtained from white noise theory that make predictions about the dynamics on time scales approximately equal to τ_n clearly do not lie within its regime of validity. Nevertheless, the results of measurements on real physical systems for which noise forces with very large effective bandwidths are encountered (at least a factor of 10 larger than the deterministic system bandwidth) are for almost all practical purposes indistinguishable from the predictions of the white noise theory. Of course, all actual noise encountered in nature has some nonzero (though perhaps small) correlation time. In Section III, corrections to the white noise theory that are necessary to describe systems driven by noise with nonnegligible correlation time, commonly known as "colored noise", are considered.

B. The Role of Colored Noise

Statistical fluctuations always reflect a lack of knowledge about the exact state of the system. Usually, the system behavior is modeled in terms of two classes of variables: state variables that change on a slow time scale, which are most often monitored directly in experiments, and those that are generally more rapidly varying and more closely related to the random forces. Moreover, the random forces themselves can be classified into two groups as "internal noise" or "external noise", though this distinction is often ambiguous depending, as it does, on how the boundary between the "system" and the "external world" is drawn. Generally, external noise can be thought of as imposed on some subsystem by a larger fluctuating environment in which the subsystem is immersed. In laboratory experiments, external noise with well characterized and immediately controllable statistical properties, such as the stationary probability density, intensity, and correlation time, is imposed by the experimenter on the system whose response he then measures. In laboratory experiments, as well as in many naturally occurring instances of nonequilibrium noise driven systems, the external noise can take on correlation times that are much smaller than, comparable to, or much larger than the characteristic relaxation times of the system. Furthermore, random forces of moderate-to-large correlation times, $\tau_n \cong \tau_s$, can also emerge with internal noise as was already clearly shown in 1962 by Kubo in the case of spin relaxation in magnetic systems [9].

In practice, for a complex system, any strongly colored noise implies a significant deviation from Markovian behavior. In the theoretical treatment of such systems it is often the case that strongly colored internal noise emerges as the result of *coarse graining over a hidden set of slow variables*. In this context, we touch upon a major problem that sooner or later confronts nearly every perplexed modelist of noisy stochastic flows: Given a nonlinear system, which and how many slow variables are needed to adequately describe the system dynamics? One generally hopes to monitor only a few, and preferably just one physical variable. There is a price to be paid for this simplification, however, precisely because such a resulting low-dimensional flow implies a loss of the Markovian properties of the original higher dimensional system. Systems that exhibit noise of moderate or large correlation time are often intrinsically high dimensionally, and can be reduced in dimension only at the expense of the Markovian character. Because multidimensional Markovian objects, of the form given by Eq. (2.4) with $n > 1$, present a rather complicated dynamics that is already difficult to study in analytical form, the study of colored noise driven flows even in one-coordinate dimension, such as

$$\dot{x} = f(x) + g(x)\xi(t) \tag{2.8}$$

where $\xi(t)$ is a stationary noise with correlation function,

$$\langle \xi(t)\xi(s) \rangle = D\gamma(t-s) \tag{2.9}$$

is thus expected to be challenging as well. However, any modeling in terms of colored noise is expected to be more physically realistic, since when a nonzero correlation time is explicitly accounted for, the realizations become differentiable as they must be for all real macroscopic systems. The white noise limits of such theories can then be compared to purely white noise theories as well as to the results of experiments performed with wide bandwidth noise. Often, the fluctuations $\xi(t)$ represent the cumulative effects of many weakly coupled environmental degrees of freedom. Outside critical neighborhoods, and in the absence of long-range correlations that induce large scale collective effects, one can invoke the *central limit theorem* and thus treat the fluctuations as Gaussian. In particular, if $\xi(t)$ is *in addition Markovian*, then Doob's theorem[2] [50, 52] states that $\xi(t)$ is necessarily an *Ornstein–Uhlenbeck*

[2] For a generalization of Doob's theorem to nonstationary processes (see [51]).

process, with exponential correlation function,

$$\langle \xi(t)\xi(s) \rangle = \left(\frac{D}{\tau}\right) e^{-|t-s|/\tau} \tag{2.10}$$

with the Lorentzian power spectrum $S_\xi(\omega) = 2D/(\tau^2\omega^2 + 1)$ [Fourier transform (FT) of the correlation function]. In the following discussion, we will often restrict the discussion to Gaussian processes with exponential correlation functions, as given by Eq. (2.10), unless stated otherwise. This exponentially correlated Gaussian noise source has been widely used in numerous recent studies. One of the earliest application dates back to 1966 by Berne et al. [53], where it has been used to model transport in simple liquids.

Pioneering studies of stochastic, nonlinear flows of the type given by Eq. (2.8), and applied to problems in electrical engineering and radiophysics, date to the late 1950s and were developed primarily by the school surrounding Stratonovich and co-workers [3, 54]. They considered corrections to the white noise theory valid for small τ, meaning, of course, that their approximate theory would apply in the range $\tau_n \ll \tau_s$, and succeeded to obtain an approximate Fokker–Planck like evolution for the probability [55],

$$\begin{aligned}\frac{\partial p_t(x,\tau)}{\partial t} &= -\frac{\partial}{\partial x}[f(x)p_t(x,\tau)] + D\frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} \\ &\quad \times \left\{ g(x)\left[1 + \tau g(x)\left(\frac{f(x)}{g(x)}\right)'\right] p_t(x,\tau)\right\}\end{aligned} \tag{2.11}$$

where the prime (′) indicates differentiation with respect to x.

This celebrated result is now commonly known as the "small τ approximation". Over the last two decades it has been applied to many different systems, rederived, commented on, and extended by many authors using a variety of methods. In particular, we mention here the method of cumulant expansions [56–63], expansions in functional derivatives [64–66], singular perturbation methods [67–71], the method of moments [72], adiabatic elimination procedures [73, 74], and projector operator techniques [75–77]. We wish to emphasize that this list of references is intended to be representative only and certainly is not complete. These different methods will not be further reviewed, instead we will return, in Section IV.A, to the small τ approximation with a discussion of the regime of its validity.

Colored noise of arbitrarily long correlation time has been considered in Kubo's cornerstone paper on the theory of line shapes and relaxation in magnetic resonance systems [9, 78]. He employed a modified Bloch equation, that is, the so-called "Kubo oscillator"[3]

$$\dot{x} = i[\omega_0 + \xi(t)]x(t) \tag{2.12}$$

Because this is inherently a linear system, and because $\xi(t)$ is Gaussian, Ornstein–Uhlenbeck noise in Eq. (2.10), Eq. (2.12) can be solved exactly for the first moment [9, 78]

$$\langle x(t)\rangle = \langle x(0)\rangle \exp\{i\omega_0 t - D[t - \tau(1 - \exp(-t/\tau))]\} \tag{2.13}$$

The transformation, $u = \ln x$ yields a linear stochastic flow with additive noise,

$$\dot{u} = i[\omega_0 + \xi(t)] \tag{2.14}$$

which in turn yields an exact master equation for $p_t(x) = p_t(u)|x|^{-1}$; see Eqs. (3.28–3.34). Defining the relaxation function $\phi(t)$,

$$\phi(-t) = \phi(t) \equiv \left\langle \exp \int_0^t \xi(s)\, ds \right\rangle = \frac{\langle x(t)\rangle}{\langle x(0)\rangle} e^{-i\omega_0 t} \tag{2.15}$$

we obtain, in the white noise limit, from Eq. (2.13) (termed "fast modulation limit" in [9]),

$$\lim_{\tau\to 0} \phi(t) = \phi_0(t) = \exp(-Dt) \qquad \tau \to 0 \tag{2.16}$$

whereas in the case of large τ we have the Gaussian (termed "slow modulation limit" in [9]), that is, with $(D\tau)^{1/2} \gg 1$

$$\lim_{\tau\to\infty} \phi(t) = \phi_\infty(t) = \exp[-(Dt^2/2\tau) + O(\tau^{-2})] \qquad \tau \to \infty \tag{2.17}$$

For the absorption spectrum,

$$I(\omega - \omega_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, \phi(t) e^{-i(\omega-\omega_0)t} \tag{2.18}$$

one obtains, using Eq. (2.16), a Lorentzian in the white noise limit,

$$I_0(\omega - \omega_0) = \frac{1}{\pi} \frac{D}{(\omega - \omega_0)^2 + D^2} \tag{2.19}$$

[3] Note that the Kubo oscillator, Eq. (2.12), is not overdamped, although it is described by a differential equation of first order. Eliminating the real or the imaginary part of the complex variable x reveals the undamped harmonic oscillator if no noise is present.

In contrast, for the limit of large τ, or the "slow modulation limit", Eq. (2.17) yields a Gaussian line shape,

$$I_0(\omega - \omega_0) = \left(\frac{2\pi D}{\tau}\right)^{-1/2} \exp\left[-\frac{(\omega - \omega_0)^2}{2D/\tau}\right] \tag{2.20}$$

Thus, with $\sigma^2 = D/\tau$ held constant, and with $D \ll \sigma$ (i.e. $D \ll \sqrt{D/\tau}$, as $\tau \ll 1$), the line shape in Eq. (2.19) becomes *narrowed* as compared with that given by Eq. (2.20). In nuclear magnetic resonance (NMR), this experimentally well-known effect is called "motional narrowing " [9, 78], whereas in paramagnetic resonance it is called "exchange narrowing" [79, 80]. It is worth emphasizing that *Kubo's* explanation of these line shapes represents an exact theory, valid for arbitrarily long correlation time τ, successfully applied to nonsubtle, experimentally observable features of many-body spin systems, which are naturally subject to *internal* colored noise. It is, historically, one of the best known examples elucidating the effect of fluctuations with nonzero τ in a macroscopic physical system.

Prior to the advent of this theory, Anderson [81] and Kubo [82] considered a Markovian modulation $\xi(t)$ in Eq. (2.12), which became known as the "Kubo–Anderson process". The noise model consisted of a discontinuous Markovian process $z(t)$, which was made up of independently occurring steps with random amplitudes $m_i(t)$ during the interval $t_i \leq t < t_{i+1}$. The amplitudes were distributed with density $\rho(m)$, and the jump times $\{t_i\}$ were determined by a Poissonian distribution. This process, too, has an exponential correlation.

$$\langle z(t)z(s)\rangle = \langle m^2\rangle e^{-\lambda|t-s|} \tag{2.21}$$

with $\langle m_i\rangle = 0$, where λ denotes the Poisson parameter in $P[n(t) = k] = [(\lambda t)^k/k!]\exp(-\lambda t)$, with $n(t)$ describing the number of jumps. When $\rho(m) = (\frac{1}{2})[\delta(m + a) + \delta(m - a)]$ one obtains as a special case the (symmetric) two-state Markov process $z(t)$ discussed, for example, in [7]

$$z(t) = a(-1)^{n(t)} \tag{2.22}$$

with

$$\langle z(t)z(s)\rangle = a^2 e^{-\lambda|t-s|} \tag{2.23}$$

This process is also known as "telegraphic noise" and "dichotomous noise", and it plays an important role in applications in radiophysics [83, 84] and in noise-induced transition phenomena [6, 85]. In particular, for nonlinear colored noise flows of the form

$$\dot{x}(t) = f(x) + g(x)z(t) \tag{2.24}$$

one obtains an exact, retarded but closed master equation, which results in stationary probabilities [83–85], and non-Markovian mean first-passage times [86–90], which can be calculated exactly. An interesting application of nonexponential correlated colored noise has been brought forth by Brissaud and Frisch [91, 92] in order to explain noise-induced Stark broadening. They make use of the "Kangaroo process", that is, a Kubo–Anderson like noise with a correlation function $\langle z(t)z(s)\rangle \propto |t-s|^{-1}$. This noise, however, is not always realistic, at least for short times (or high frequencies), since it clearly does not have an integrable FT. In this chapter, we confine the discussion largely to Gaussian noise. For the many results and applications of stochastic flows driven by, for example, two-state or "dichotomous" noise as given by Eq. (2.24), we refer the reader to Section IV.

III. COLORED NOISE THEORY

A. Characterization of Colored Noise

In the following sections we shall elaborate on various theoretical methods being tailored to investigate stochastic differential equations driven by colored noise sources, Eq. (2.8). These dynamical flows are rather difficult to study because the statistical properties of such flows depend at least on two intrinsic parameters which, apart from the statistical nature of the random force (i.e., Gaussian versus non-Gaussian noise) characterize the correlation function of the noise. The first parameter is its overall *noise intensity* D, which we identify with the zero-frequency part of the power spectrum of the (stationary) noise source $\xi(t)$,

$$2D \equiv \int_{-\infty}^{\infty} |\langle \xi(t)\xi(0)\rangle|\, dt \equiv S_\xi(\omega = 0) \tag{3.1}$$

The second parameter refers to the intrinsic *correlation time* τ of $\xi(t)$,

$$\tau \equiv \frac{\int_0^\infty |\langle \xi(t)\xi(0)\rangle|\, dt}{\langle \xi^2\rangle} \tag{3.2}$$

Thus, the complete theoretical analysis of the noisy dynamical flow involves a study in terms of a two-parameter space (D, τ); certainly a

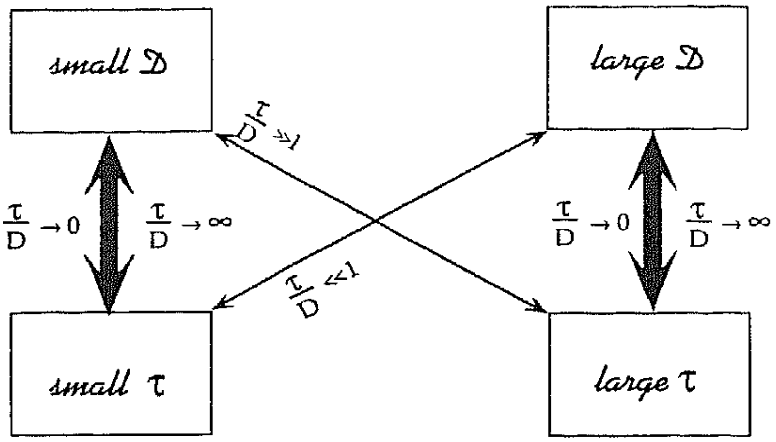


Figure 3.1. Study of colored noise: The various asymptotic regimes in parameter space (τ, D) are indicated by the set of boxes. The arrows connect mutually asymptotic regimes.

rather formidable task. Accurate approximation schemes for colored noise are thus expected only in the asymptotic limits of one or both parameters D and/or τ. These possible asymptotic regimes are depicted in Fig. 3.1. Note, that for the asymptotic regimes indicated in Fig. 3.1 by vertical double arrows, one must also distinguish between the limiting behaviors $\tau/D \to 0$, or $\tau/D \to \infty$, that is, one has to account for the relative change of one parameter compared to the relative change of its accompanying second parameter. Because most of the practical applications are driven by weak noise intensities our primary goal has been to develop workable approximation schemes that hold for weak noise D (see Section V). For a state vector $\mathbf{x} = (x_1, \ldots, x_n)$ we shall assume in the following a noisy, multidimensional dynamical law, which is of the form

$$\dot{x}_\alpha = f_\alpha(\mathbf{x}, \lambda) + \sum_{i=1}^{n} g_{\alpha i}(\mathbf{x}, \lambda)\xi_i(t) \qquad \alpha = 1, \ldots, n \tag{3.3}$$

wherein λ denotes a set of external control parameters, and $\{\xi_i(t)\}$ are *colored* noise forces. In particular, we restrict ourselves to memory-free dynamical laws for $\dot{\mathbf{x}}(t)$, which generally model nonequilibrium phenomena. For the vast literature on colored thermal equilibrium fluctuations obeying a fluctuation–dissipation theorem in terms of a memory friction we refer the reader to the references given in Section VIII. The dynamics in Eq. (3.3) constitute a *non-Markovian process* $\mathbf{x}(t)$; this occurs because $\mathbf{x}(t)$ is with colored noise $\{\xi_i(t)\}$ *not* made up of independent, infinitesimal increments that are statistically uncorrelated [93]. Moreover, it should be noted that the drift vectors $\{f_\alpha(\mathbf{x}, \lambda)\}$ are generally *not* identical with the deterministic flow F_α

$$\dot{x}_\alpha = F_\alpha(\mathbf{x}, \lambda) \tag{3.4}$$

which is the result of the motion of the conditional average $\langle x_\alpha(t)|\mathbf{x}(t)=\mathbf{x}\rangle$, with the noise intensity approaching zero; but $f_\alpha(x, \lambda)$ contain, in general, nonlinear, noise induced effects of the fluctuations $\xi_i(t)$. This constitutes one of the major problems of any phenomenological modeling if partially internal noise sources are involved: The deterministic flow does not even determine the drift term uniquely [94]. The basis for the form in Eq. (3.3) is related to the notion already discussed in Section II. Usually, one assumes that the variables of the system separate into two classes: One class of slowly varying macroscopic variables $\mathbf{x}(t)$, the motion of which is determined by the drift vector $\mathbf{f}(\mathbf{x}, \lambda)$ varying on a time scale τ_s, and the small perturbation around this relevant motion, that is, its irrelevant motion varying on a much shorter time scale $\tau \ll \tau_s$. In this case, the random forces $\xi_i(t)$ are almost white noise forces. This limit is referred to in the literature as (see Section II) short-correlation time limit, "off-white noise", or "pink noise", respectively. In general, however, the experimenter does not monitor the complete set of all slowly varying variables. Then the noise term $\xi_i(t)$ can be correlated on the same, or on an even larger time scale, that is, $\tau \geq \tau_s$. We will refer to this situation as "moderate-to-strong noise color". Before we engage in the study of colored noise driven flows, we first elaborate on general properties of propagating non-Markovian processes

B. Time Evolution of Non-Markovian Processes

The stochastic flow generated by Eq. (3.3) is non-Markovian, whose conditional probability $R(\mathbf{x}t\,|\,\mathbf{y}s)$, $t>s$, depends on the previous history. Thus, the operator $\mathbf{R}(t\,|\,s)$ with the kernel $R(\mathbf{x}t\,|\,\mathbf{y}s)$ is not a linear operator, but depends in a nonlinear way on the initial probability $p_0(x)$ at time t_0 of preparation [95]. Consequently, this operator fails to satisfy the celebrated Bachelier–Smoluchowski–Kólmogorov–Chapman equation [95]. Therefore, an important question is whether it is possible to construct an operator $\mathbf{G}(t\,|\,s)$ for the single-event probability $p(t)[=p(\mathbf{x}, t)]$,

$$p(t) = \mathbf{G}(t\,|\,s)p(s) \tag{3.5}$$

which satisfies the property of a propagator,

$$\mathbf{G}(t\,|\,t_1) = \mathbf{G}(t\,|\,s)\mathbf{G}(s\,|\,t_1) \qquad t \geq s \geq t_1 \tag{3.6}$$

with

$$\mathbf{G}(t^+\,|\,t) = \mathbf{1} \tag{3.7}$$

This concept allows the derivation of a master equation

$$\dot{p}(t) = \mathbf{\Gamma}(t)p(t) \tag{3.8}$$

with the operator $\mathbf{\Gamma}(t)$ given by

$$\mathbf{\Gamma}(t) = \frac{d}{du}\mathbf{G}(u \mid t)\big|_{u=t^+} \tag{3.9}$$

The existence of such propagators $\{\mathbf{G}(s \mid t)\}$, or corresponding (pseudo-Markovian) generators $\mathbf{\Gamma}(t)$, which *yield the identical propagation behavior of the non-Markovian single event probability has been studied some time ago* [96–99]. It has been shown that Eq. (3.5) does not determine the propagator set $\{\mathbf{G}(t \mid s)\}$ uniquely, but there exists many such sets, which in general depend on the initial probability p_0. We must stress that the kernels $\mathbf{G}(\mathbf{x}t \mid ys)$ of these propagator sets are in general *different* from the conditional probability $R(\mathbf{x}t \mid \mathbf{y}s)$; thus they cannot be used to calculate correlation functions such as $C_{ij}(t,s) = \langle x_i(t)x_j(s)\rangle$, or conditional averages, $\langle \mathbf{x}(t) \mid \mathbf{x}(s) = \mathbf{x}\rangle$. There is one important exception, however: If the system has been prepared at time t_0 without any memory of the past, and for time sets for which $R(t \mid t_0)$ is nonsingular for $t > t_0$, the operator

$$\mathbf{G}(t \mid s) = \mathbf{R}(t \mid t_0)\mathbf{R}(s \mid t_0)^{-1} \tag{3.10}$$

or

$$\mathbf{\Gamma}(t) = \dot{\mathbf{R}}(t \mid t_0)\mathbf{R}(s \mid t_0)^{-1} \tag{3.11}$$

is independent of p_0; that is, $\mathbf{G}(t \mid t_0) = \mathbf{R}(t \mid t_0)$. Therefore, $\mathbf{G}(t \mid t_0)$ can be invoked to calculate *initial*, non-Markovian correlations, such as $\langle \mathbf{x}(t)\mathbf{x}(t_0)\rangle$. For the dynamical flow in Eq. (3.3) we shall *further* assume for the following that the system has been prepared at time t_0 without any memory of the past and *without correlations between system and environment*, that is, $p_0(x, \mathit{environment}) = p_0(x)p_0(\mathit{environment})$. This preparation scheme (π) will be termed "correlation-free preparation", with the conditional preparation function $W_\pi[\text{environment} \mid \mathbf{x}(t_0)] = p_0(\text{environment})$ [97].

This concept can be generalized [100] for any preparation scheme characterized by a preparation function $W_\pi[\text{environment} \mid \mathbf{x}(t_0)]$, characterizing the distribution of microstates of the environment for given macrostate $\mathbf{x}_0 = \mathbf{x}(t_0)$ [100]. Of particular importance is the stationary preparation W_s for which $W_s[\text{environment} \mid \mathbf{x}_0]p_0(\mathbf{x}_0)$ represents the *stationary* probability of the total system (system plus bath). Then

$\mathbf{R}_s(t\,|\,t_0)$ becomes time homogeneous, that is, $\mathbf{R}_s(t\,|\,t_0) = \mathbf{R}_s(t+s\,|\,t_0+s)$, and thus can be used to calculate stationary correlation properties [95–100].

C. Correlation Formulas between Noise Functionals

In this section we restrict for the sake of clarity and simplification only, the further discussion to 1-D stochastic flows in Eq. (2.8), that is,

$$\dot{x} = f(x) + g(x)\xi(t) \tag{3.12}$$

Moreover, for a multiplicative noise function $g(x)$, which does not vanish, we can use the transform: $x \to y = \int^x [dz/g(z)]$, $h(x) = f(x)/g(x)$, to obtain the simplified, additive noise equation

$$\dot{y} = h(y) + \xi(t) \tag{3.13}$$

Its single event probability $p_t(x)$ is given in terms of an average over the noise realizations of $\xi(t)$, that is,

$$p_t(x) = \langle \delta(x(t) - x) \rangle \tag{3.14}$$

The rate of change of $p_t(x)$ then obeys

$$\begin{aligned} \dot{p}_t(x) = -\frac{\partial}{\partial x}\langle \delta(x(t)-x)\dot{x}(t)\rangle = -\frac{\partial}{\partial x}[f(x)p_t(x)] - \frac{\partial}{\partial x}\, g(x) \\ \times \langle \xi(t)\delta(x(t)-x)\rangle \end{aligned} \tag{3.15}$$

We note that a colored noise master equation for the probability $p_t(x)$ introduces a correlation between the noise $\xi(t)$ and the functional $\mathscr{F}\{\xi\} = \delta(x(t)-x)$ of the colored noise source $\xi(s)$, $t \geq s \geq t_0$ (t_0 is the time of preparation). The expression in Eq. (3.15) can only be disentangled further if we explicitly invoke the statistical properties of the noise $\xi(t)$. We now give (without proof) some important relations that are needed for the derivation of a colored noise master equation. For an explicit derivation of these relations the readers are referred to the original paper [101] and the reviews [102, 103]. Moreover, we shall explicitly assume that the random force $\xi(t)$ is of vanishing mean,

$$\langle \xi(t) \rangle \equiv C_1(t) = 0 \tag{3.16}$$

Let $F\{\xi\}$, $G\{\xi\}$ denote two functionals of $\xi(t)$. Then we have with

$F\{\xi\} = \xi(t)$ and $G\{\xi\} = \delta(x(t) - x)$ the important relation [101, 102]

$$\langle \xi(t)\delta(x(t) - x) \rangle = \sum_{n=1}^{\infty} \left(\frac{1}{n!}\right) \int_{t_0}^{t} \cdots \int_{t_0}^{t} dt_1 \cdots dt_n C_{n+1}(t, t_1, \ldots, t_n) \times \left\langle \frac{\delta^n[\delta(x(t) - x)]}{\delta\xi(t_1) \cdots \delta\xi(t_n)} \right\rangle \tag{3.17}$$

Here, $C_m(t_1, \ldots, t_m)$ denotes the mth order cumulant of the noise $\xi(t)$. The notation $\delta F\{\xi\}/\delta\xi(s)$ denotes the functional derivative; it can be viewed of as a usual derivative if we set

$$\frac{\delta F\{\xi\}}{\delta\xi(s)} = \left. \frac{dF\{\xi(t) + \lambda\delta(t - s)\}}{d\lambda} \right|_{\lambda=0} \tag{3.18}$$

assuming that both sides exist.

For a stationary *Gaussian random force* $\xi(t)$ one then finds with $C_2(t, s) \equiv C(t - s) = \langle \xi(t)\xi(s) \rangle - \langle \xi(t) \rangle \langle \xi(s) \rangle = \langle \xi(t)\xi(s) \rangle$, the useful result [104, 105]

$$\langle \xi(t)G\{\xi\} \rangle = \int_{t_0}^{t} C(t - s) \left\langle \frac{\delta G\{\xi\}}{\delta\xi(s)} \right\rangle ds \tag{3.19}$$

For two functionals $F\{\xi\}$, $G[\xi]$ one obtains for stationary Gaussian noise [102, 103]

$$\langle F\{\xi\}G\{\xi\} \rangle = \langle F\{\xi\} \rangle \langle G\{\xi\} \rangle + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{t_0}^{t} \cdots \int_{t_0}^{t} \left\langle \frac{\delta^n F\{\xi\}}{\delta\xi(t_1) \cdots \delta\xi(t_n)} \right\rangle \times \left\langle \frac{\delta^n G\{\xi\}}{\delta\xi(s_1) \cdots \delta\xi(t_n)} \right\rangle \prod_{i=1}^{n} C(t_i - s_i)\, dt_i\, ds_i \tag{3.20}$$

With these results in hands we are well equipped to tackle the master equation for colored noise in Eq. (3.15) in greater detail.

D. The Colored Noise Master Equation

Here we only consider the case of stationary Gaussian noise $\xi(t)$ of vanishing mean $\langle \xi(t) \rangle = 0$ and correlation $\langle \xi(t)\xi(s) \rangle = C(t - s)$ [see Eq. (3.19)]. Then, the rate of change of the single-event probability $p_t(x) =$

$\langle \delta(x(t)-x) \rangle$ from Eqs. (3.15) and (3.19) is given by

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}[f(x)p_t(x)] - \frac{\partial}{\partial x} g(x) \int_{t_0}^{t} C(t-s)\left\langle \frac{\delta[\delta(x(t)-x)]}{\delta\xi(s)} \right\rangle ds \tag{3.21}$$

With

$$\frac{\delta}{\delta\xi(s)}\delta(x(t)-x) = \left[-\frac{\partial}{\partial x}\delta(x(t)-x)\right]\frac{\delta x(t)}{\delta\xi(s)} \tag{3.22}$$

we have from the dynamical law in Eq. (3.12) for the functional derivative $\delta x(t)/\delta\xi(s)$ the integral equation [101, 102]

$$\frac{\delta x(t)}{\delta\xi(s)} = \theta(t-s)\left\{ g(x(s)) + \int_s^t du \left(\frac{\partial \dot{x}(u)}{\partial x(u)}\right)\frac{\delta x(u)}{\delta\xi(s)} \right\} \tag{3.23}$$

Here $\theta(t-s)$ is the unit step function expressing causality. Its solution is readily found to read

$$\frac{\delta x(t)}{\delta\xi(s)} = \theta(t-s)g(x(s)) \exp \int_s^t \frac{\partial \dot{x}(u)}{\partial x(u)} du \tag{3.24}$$

$$= \theta(t-s)g(x(s)) \exp \int_s^t \{f'(x(u)) + g'(x(u))\xi(u)\}\, du \tag{3.25}$$

where $h'(x)$ denotes differentiation $dh(x)/dx$. Equation (3.25) can be recast in alternative, and more appealing form [103, 106]

$$\frac{\delta x(t)}{\delta\xi(s)} = \theta(t-s)g(x(t)) \exp \int_s^t \left\{ f'(x(u)) - f(x(u))\frac{g'(x(u))}{g(x(u))} \right\} du \tag{3.26}$$

Combining Eq. (3.22) with Eqs. (3.21) and (3.26) we then have for *Gaussian noise* $\xi(t)$ the formally exact result [101–103],

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}[f(x)p_t(x)] + \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) \cdot \int_{t_0}^{t} ds C(t-s) x \Big\langle \delta(x(t)-x)$$
$$\times \exp \int_s^t [f' - (fg'/g)]\, du \Big\rangle \tag{3.27}$$

which with different notation has been given first in [101]. At this point, the exact relation in Eq. (3.27) cannot be generally simplified further. Because of the function $\delta(x(t)-x)$, a closed expression that involves only the single-event probability $p_u(x), t \geq u \geq t_0$ only results if either $\delta x(t)/\delta\xi(s)$ does not depend on the process $x(t)$, or if it depends on $x(s)$ solely on its endpoint time t. Classes of such exact, closed colored noise master equations (e.g., all linear processes, $\dot{x} = a + bx + c\xi(t)$, nonlinear processes driven by two-state noise and/or white noise) have been discussed in [101–103]. The main result obtained in Eq. (3.27) will serve as our appropriate starting point in Section V to construct various approximation schemes.

E. Master Equation for a Linear Process Driven by Gaussian Colored Noise

We shall illustrate the result in Eq. (3.27) for a linear colored noise process. Let

$$\dot{x} = a - bx + \xi(t) \tag{3.28}$$

From Eq. (3.26) we obtain

$$\frac{\delta x(t)}{\delta\xi(s)} = \theta(t-s)\exp[-b(t-s)] \tag{3.29}$$

Thus, the master equation, Eq. (3.27), takes on a Fokker–Planck like form, that is,

$$\dot{p}_t(x) = -a\frac{\partial}{\partial x}p_t(x) + b\frac{\partial}{\partial x}[xp_t(x)] + \left\{\int_{t_0}^{t} ds C(t-s)\exp[-b(t-s)]\right\} \times \frac{\partial^2}{\partial x^2}p_t(x) \tag{3.30}$$

Note that with time-independent drift coefficients in Eq. (3.28) and stationary Gaussian noise $\xi(t)$ the effective diffusion in Eq. (3.30) is *time dependent*, and it may even take on negative values when $C(t-s)$ is, for example, an oscillatory-like function. The solution of Eq. (3.30) constitutes with $p_0(x) = \delta(x - x_0)$ a Gaussian, non-Markovian probability, which explicitly reads [96, 107]

$$p_t(x) \equiv R_t(x\,|\,x_0) = \frac{\exp\{-\frac{1}{2}\alpha^{-1}(t,t_0)[x-\beta(t,x_0,t_0)]^2\}}{[2\pi\alpha(t,t_0)]^{1/2}} \tag{3.31}$$

where

$$\alpha(t, t_0) = \int_{t_0}^{t} \exp[-2b(t-s)]D(s)\, ds \tag{3.32}$$

$$D(s) = 2\int_{t_0}^{s} drC(s-r)\exp[-b(s-r)] \tag{3.33}$$

$$\beta(t, x_0, t_0) = x_0 \exp[-b(t-t_0)] + a\int_{t_0}^{t} \exp[-b(t-s)]\, ds \tag{3.34}$$

Thus, $\mathbf{R}(t \mid t_0)$ is a Gaussian, and with the initial probability p_0 also Gaussian, the time evolution of $p(t) = \mathbf{R}(t \mid t_0)p_0$ will remain a Gaussian. This clearly no longer holds for a non-Gaussian initial probability $p_0(x_0)$. We close this section with some general observations about non-Markovian master equations as exhibited by Eq. (3.27). The non-Markovian character of the process $x(t)$, Eq. (3.12), is reflected by the dependence of $\dot{p}_t(x)$ on the initial time of preparation $t = t_0$ in Eq. (3.27). Moreover, the initial rate of change of $p_t(x)$ is given by

$$\dot{p}_{t=t_0}(x) = -\frac{\partial}{\partial x}[f(x)p_{t=t_0}(x)] \tag{3.35}$$

Equation (3.35) holds true for any noise statistics with nonsingular cumulants $C_n(t_1, \ldots, t_n)$.

IV. COLORED TWO-STATE NOISE

In Section III the emphasis has been put on the equation of motion for the time evolution of the single-event probability, that is, the master equation. The environmental colored noise fluctuations are frequently based on the cumulative effect of an abundance of environmental factors. The central limit theorem implies then that the fluctuations are distributed Gaussian. As demonstrated in Section III, any Gaussian process leads to a closed equation of motion of the probability. Moreover, its solution for the conditional probability is solely determined by the vector of mean values and the covariance matrix [51]. Moreover, any process resulting from a linear transformation of Gaussian processes (Markovian or non-Markovian) is again Gaussian. Colored noise processes that are the result of a *nonlinear* transformation of a Gaussian process can also be considered to be exactly solvable. A set of criteria, which show when a process $y(t)$ can be related (via a nonlinear transformation) to a linear transformation of a Gaussian process, can be inferred from the literature [108]. The models of Hongler [109] are precisely of this form, being a nonlinear transformation of a Gaussian process [108, 110]. For Gaussian colored noise sources that result from an embedding of an n-dimensional Gauss–Markov process the statistical properties together with the spectral behavior are known explicitly [111]. Likewise, colored noise Markovian processes, which via the Darboux procedure, the Abraham–Moses procedure, the Pursey procedure, or the supersymmetry procedure, are isospectral with the quantum harmonic oscillator [112–117] can be considered as exactly solvable. Such specific examples are the models by Hongler and Zheng [118, 119] and by Razavy [120]. Other examples of noise sources, described by a Fokker–Planck process that can be related to exactly solvable 1-D Schrödinger equations [5, 7, 121], can be found in [122]. Next we shall focus on a class of exactly solvable colored noise driven nonlinear systems, whose stationary probability and mean first-passage times can be obtained (up to quadratures) in exact closed form for an arbitrarily chosen nonlineary $f(x)$ [see Eq. (2.8)].

A. Correlated Two-State Noise

A class of correlated noise that has found applications in numerous systems is *two-state noise*, that is, a noise that switches back and forth between two prescribed state values with a waiting time probability that is Poissonian. Note that within any switching process in which intradomain-of-attraction motion is filtered out can satisfactorily be modeled by such two-state noise. For the sake of simplicity we confine the discussion here to symmetric two-state noise (for asymmetric two-state noise the reader may consult chapter 9 in [6]), which switches back and forth between the state $\xi = a$ and $\xi = -a$; that is,

$$\xi(t) = a(-1)^{n(t)} \tag{4.1}$$

where $n(t) \equiv n(0, t)$ is a Poisson process with parameter λ. Put differently, $\xi(t)$ in Eq. (4.1) is a two-state Markovian process [6, 7]. Let us investigate its statistical properties. With $\xi(0) = a$, its mean reads,

$$\begin{aligned} \langle \xi(t) \rangle &= a \sum_{m=0}^{\infty} (-1)^m P[n(t) = m] \\ &= a \exp(-\lambda t) \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda t)^m}{m!} \\ &= a \exp[-2\lambda t] \qquad t > 0 \end{aligned} \tag{4.2}$$

Likewise one readily evaluates the correlation as

$$\begin{aligned}\langle \xi(t)\xi(s)\rangle &= a^2\langle(-1)^{n(0,t)+n(0,s)}\rangle \qquad t<s\\ &= a^2\langle(-1)^{2n(0,s)+n(s,t)}\rangle\\ &= a^2\langle(-1)^{n(s,t)}\rangle\\ &= a^2\exp(-2\lambda|t-s|)\end{aligned} \tag{4.3}$$

In particular, note that the correlation is time homogeneous although $\xi(t)$ in Eq. (4.2) is not stationary. Let us now distribute the initial value. By use of the symmetric initial probability for the state variable $\rho_0(u) = \frac{1}{2}\{\delta(u+a)+\delta(u-a)\}$ the noise $\xi(t)$ assumes a zero mean. For the sequence of time instants $[t_1 \geq t_2 \geq \ldots \geq t_n]$ we then find for the nth correlation m_n

$$m_n = \langle \xi(t_1)\cdots\xi(t_n)\rangle = \langle \xi(t_1)\xi(t_2)\rangle\langle \xi(t_3)\cdots\xi(t_n)\rangle \tag{4.4}$$

or

$$m_n = a^2\exp[-2\lambda(t_1-t_2)]m_{n-2} \tag{4.5}$$

Here we used the fact that the statistics of nonoverlapping time intervals are independent of each other. The result in Eq. (4.5) can be generalized to yield

$$\langle \xi(t_1)\xi(t_2)G[\xi(s)]\rangle = \langle \xi(t_1)\xi(t_2)\rangle\langle G[\xi(s)]\rangle \tag{4.6}$$

where with $s \leq t_2 \leq t_1$, $G[\xi(s)]$ is a functional of the two-state noise. The analogue of the Furutsu [104] and Novitrov [105] correlation formula for the case of two-state noise has been derived by Klyatskin [83, 84]; that is, with $\theta(x)$ being the step function

$$\langle \xi(t)G[\xi]\rangle = a^2\int_0^t ds\,\exp[-2\lambda(t-s)]\cdot\left\langle \frac{\delta}{\delta\xi(s)}G[\xi(u)\theta(s-u^-)]\right\rangle \tag{4.7}$$

where the noise dependence in $G[\xi]$ is *switched-off* for times $u>s$; that is, $\xi \equiv 0$ for $u>s$.

B. Master Equation for Colored Two-State Noise Driven Nonlinear Flows

Give the nonlinear flow in Eq. (3.12), that is, $\dot{x} = f(x) + g(x)\xi(t)$, with $\xi(t)$ being *correlated two-state noise* in Eq. (4.1), the rate of change of $p_t(x)$ in

Eq. (3.15) can, by use of Eq. (4.7), be recast as

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}[f(x)p_t(x)] - \frac{\partial}{\partial x} g(x)a^2 \int_0^t ds[\exp -2\lambda(t-s)]\left\langle \frac{\delta}{\delta\xi(s)} \hat{\delta}_t(x(t)-x)\right\rangle \tag{4.8}$$

where $\hat{\delta}_t = \delta_t[\xi(u)\theta(s-u)]$ and $t_0 = 0$. From the dynamical equation of motion we find that $\delta_t \equiv \delta(x(t)-x)$ satisfies

$$\dot{\delta}_t = -\frac{\partial}{\partial x}[f(x)\delta_t] - \frac{\partial}{\partial x}[g(x)\xi(t)\delta_t] \tag{4.9}$$

and $\hat{\delta}_{t=s} = \delta(x(s)-x)$. With $\xi(u)$ being switched off for times $u > s$ we thus find for $\hat{\delta}_t$ the differential equation

$$\frac{\partial}{\partial t}\hat{\delta}_t = -\frac{\partial}{\partial x}[f(x)\hat{\delta}_t] \qquad t > s \tag{4.10}$$

Therefore, its solution can be cast in operator form to read

$$\hat{\delta}_t = \exp\left\{-\frac{\partial}{\partial x} f(x)(t-s)\right\}\delta(x(s)-x) \tag{4.11}$$

Observing that $(\delta/\delta\xi_s)\delta(x(s)-x) = -\frac{\partial}{\partial x} g(x)\delta(x(s)-x)$ we end up with a closed-form master equation

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}[f(x)p_t(x)] + a^2\frac{\partial}{\partial x} g(x) \int_0^t ds \exp\left\{-(t-s)[2\lambda + \frac{\partial}{\partial x} f(x)]\right\}\frac{\partial}{\partial x} g(x)p_s(x) \tag{4.12}$$

The stationary probability $p(x;\lambda)$ is obtained from Eq. (4.12) if we integrate between $s\varepsilon[0,\infty]$, and equate the probability current at zero value, that is,

$$f(x)p(x;\lambda) = a^2 g(x)\left[\frac{1}{2\lambda + (d/dx)f(x)}\right]\frac{d}{dx}\{g(x)p(x;\lambda)\} \tag{4.13}$$

After multiplication from left with $g^{-1}[2\lambda + (d/dx)f]$ one finds an ordinary first-order equation for $p(x;\lambda)$. Its solution therefore is readily found as [6, 83–85, 102]

$$p(x;\lambda) = Z^{-1}\frac{|g(x)|}{[a^2g^2(x)-f^2(x)]}\exp\left\{2\lambda\int^x dy\,\frac{f(y)}{[a^2g^2(y)-f^2(y)]}\right\} \tag{4.14}$$

where Z^{-1} is the normalization constant. Note that $p(x;\lambda)$ has a support on all those x values for which the term $[a^2g^2(x)-f^2(x)]$ takes on a positive value! With the correlation time $\tau \equiv (2\lambda)^{-1}$, $p(x;\lambda)$ depends exponentially on the colored noise correlation τ. We conclude this section by presenting (without proof) a few further relations that are of use in applying two-state noise in colored noise driven flows.

The curtailed characteristic functional

$$\phi_t[v] = \langle \exp i\int_0^t ds\xi(s)v(s)\rangle \tag{4.15}$$

obeys the exact second ordinary differential equation

$$\frac{d^2}{dt^2}\phi_t + \left[2\lambda - \frac{1}{v(t)}\frac{dv(t)}{dt}\right]\frac{d\phi_t}{dt} + a^2v^2(t)\phi_t = 0 \tag{4.16}$$

which with $\phi_0 = 1$, and $\dot{\phi}_{t=0} = 0$ generally is not explicitly solvable. Equivalently, Eq. (4.16) can be recast as an integrodifferential equation

$$\frac{d}{dt}\phi_t = -a^2v(t)\int_0^t dsv(s)\exp[-2\lambda(t-s)]\phi_s \tag{4.17}$$

From a nonequilibrium Brownian motion driven by correlated two-state noise $\xi(t)$, that is, with $a^2 \equiv 2kT\gamma\lambda$

$$\begin{aligned} \dot{x} &= u \\ \dot{u} &= f(x) - \gamma u + \xi(t) \end{aligned} \tag{4.18}$$

we obtain for the stationary probability $p(x,u;\lambda)$ the exact equation

$$\Gamma p(x,u;\lambda) = \frac{\partial}{\partial u}\left(\frac{2kT\gamma\lambda}{2\lambda+\Gamma}\right)\frac{\partial}{\partial u}p(x,u;\lambda) \tag{4.19}$$

where $\Gamma \equiv u(\partial/\partial x) + f(x)(\partial/\partial u) - \gamma(\partial/\partial u)u$ is the deterministic drift operator. With the parameter $\lambda \to \infty$, telegraphic noise approaches Gaussian white noise of vanishing mean and correlation $2kT\gamma\delta(t)$. With $\lambda \to \infty$, Eq. (4.19) reduces to the usual equation for Brownian motion in a force field $f(x) = -[dU(x)/dx]$. Equation (4.19) can further be recast as a partial differential equation, that is,

$$\left\{(2\lambda + \Gamma)^2\Gamma - 2kT\gamma\lambda(2\lambda + \Gamma)\frac{\partial^2}{\partial u^2}\right.$$
$$\left. - \gamma(2\lambda + \Gamma)\Gamma + 2kT\gamma\lambda\frac{\partial^2}{\partial x\,\partial u} - \frac{df}{dx}\Gamma\right\}p(x, u; \lambda) = 0 \quad (4.20)$$

which generalizes the usual Klein–Kramers equation [45, 46, 123]. With Eq. (4.18) violating the fluctuation–dissipation relation for any finite λ, the solution of Eq. (4.20) clearly no longer exhibits the Boltzmann form, but the coordinate x and its velocity $\dot{x} = u$ now become statistically dependent variables.

C. Mean First-Passage Times

A quantity that carries valuable dynamic information is the first average of the first-passage time random variable, the so-called mean first-passage time (MFPT). The MFPT can be used to characterize relevant time scales in nonlinear dynamical problems such as they originate in chemical kinetics, decay of arbitrary metastable states, decay of unstable states, and nucleation [123], to name only a few. With a colored noise driven flow, the concept of the MFPT becomes rather nontrivial [124, 125]. For two-state noise with exponentially distributed waiting time, however, the complexity can be handled in analytical closed form.

In Section IV.B we already made extensive use of the fact that the stationary probability obeys an ordinary differential equation being of first order. Not totally surprising, this fact also holds true for the derivative of the MFPT [86].

Let $T_+(y)$ denote the MFPT for a particle, which started out at initial time $t_0 = 0$ at $x = y$, with initial velocity $\xi(0) = +a$, i.e. $\rho_0(u) = \delta(a - u)$. Here y is restricted to some a priori prescribed interval $I = [x_A, x_B]$. $T_-(y)$ is the MFPT for a particle starting out with initial velocity $\xi(0) = -a$, i.e., $\rho_0(u) = \delta(a + u)$. For the dynamical flow

$$\dot{x} = f(x) + g(x)\xi(t) \quad (4.21)$$

The coupled equation for $T_\pm(y)$ is explicitly given by [86, 90]

$$\begin{aligned}(f + ag)T'_+ - \lambda T_+ + \lambda T_- &= -1 \\ (f - ag)T'_- - \lambda T_- + \lambda T_+ &= -1\end{aligned} \quad (4.22)$$

Here the prime denotes differentiation after y, that is, $T'(y) = dT/dy$.

Upon eliminating T_+ or T_- one obtains an ordinary first-order differential equation for T'_+ or $T^{-\prime}$, respectively. The MFPT is therefore readily integrated if only the boundary conditions are known. For absorbing boundaries at x_A and x_B the exact MFPT has been obtained first in [86], and has been reobtained by use of alternative techniques in [87–89, 126, 127]. The essential difficulty in obtaining the MFPT for non-Markovian processes is the incorporation of the correct boundary conditions [86, 90, 124, 125]. For a detailed discussion of implementing the correct absorbing and/or reflecting boundary conditions we refer the interested reader to the original literature [86, 90]. For the important problem of escape from a domain of attraction, where with $x_S > x_A$, x_S being a metastable state and $x_* < x_B$, x_* denoting an unstable (barrier toplike) state, the MFPT with x_A being *reflecting* and x_B being *absorbing* yields the time scale for the escape; and its inverse yields the reaction rate, respectively. This MFPT can then be obtained in closed form, that is, from Eq. (4.14) in [90] we have with $D \equiv a^2/2\lambda$ and $p \equiv p(x; \lambda)$ given in Eq. (4.12)

$$\begin{aligned} T_+^{\mathrm{ref}}(y) = {} & \int_y^{x_B} dz \frac{g}{D[g + (f/a)]^2[(g - (f/a)]p} \int_{x_A}^{z} du g^{-1}(g + f/a)p \\ & + \frac{D}{a} p(x_A; \lambda)[g^2 - (f/a)^2]_{x=x_A} g^{-1}(x_A) \\ & \cdot \int_y^{x_B} du \frac{g}{D[g + (f/a)]^2[g - (f/a)]p} \end{aligned} \tag{4.23}$$

and a similar expression holds for $T_-^{\mathrm{ref}}(y)$ [90]. At weak noise strength $D \ll 1$, the use of the steepest descent approximation yields the reaction rate $k \equiv 1/T_+^{\mathrm{ref}}$ as [90, 128–131]

$$k = \frac{1}{2\pi} [|f'(x_1)|f'(x_*)]^{1/2} \exp(-\Delta\phi/D) \tag{4.24}$$

with the "Arrhenius energy" given by

$$\Delta\phi = -\int_{x_1}^{x_*} du \frac{f}{[g - (f/a)][g + (f/a)]} \tag{4.25}$$

Alternatively, this reaction rate can be evaluated directly (via the method of flux over population), if one solves Eq. (4.12) for a constant, vanishing probability flux [128, 129]. The result again can be cast into a closed form involving only two quadratures in terms of the stationary probability. This exact quadrature expression thus intrinsically incorporates all the corrections to the steepest-descent expression in Eq. (4.24). These latter corrections are of relevance for finite but small effective barriers $\Delta\phi/D$.

V. COLORED NOISE THEORY: APPROXIMATION SCHEMES

Apart from the specific set of classes of systems (see Sections III.D and IV) that yield a closed-form master equation, the relation in Eq. (3.27) cannot be evaluated explicitly. Further theoretical progress must therefore invoke some form of approximation. In practice, such approximate schemes become useful only if the approximation reduces to an approximate Fokker–Planck process, or at best, a Fokker–Planck like master equation for the single event probability $p_t(x)$. The tacit assumption with such a procedure is that the resulting approximate solution in fact presents a useful estimate for the actual non-Markovian process in Eq. (3.12). Of course, such an approximation is not expected to describe *all* of the statistical information of the true non-Markovian process but only some limited statistical quantities such as its stationary probability, or its transient initial correlation function. As it will become clear below, approximation schemes that also approximate the dynamics equally well, such as the stationary two-point correlation function, the relaxation time, or its mean first-passage time, are much more difficult to obtain. Next, we shall report, extend, and interpret various novel approximation schemes for colored noise driven Langevin equations. Particular emphasis will be put, wherever possible, on a study of the regime of validity of such corresponding approximation schemes.

A. Small Correlation Time Expansion

If the noise color is "off-white", that is, close to the white noise limit, it seems appropriate to search for an effective Fokker–Planck like equation. Our starting point for this approximation is the formally exact master equation in Eq. (3.21) or (3.27). If we expand $\delta x(t)/\delta\xi(s)$ into a Taylor series around the latest time t,

$$\frac{\delta x(t)}{\delta\xi(s)} = \frac{\delta x(t)}{\delta\xi(t)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}(t-s)^n \left[\frac{d^n}{ds^n}\frac{\delta x(t)}{\delta\xi(s)}\right]_{s=t^-} \tag{5.1}$$

one finds from Eqs. (3.12) and (3.23)

$$\frac{\delta x(t)}{\delta \xi(s)} = \theta(t-s)\Big\{ g(x(t)) + [g'(x(t))f(x(t)) - g(x(t))f'(x(t))](t-s)$$
$$+ \Big[\Big\{ f^2 \Big[f\Big(\frac{g}{f}\Big)' \Big]' - g^2 \Big[g\Big(\frac{f}{g}\Big)' \Big]' \xi(t) \Big\}_{x=x(t)} (t-s)^2 \Big] + \cdots \Big\} \tag{5.2}$$

Here, the prime (′) again denotes differentiation with respect to x. Obviously, this expansion involves the noise $\xi(t)$ already in second order. This leads to new correlations that again must be disentangled with relations such as Eq. (3.19). We will now specify the Gaussian noise $\xi(t)$ to the Ornstein–Uhlenbeck process,

$$C_2(t-s) = \frac{D}{\tau} \exp(-|t-s|/\tau) \tag{5.3}$$

If we *truncate* Eq. (5.1) at first order (i.e., $n = 1$) one finds for the master equation in Eq. (3.27) with $t_0 \equiv 0$ [65, 66]

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}[f(x)p_t(x)] + D\frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x}[g(x)h(x,t)p_t(x)] \tag{5.4}$$

where

$$h(x,t) = [1 - \exp(-t/\tau)] + \tau g(x)\Big(\frac{f(x)}{g(x)}\Big)' \Big\{ [1 - \exp(-t/\tau)] - \frac{t}{\tau}\exp(-t/\tau) \Big\} \tag{5.5}$$

This is the time-dependent small τ approximation describing the time evolution of $p_t(x)$. Next we address the long-time limit, that is, we neglect the transients in Eq. (5.5) to obtain the *standard small τ approximation*

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}[f(x)p_t(x)] + D\Big\{ \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x)[1 + \tau g(x)\{f(x)/g(x)\}'] \Big\} p_t(x) \tag{5.6}$$

By far this presents the most often used small correlation time approximation [55–77, 132, 133]. The stationary probability $p(x;\tau)$ is given by

$$p(x;\tau)=\frac{Z^{-1}}{|g(x)\{1+\tau g(x)[f(x)/g(x)]'\}|}\cdot\exp\int^{x}\frac{f(y)\,dy}{Dg^{2}(y)\{1+\tau g(y)[f(y)/g(y)]'\}}\tag{5.7}$$

where Z is the normalization constant. This very result has been repeatedly derived in the literature by a variety of methods mentioned in Section II.B below Eq. (2.11). Some authors [57, 61–63, 66, 67, 132, 133] also consider higher order corrections to Eq. (5.6) being proportional to $D\tau^{n}$. By doing so, however, one simply neglects the noise-dependent contributions of the type in Eq. (5.2), which also yield additional Fokker–Planck terms together with non-Fokker–Planck terms: As first pointed out in [134], and later reiterated in [135, 136–138], such a formal ordering of the τ expansions is fictitious, and does not improve the approximation consistently. In short, these higher order terms are of the same order as other neglected Fokker–Planck and non-Fokker–Planck terms. We next state a few properties of the approximation in Eq. (5.6).

1. For $\tau=0$, one recovers the white noise result from both Eqs. (5.4) and (5.6); that is, the white noise Fokker–Planck equation for a white noise Langevin equation Eq. (3.12), being interpreted in the Stratonovich sense.
2. The drift and diffusion coefficients in Eq. (5.6) differ in order τ from the corresponding Markovian Fokker–Planck equation. In particular, with increasing τ the diffusion coefficient in Eq. (5.6) may take on zeros, and negative values, thereby introducing unphysical, approximation-related boundaries for the non-Markovian process.
3. With the diffusion in Eq. (5.6) generally not satisfying strict positivity, there exists no white noise Langevin equation which is stochastically equivalent with Eq. (5.6), that is, $p_t(x)$ cannot be sampled in terms of random trajectories.
4. The solution $p_t(x\,|\,x_0)$ of the time-dependent equation Eq. (5.4) with initial condition $p_0=\delta(x-x_0)$ represents an approximation to the conditional probability $R(xt\,|\,x_0t_0=0)$ of the non-Markovian process with correlation-free initial preparation (see Section III.B). Thus, Eq. (5.4) can be utilized for the calculation of *initial* correlations in the regime of small noise color τ.
5. As demonstrated below, the regime of validity of the small τ

approximation is limited to small correlation times $\tau \to 0$, with (τ/D) being a small quantity, and to regimes in state space where $\tau g(x)[f(x)/g(x)]' < 1$. In particular with (τ, D) both understood as being dimensionless, the weak noise asymptotic regime $\tau \to 0$; $\tau/D \gg 1$ is not within the regime of applicability of the small τ approximation in Eqs. (5.4) and (5.6).

Now, let us consider the contribution of the second Taylor coefficient in Eq. (5.2) in greater detail. This part contributes, with $g(x) \equiv 1$, to the master equation in Eq. (5.5) the term (see in [134])

$$D\tau^2 \frac{\partial^2}{\partial x^2} \left\{ [(f'(x))^2 - f(x)f''(x)]p_t(x) \right.$$
$$\left. + \frac{D}{\tau} f''(x) \frac{\partial}{\partial x} \int_0^t \left\langle \delta(x(t) - x) \frac{\delta x(t)}{\delta \xi(s)} \right\rangle \exp\left(-\frac{(t-s)}{\tau} \right) ds \right\} \quad (5.8)$$

If we approximate $\delta x(t)/\delta\xi(s)$ by its first term [see Eq. (3.23)], that is, $\delta x(t)/\delta\xi(s) \simeq 1$, we find, upon neglect of transients, the following third-order non-Fokker–Planck contribution to Eq. (5.6)

$$D^2\tau^2 \frac{\partial^2}{\partial x^2} f(x) \frac{\partial}{\partial x} p_t(x) \quad (5.9)$$

By use of a nonequilibrium potential $\phi_t(x, \tau)$; that is, $p_t \propto \exp(-\phi_t(x, \tau)/D)$, we note that each Kramers–Moyal moment yields a contribution of order D^{-1}, D^0, and higher to $\dot{p}_t(x)$. If we collect the singular terms we find the following contributions to $\dot{p}_t(x)$ [134] with Eq. (5.8)

$$\dot{p}_t(x) \sim p_t \left\{ \frac{A_1^{(1)}}{D} + \frac{A_1^{(2)}}{D} + \frac{\tau}{D} [A_2^{(2)} + \tau A_3^{(2)} + \tau A_1^{(3)}] \right\} + O(D^0) \quad (5.10)$$

Here, the superscript in the functions $\{A_n^{(i)}\}$ indicates a contribution stemming from the i-th order Kramers–Moyal moment. Thus, we immediately see that it is not consistent to keep contributions of order $D\tau^n$, $n > 1$ in the Fokker–Planck like equation in Eq. (5.5), while at the same time neglecting non-Fokker–Planck terms. Moreover, with $\tau \neq 0$, the correction to the white noise limit should be small, that is, the *parameter* (τ/D) *must be small* in order for Eq. (5.6) to present a meaningful correction to the white noise limit! In recent work on small noise color $\tau \ll 1$, Fox [106, 137] attempted to patch up some of the shortcomings inherent in Eq. (5.6), such as the problem with unphysical boundaries. In this approximate treatment he obtains for the effective diffusion operator the result [106]

$$D\frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}\frac{g(x)}{[1-\tau g(x)(f(x)/g(x))']} \tag{5.11}$$

It corresponds to formally summing up a geometric series, that is, $[1+\tau h(x)]\rightarrow 1/[1-\tau h(x)]$. This expression has the advantage that the small τ theory in Eq. (5.6) with the diffusion coefficient substituted by Eq. (5.11) yields the exact (Gaussian) stationary probability for a linear process. However, the diffusion in Eq. (5.11) is in general still not strictly positive for all x values. With Eq. (5.11) the corresponding stationary probability $p^{\mathrm{Fox}}(x;\tau)$ reads

$$p^{\mathrm{Fox}}(x;\tau)=Z^{-1}\frac{|[1-\tau g(x)(f(x)/g(x))']|}{|g(x)|} \times \exp\int^{x} dy\,\frac{f(y)[1-\tau g(y)(f(y)/g(y))']}{Dg^{2}(y)} \tag{5.12}$$

Here we stress that the validity of the Fox approximation in Eq. (5.11) is *restricted to the very same regime of validity as the standard small τ approximation* in Eq. (5.6); that is, $\tau\rightarrow 0$ with $\tau/D \ll 1$.

We bring out further complications not present in the 1-D non-Markovian flow Eq. (3.12) by turning to the multidimensional stochastic flow in Eq. (3.3). Use of the functional methods in Section III.C for the multidimensional flow yields in terms of the functional derivative in Eq. (3.23) (we use, apart from the index i, the summation convention over equal indexes)

$$\frac{\delta x_{\alpha}}{\delta\xi_{i}(s)}=\theta(t-s)\left\{\left(\int_{s}^{t} du\left[\frac{\partial f_{\alpha}}{\partial x_{\beta}}+\frac{\partial g_{\alpha i}}{\partial x_{\beta}}\right]\frac{\delta x_{\beta}(u)}{\delta\xi_{i}(s)}\right)+g_{\alpha i}(x(s))\right\} \tag{5.13}$$

being an analogue for Eq. (3.23) for the multivariable case [139–142]. One finds that *generally there does not even exist* a consistent Fokker–Planck like structure in first order in the correlation time τ. Such a small τ multidimensional Fokker–Planck like approximation does exist, however, if the Gaussian correlations $\langle\xi_{i}(t)\xi_{j}(s)\rangle=C_{ij}(t-s)\equiv D_{ij}\gamma_{ij}(t-s)$, with correlation time τ_{ij}, are diagonal and all are of equal correlation time, $\tau_{ii}=\tau_{i}\equiv\tau$ for all i. It can also be obtained whenever the antisymmetric tensor $K_{\beta ij}$ vanishes, that is, if [141]

$$K_{\beta ij} \equiv g_{\alpha i}\left(\frac{\partial g_{\beta j}}{\partial x_\alpha}\right) - \left(\frac{\partial g_{\beta i}}{\partial x_\alpha}\right) g_{\alpha j} = 0 \tag{5.14}$$

Here again, a summation convention over α is implied. Moreover, if $\{g_{\alpha i}\}$ has an inverse obeying

$$\frac{\partial g^{-1}_{\alpha\mu}}{\partial x_\nu} = \frac{\partial g^{-1}_{\alpha\nu}}{\partial x_\mu} \tag{5.15}$$

one can transform multiplicative noise in Eq. (3.3) into additive noise; therefore trivially obeying Eq. (5.14). This multidimensional, small τ Fokker–Planck like approximation, whose precise form is given in [139–142], has, of course, the same regime of validity discussed above, that is,

$$\tau_i \to 0 \qquad \frac{\tau_i}{D_i} \ll 1 \qquad \frac{\tau_{ij}}{D_{ij}} \ll 1 \tag{5.16}$$

B. Decoupling Approximation

As noted in Section V.A, there is a definite need to consider approximation schemes that *do not, a priori restrict* the noise correlation time to small values, $\tau_n \ll \tau_s$, only. Let us go back to Eq. (3.27): On inspecting the structure in Eq. (3.27) we note that a Fokker–Planck like master equation results if we *decouple* the correlation entering the second part of Eq. (3.27), that is,

$$\left\langle \delta(x(t) - x) \exp \int_s^t [f' - (fg'/g)]\, d\tau \right\rangle \to \left\langle \exp \int_s^t [f' - (fg'/g)]\, d\tau \right\rangle p_t(x) \tag{5.17}$$

Consistent use of this decoupling procedure yields for Eq. (3.27) the approximation

$$\begin{aligned} \dot{p}_t(x) = {} & -\frac{\partial}{\partial x}[f(x)p_t(x)] + \left(\int_{t_0}^t ds C(t-s) \exp \int_s^t [\langle f' \rangle - \langle fg'/g \rangle]\, d\tau\right) \\ & \cdot \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) p_t(x) \end{aligned} \tag{5.18}$$

Next, we also only consider the long-time limit of Eq. (5.18); that is, with $t \to \infty$ we neglect transients consistently and use the stationary average in all occurring averaging prescriptions. This yields the Fokker–Planck approximation for a general stationary Gaussian colored noise with correlation $C(t)$, that is,

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}[f(x)p_t(x)] + \left(\int_0^\infty dt C(t) \exp\{t[\langle f' \rangle - \langle fg'/g \rangle]\}\right) \cdot \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) p_t(x) \tag{5.19}$$

For Ornstein–Uhlenbeck noise in Eq. (5.3) this reduces to [143–145],

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}[f(x)p_t(x)] + \left(\frac{D}{\{1 - \tau[\langle f' \rangle - \langle fg'/g \rangle]\}}\right) \times \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) p_t(x) \tag{5.20}$$

This approximation thus retains the white noise Fokker–Planck form wherein the diffusion strength is substituted by the effective diffusion $D_{\text{eff}}(\tau)$

$$D \to D_{\text{eff}}(\tau) \equiv \frac{D}{\{1 - \tau[\langle f' \rangle - \langle fg'/g \rangle]\}} \tag{5.21}$$

which must be determined from Eq. (5.20) self-consistently. In practice, however, it is usually sufficient to evaluate the stationary averages in Eqs. (5.19–5.21) within the white noise approximation for the stationary probability. Note also that with the neglect of transients and the consistent replacement of averages by stationary averages, the Fokker–Planck equation in Eqs. (5.19) and (5.20) *is restricted to yield reliable information about the stationary probability* $p(x; \tau)$ only. The stationary solution of Eq. (5.20) explicitly reads

$$p(x; \tau) = \frac{Z^{-1}}{|g(x)|} \exp\left\{\frac{[1 - \tau(\langle f' \rangle - \langle fg'/g \rangle)]}{D} \int^x \frac{f(y)}{g^2(y)} dy\right\} \tag{5.22}$$

where Z denotes a normalization constant. For globally stable physical systems, that is, $\langle f' \rangle$ *is less than zero*, we find the relation $0 < D_{\text{eff}}(\tau) < D_{\text{eff}}(\tau = 0) = D$. Thus, the stationary probability in Eq. (5.22) generally [e.g., for $g(x) = const$] exhibits a *sharpening* of the probability peaks upon increasing the noise color τ. Indeed, numerical studies verify this typical colored noise effect (see Section VI). The approximation scheme in Eq. (5.17) *does not restrict the value of the noise color* τ. The decoupling ansatz in Eq. (5.17), however, neglects correlations, and thus is expected to be a valid procedure only for narrow distributions, that is, generally $D \ll 1$. Normally, the decoupling approximation is not suitable to approximate multidimensional features such as multidimensional probabilities that may exhibit color dependent correlations among the state variables [146–148]. Thus, the approximation in Eqs. (5.19–5.22) can be viewed as a weak noise approximation to the colored noise flow in Eq. (3.12), that is, the (dimensionless) noise intensity must be small, $D \ll 1$. This latter weak noise condition is fulfilled in most physical applications, see, for example, in [148, 149]. It is not straightforward to evaluate a generally valid estimate on the error induced by the decoupling ansatz. In principle, the decoupling procedure in Eq. (5.17) can be corrected to higher order if we observe the exact relation in Eq. (3.20). The approximation in Eq. (5.17) just presents the first term in Eq. (3.20). For example, we have with $g(x) = 1$ from Eq. (3.20) to second order ($n = 0$ and $n = 1$)

$$\begin{aligned}\left\langle \delta(x(t) - x) \exp \int_s^t f' \, d\tau \right\rangle &= \left\langle \exp \int_s^t f' \, d\tau \right\rangle p_t(x) \\ &\quad - \frac{\partial}{\partial x} \int_{t_0}^t du \int_{t_0}^t dv C(u - v) \left\langle \delta(x(t) - x) \exp \int_u^t f' \, d\tau \right\rangle \\ &\quad \cdot \left\langle \left(\exp \int_s^t f' \, dr \right) \int_s^t f'' \left(\exp \int_v^\tau f' \, dr \right) d\tau \right\rangle \end{aligned} \tag{5.23}$$

Thus, with a repeatedly applied decoupling procedure as outlined above we obtain the result that non-Fokker–Planck contributions already enter at second order. Such an improved approximation is thus not tractable from a practical viewpoint. Nevertheless, the decoupling approximation to lowest order in Eqs. (5.20–5.22) has successfully been applied to model moderate-to-large noise color in a dye laser [138, 149, 190], the optical bistability [135, 144], and the ring-laser gyroscope [150].

C. Unified Colored Noise Approximation

In Section V.B we belabored an approximation for weak noise $D \ll 1$ which, however, does not restrict the noise correlation time τ. The decoupling scheme, however, involves the averaging of state functions. This means that the approximation is of a global character. In other words, local effects such as colored noise induced shifts of probability extrema are likely not sensitively accounted for. With weak noise intensity $D \ll 1$, such effects are generally strongly suppressed; nevertheless, the local character can be substantially misrepresented with the decoupling ansatz in regions of small probability as it occurs with the tails of the probability, or with minima of the probability in bistable situations. Moreover, neither the small τ approximation in Section V.A, nor the decoupling approximation in Section V.B can be used to evaluate the stationary dynamics such as the stationary correlation function $\langle x(t)x(0)\rangle$, the relaxation time T,

$$T \equiv \frac{\int_0^\infty \langle x(t) - \langle x\rangle\rangle(x(0) - \langle x\rangle)\, dt}{\langle x^2\rangle - \langle x\rangle^2} \tag{5.24}$$

or other quantities of dynamical origin. The authors recently have put forward an approximation scheme that effectively overcomes most of these restrictions. We have termed it the unified colored noise approximation (UCNA) (see [151]).

1. *UCNA for Colored One-Dimensional Flows*

Let us consider Eq. (5.25),

$$\dot{x} = f(x) + g(x)\xi(t) \tag{5.25}$$

where $\xi(t)$ is an exponentially correlated Gaussian noise [see Eq. (2.10)] of vanishing mean. First, let us consider additive noise, that is,

$$\dot{x} = f(x) + \xi(t) \tag{5.26}$$

which with $\xi(t)$, an Ornstein–Uhlenbeck process, constitutes a 2-D Markovian process driven by Gaussian white noise $\zeta_w(t)$,

$$\dot{x} = f(x) + \xi \tag{5.27}$$

$$\dot{\xi} = -\frac{1}{\tau}\xi + \frac{D^{1/2}}{\tau}\zeta_w(t) \tag{5.28}$$

where $\langle \zeta_w(t)\zeta_w(s)\rangle = 2\delta(t-s)$. If we follow our original work [151] we eliminate ξ in Eq. (5.27) by use of Eq. (5.28). Then we obtain a Langevin equation for a noisy nonlinear oscillator,

$$\ddot{x} + \dot{x}[\tau^{-1} - f'(x)] - f(x)/\tau = \frac{D^{1/2}}{\tau}\zeta_w(t) \tag{5.29}$$

with a nonlinear damping function. On the new time scale $s = t\tau^{-1/2}$ this nonlinear oscillator dynamics is recast as (a dot indicates the differentiation with respect to time s)

$$\ddot{x} + \gamma(x, \tau)\dot{x} - f(x) = \frac{D^{1/2}}{\tau^{1/4}} \zeta_w(s) \tag{5.30}$$

where $\langle \zeta_w(s)\zeta_w(s')\rangle = 2\delta(s - s')$. The nonlinear damping γ explicitly reads

$$\gamma(x, \tau) = \tau^{-1/2} + \tau^{1/2}[-f'(x)] \tag{5.31}$$

With multiplicative noise, $g(x)\xi(t)$, see Eq. (5.25), the corresponding nonlinear friction would read

$$\gamma(x, \tau) = \tau^{-1/2} + \tau^{1/2}\left[-f'(x) + f(x)\frac{g'(x)}{g(x)}\right] \tag{5.32}$$

If the expression in the squared brackets in Eq. (5.31) or (5.32), is positive, the damping will become *large for both small* and large correlation times τ. The positivity condition is with $f'(x) < 0$, obeyed in regions of state space, where the noise-free flow is locally stable. The condition of large positive damping $\gamma(x, \tau) \gg 1$, allows the adiabatic elimination of $\dot{x} = v$. Setting $\dot{v} = \ddot{x} = 0$ then yields a truely Markovian approximation of the colored noise flow in Eq. (5.26),

$$\dot{x} = \frac{f(x)}{\gamma(x, \tau)} + \frac{D^{1/2}}{\tau^{1/4}\gamma(x, \tau)} \zeta_w(s) \tag{5.33}$$

Within the original time variable $t = \tau^{1/2}s$, and with multiplicative noise $g(x)\xi(t)$, the analogue of Eq. (5.33) reads [144, 151]

$$\begin{aligned}\dot{x} &= f(x)[1 - \tau(f'(x) - f(x)g'(x)/g(x))]^{-1} \\ &\quad + D^{1/2}g(x)[1 - \tau(f'(x) - f(x)g'(x)/g(x))]^{-1}\zeta_w(t)\end{aligned} \tag{5.34}$$

which is to be *interpreted in the Stratonovich sense* [3–7]. Equations (5.33) and (5.34) define a truely 1-D (Stratonovich) Fokker–Planck process, whose equation is readily written down [7]. We must emphasize the true Markovian (approximate) description in Eq. (5.34) of the original non-Markovian process. This feature has a striking advantage over the small correlation time theories outlined in Section V.A. Not only does Eq. (5.34) *become exact* both at correlation time $\tau = 0$ and $\tau \to \infty$, and hence is expected to be a useful approximation for intermediate noise color, it also provides an approximation for the time-homogeneous conditional probability of $x(t)$, that is, the stationary conditional probability $R(x, t \mid y, t_0) = R(x, t+\tau \mid y, t_0+\tau)$ obeys the very same Fokker–Planck equation. Thus, Eq. (5.34) and its corresponding Fokker–Planck equation, can be used to evaluate approximate stationary correlation functions, and so on. The approximation scheme is valid for both small and large correlation times τ, and in parts of the state space where the nonlinear damping $\gamma(x, \tau)$ is positive. In contrast to the dynamics of the small correlation time approximation, which corresponds to the correlation-free preparation (see Section III.B), the UCNA in Eqs. (5.33) and (5.34), closely models the stationary preparation class of the non-Markovian process $x(t)$ [144].

We now discuss the *regime of validity of the UCNA* in more detail. We recall that the UCNA is valid only for regimes in state space where $\gamma(x, \tau)$, Eq. (5.32), is positive. Based on the noise intensity D we form the characteristic length scale L

$$L = \frac{D^{1/2}}{\gamma(x, \tau)} \tag{5.35}$$

Then the adiabatic elimination procedure $\dot{v} = \ddot{x} = 0$ implies that Eq. (5.33) or (5.34), respectively, is a good approximation only on time scales $t > \tau^{1/2}\gamma^{-1}$, that is, with $\gamma > 0$

$$t > \tau[1 + \tau(-f' + fg'/g)]^{-1} \tag{5.36}$$

and if on the characteristic length scale L the drift force is not varying appreciably, that is, $L|f'| \ll |f|$ [144, 151]. This latter condition is the analogue of the condition for the validity of the Smoluchowski approximation in Brownian motion theory [151], wherein $L = (kTm^{-1}\gamma^{-2})^{1/2}$ denotes the thermal length scale. Let $\hat{x}$ denote a characteristic value within the regime where $\gamma > 0$. Then, we obtain for the validity of the UCNA the relation

$$\gamma(\hat{x}, \tau) \gg D^{1/2}\left|\frac{f'(\hat{x})}{f(\hat{x})}\right| \tag{5.37}$$

Thus, we deduce from Eq. (5.37) that the UCNA improves in accuracy for *increasing nonlinear damping* $\gamma \to \infty$, and *decreases in accuracy* with increasing noise intensity. Keeping the restrictions in Eqs. (5.36) and (5.37) in mind we study the solution of Eqs. (5.33) and (5.34). With the effective multiplicative noise function $g_{\mathrm{UCNA}}(x, \tau)$

$$g_{\mathrm{UCNA}}(x, \tau) = g(x)[1 - \tau(f' - fg'/g)]^{-1} \tag{5.38}$$

the (Stratonovich) Fokker–Planck equation for the UCNA in Eq. (5.34) reads[4]

$$\dot{p}_t^{\mathrm{UCNA}}(x) = -\frac{\partial}{\partial x}[f(x)g^{-1}(x)g_{\mathrm{UCNA}}(x,\tau)p_t(x)] + D\frac{\partial}{\partial x}g_{\mathrm{UCNA}}(x,\tau)\frac{\partial}{\partial x}[g_{\mathrm{UCNA}}(x,\tau)p_t(x)] \tag{5.39}$$

Its stationary solution $p^{\mathrm{UCNA}}(x,\tau)$ reads

$$p^{\mathrm{UCNA}}(x,\tau) = \frac{Z^{-1}}{|g(x)|}\,|[1-\tau g(x)(f(x)/g(x))']| \times \exp\left\{\int^x \frac{f(y)[1-\tau g(y)(f(y)/g(y))']\,dy}{Dg^2(y)}\right\} \tag{5.40}$$

being valid both for small and moderate-to-large correlation times τ. Note also that the Fokker–Planck equation in Eq. (5.39) substantially *differs* from the small τ Fox theory [138] in Eq. (5.11). Nevertheless, the stationary probability $p^{\mathrm{Fox}}(x,\tau)$ in Eq. (5.12) precisely coincides with $p^{\mathrm{UCNA}}(x,\tau)$ in Eq. (5.40). Keep in mind, however, that [in *clear contrast* to UCNA in Eq. (5.40)] the theory of Fox, that is, its dynamics, is *restricted* nevertheless to the small τ regime discussed in Section V.A. The extrema of $p^{\mathrm{UCNA}}(x,\tau)$ are located at position $\{\bar{x}\}$, which obey

$$[1-\tau g(f/g)']\{[1-\tau g(f/g)']f - Dg'g\} + Dg^2[1-\tau g(f/g)']' = 0 \tag{5.41}$$

In particular, even in the case where the noise is additive only, one obtains a colored noise induced shift of extrema of $p^{\mathrm{UCNA}}(x,\tau)$ located at

$$-D\tau f''(\bar{x}) + [1-\tau f'(\bar{x})]^2 f(\bar{x}) = 0 \tag{5.42}$$

By use of the Markov character in Eq. (5.39) we can also give the explicit formula for the relaxation time T in Eq. (5.24) [144, 152]

$$T(D,\tau) = \frac{1}{\langle x^2\rangle - \langle x\rangle^2}\int_0^\infty dy\left(\frac{f^2(y)}{Dg_{\mathrm{eff}}^2(y)p^{\mathrm{UCNA}}(y,\tau)}\right) \tag{5.43}$$

where $f(y) \equiv -\int_0^y dx(x-\langle x\rangle)p^{\mathrm{UCNA}}(x)$.

[4] If in Eq. (5.25) additional Gaussian white noise $\eta(t)$ is present, that is, if $\dot{x} = f(x) + g(x)\xi(t) + \eta(t)$, with $\langle \eta(t)\eta(s)\rangle = 2T\delta(t-s)$, the UCNA can be generalized by introducing the auxiliary process $u = \xi + (f/g)\{1 + (T/Dg^2)[1-\tau g(f/g)']\}^{-1}$. An adiabatic elimination of $u(t)$; i.e. $\dot{u}(t) = 0$, then provides the UCNA with the correct behavior as $\tau \to 0$, and which with $u \to 0$ as $\tau \to \infty$ is corrected also for $\tau \to \infty$; for applications cf. Section VII.

D. Remarks on Sundry Colored Noise Approximation Schemes

The small noise color approximation reviewed in Section V.A, the decoupling theory (often also termed Hänggi–Ansatz [135, 138, 149, 190]) in Section V.B, and the UCNA in Section V.C.1, are by and large the most often employed perturbation schemes in the study of dynamical flows driven by correlated random forces. There exist, of course, other possibilities that might be preferred from time to time. For example, for flows driven by Markovian colored noise, such as the exponentially correlated Ornstein–Uhlenbeck noise, the physics can be studied in terms of an enlarged phase space, which renders the dynamics Markovian again [69, 70, 134, 153]. As pointed out in [134] some care, however, must be observed if one compares the dynamics in full space with the one in reduced space; because the correlations in the enlarged space are richer as compared to the reduced, non-Markovian dynamics. Nevertheless, the approximation schemes available for the study of higher dimensional Markovian systems, which unfortunately are rather sparse indeed, can be invoked. Usually, this reasoning has been utilized thus far only for the investigation of stationary quantities, such as the stationary probability [69, 70, 153]. The study of colored noise in the asymptotic regime of large correlation time is another topic that has attracted considerable interest in recent years [154–158]. The UCNA approach clearly does cover this regime, as demonstrated in [158]. The very asymptotic extreme large colored noise regime, can also be more directly addressed by noting that the noise with correlation time $\tau \to \infty$ becomes extremely slowly varying. This then leads to the quasistatic "switching-curve reasoning" originally put forward by Horsthemke and Lefever [159]: With $\tau \to \infty$, the variable x is in a quasi-stationary state with respect to the instantaneous value of the fluctuation force $\xi(t)$, that is, one finds with $\dot{x} \cong 0$ from Eq. (2.7)

$$f(x) + g(x)\xi(t) = 0 \tag{5.44}$$

Setting $\xi = y^{-1}(x)$, and observing the identity between the probabilities,

$$p(x)\,dx = \rho(\xi)\,d\xi \tag{5.45}$$

one therefore finds

$$p(x, \tau \to \infty) = \rho[y^{-1}(x)]\left|\frac{d\xi}{dx}\right| \tag{5.46}$$

where $\rho(\cdots)$ denotes the stationary probability for the noise ξ.

Finally, let us take another look at Eq. (3.27): With Ornstein–Uhlenbeck noise in mind [see Eq. (5.3)] the exact master equation in Eq. (3.27) reads

$$\dot{p}_t(x) = \frac{-\partial}{\partial x}[f(x)p_t(x)] + \frac{D}{\tau}\frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) \int_{t_0}^{t} ds$$
$$\times \exp\left(-\frac{t-s}{\tau}\right) \left\langle \delta[x(t)-x] \exp \int_s^t \left[f' - \left(\frac{fg'}{g}\right)\right] du \right\rangle \tag{5.47}$$

To obtain a workable equation for the probability we must close Eq. (5.47). First we let $t_0 \to -\infty$, so that we can safely neglect transient effects. We observe that Eq. (5.47) can be closed in a variety of different ways:

1. We recover the decoupling theory if we make the "Hanggi–Ansatz", thereby the average in Eq. (5.47) is decoupled. This yields a Fokker–Planck equation with a diffusion operator

$$(i)\ D(x,\tau) = \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) \frac{D}{1-\tau[\langle f' \rangle - \langle fg'/g \rangle]} \tag{5.48}$$

2. If we approximate the stochastic process $x(u)$, $t_0 \leq u \leq t$, for all times by the final value $x(u) = x(t)$, the δ function in Eq. (5.47) implies a closure with the state-dependent diffusion operator given by

$$(ii)\ D(x,\tau) = \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) \frac{D}{1-\tau[f'(x) - f(x)g'(x)/g(x)]} \tag{5.49}$$

Clearly, this approximation implicitly requires a small noise correlation time τ. This form of approximation actually coincides precisely with the approximation put forward by Fox [106], see in Eq. (5.11).

3. Instead of using the small τ approximation $x(u) \simeq x(t)$, we could instead follow the reasoning inherent in the decoupling theory and replace the stochastic process $f[x(u)]$ not by a stationary average, but by a time-dependent (deterministic) solution $f[x(u)] \to f[x_{\text{det}}(u)]$. A good candidate would be to use the path $x_e(t)$, which extremalizes the action of the corresponding Onsager–Machlup functional in a path integral solution of the corresponding non-Markovian process [156, 157, 160, 161].

With $x_e(t_0) = x_s$, being chosen as an attractor, so that $x_e(t) = x$ is attained only at very long times, we obtain from Eq. (3.27) upon a change of variables $dt' = dx_e/\dot{x}_e$ an x dependent, effective diffusion operator given by

$$(iii)\ D(x,\tau) = \frac{D}{\tau}\frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) \int_{x_s}^{x} \frac{dy}{\dot{y}} \exp \int_{y}^{x} \left\{ -\frac{1}{\tau} + g(z)\left(\frac{f}{g}\right)' \right\} \frac{dz}{\dot{z}} \tag{5.50}$$

Hereby we used stationary Ornstein–Uhlenbeck noise [see Eq. (5.3)] and the velocities are determined from the extremal action path. The reasoning to obtain this effective diffusion has recently been applied by Venkatesh and Patniak [162] in a study of colored noise driven bistability. An appealing feature of Eq. (5.50) must be pointed out: The effect of noise color enters the effective diffusion in Eq. (5.50) in two ways: First, there is the influence of the noise correlation $C(t) = (D/\tau)\exp(-t/\tau)$; second, there is the τ dependence in the extreme action path $x_e(t)$. Moreover, just as with UCNA, the approximation in Eq. (5.50) is not restricted solely to small noise color.

VI. COLORED NOISE DRIVEN BISTABLE SYSTEMS

Noisy bistable dynamics is an archetype phenomenon in many areas of physics, chemistry, and biology. It is therefore important to develop a detailed understanding of the fluctuation-related statistical characteristics, such as lifetimes of metastable states. Realistic modeling of noise sources, however, requires us to take into account finite correlation times. An important application of the theoretical framework of colored noise driven dynamical systems, provided in Sections II–V is therefore bistable dynamics. In this section we review key results obtained for probability densities and escape rates in colored noise driven bistable systems. Special focus is the dependence of those characteristics on the correlation time of the noise.

As a model, we are using a Ginzburg–Landau type potential and an exponentially correlated noise. The equation of motion reads [134]

$$\dot{x} = -V'(x) + \varepsilon(t) \tag{6.1}$$

with $\varepsilon(t)$ being Gaussian, exponentially correlated noise, that is,

$$\begin{aligned} \langle \varepsilon(t)\varepsilon(t') \rangle &= \frac{D}{\tau}\exp\left(-\frac{|t-t'|}{\tau}\right) \\ \langle \varepsilon(t) \rangle &= 0 \end{aligned} \tag{6.2}$$

The potential $V(x)$ is given by

$$V(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4 \tag{6.3}$$

with positive constants a and b. The potential $V(x)$ is bistable with minima at $x_{1,2} = \pm\sqrt{a/b}$, and a relative maximum at $x = 0$. Introducing scaled variables $\bar{x} = x\sqrt{b/a}$, $\bar{t} = at$, $\bar{\varepsilon} = \varepsilon\sqrt{b/a^3}$, $\bar{\tau} = a\tau$, $\bar{D} = (b/a^2)D$, the normalized Langevin equation reads

$$\dot{\bar{x}} = \bar{x} - \bar{x}^3 + \bar{\varepsilon}(t) \tag{6.4}$$

where the autocorrelation function of the noise variable $\bar{\varepsilon}$ is given by

$$\langle \varepsilon(t)\bar{\varepsilon}(t')\rangle = \frac{\bar{D}}{\bar{\tau}}\exp\left(-\frac{|t-t'|}{\bar{\tau}}\right) \tag{6.5}$$

The potential is shown in scaled variables in Fig. 6.1. In the case of a large typical system time scale $1/a$ in comparison to the correlation time of the noise τ, that is, $\bar{\tau} = \tau/(1/a) \rightarrow 0$, the correlation function $\langle \varepsilon(t)\varepsilon(t')\rangle$ approaches a δ function. The variance of the noise, $\langle \varepsilon^2\rangle = D/\tau$, which up to a factor of two equals the total power of the noise, diverges in this *white noise* limit. In the opposite limit, $\bar{\tau} \rightarrow \infty$, the variance vanishes, that is, the total power of the noise vanishes.

From now on, we will only use the normalized variables, but drop the bar for the sake of convenience.

A. Embedding in a Two-Dimensional Markovian Process

The stochastic process Eq. (6.4) defines a non-Markovian stochastic process. The time evolution of the probability distribution is thus not given by a Fokker–Planck equation for the state variable x (see Section

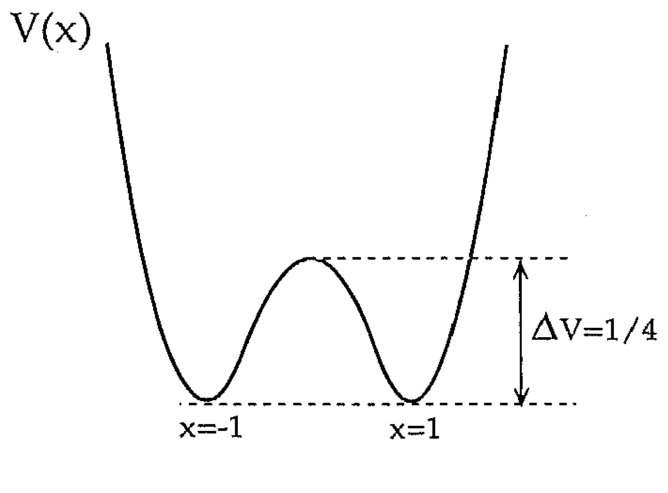


Figure 6.1. The double-well potential [Eq. (6.3)] is shown in normalized coordinates.

III). Nevertheless, one can find an equivalent 2-D pair process $(x(t), \varepsilon(t))$, with the auxiliary variable ε, obeying the linear white noise driven stochastic differential equation [134, 163]

$$\dot{\varepsilon} = -\frac{1}{\tau}\varepsilon + \frac{\sqrt{D}}{\tau}\xi(t) \tag{6.6}$$

with the Gaussian white noise

$$\begin{aligned} \langle \xi(t) \rangle &= 0 \\ \langle \xi(t)\xi(t') \rangle &= 2\delta(t-t') \end{aligned} \tag{6.7}$$

where the stationary autocorrelation function of $\varepsilon(t)$ is given by the first equation of Eq. (6.5). The pair process $(x(t), \varepsilon(t))$ is a Markovian stochastic process and the time evolution of the joint probability density, $W(x, \varepsilon, t)$, is given by the two-variable Fokker–Planck equation,

$$\begin{aligned} \frac{\partial}{\partial t} W(x, \varepsilon, t) &= \mathbf{L}_{\mathrm{em}}(x, \varepsilon) W(x, \varepsilon, t) \\ \mathbf{L}_{\mathrm{em}}(x, \varepsilon) &= -\frac{\partial}{\partial x}(x - x^3 + \varepsilon) + \frac{1}{\tau}\frac{\partial}{\partial \varepsilon}\varepsilon + \frac{D}{\tau^2}\frac{\partial^2}{\partial \varepsilon^2} \end{aligned} \tag{6.8}$$

In order to guarantee that the correlation function for ε is stationary for all times, we have to require that at the preparation time $t = 0$ the probability distribution in ε is stationary [134, 144], that is,

$$\int_{-\infty}^{\infty} W(x, \varepsilon, t=0)\, dx = \frac{1}{\sqrt{2\pi\tau/D}} \exp\left(-\frac{\tau\varepsilon^2}{2D}\right) \tag{6.9}$$

1. *Basic Properties of the Embedding Fokker–Planck Operator*

Since the Fokker–Planck (FP) operator $\mathbf{L}_{\mathrm{em}}$ is symmetric with respect to inversion,

$$\mathbf{L}_{\mathrm{em}}(x, \varepsilon) = \mathbf{L}_{\mathrm{em}}(-x, -\varepsilon) \tag{6.10}$$

the corresponding eigenfunctions

$$\mathbf{L}_{\mathrm{em}}(x, \varepsilon)\psi_\lambda(x, \varepsilon) = -\lambda\psi_\lambda(x, \varepsilon) \tag{6.11}$$

can be classified into even and odd eigenfunctions [163],

$$\begin{aligned}\psi_\lambda^{(e)}(x, \varepsilon) &= \psi_\lambda^{(e)}(-x, -\varepsilon)\\ \psi_\lambda^{(o)}(x, \varepsilon) &= -\psi_\lambda^{(o)}(-x, -\varepsilon)\end{aligned} \tag{6.12}$$

The stationary probability, being the eigenfunction corresponding to the vanishing eigenvalue, is an even eigenfunction

$$W_{st}(x, \varepsilon) = W_{st}(-x, -\varepsilon) \tag{6.13}$$

Since the stochastic differential equation for x does not couple to the equation for ε, the eigenvalues of the Ornstein–Uhlenbeck process for ε are also eigenvalues of the stochastic pair process. This result can be seen more clearly at the adjoint eigenvalue equation

$$\begin{aligned}\mathbf{L}_{\mathrm{em}}^{\dagger}(x, \varepsilon)\psi_\lambda^{\dagger}(x, \varepsilon) &= -\lambda\psi_\lambda^{\dagger}(x, \varepsilon)\\ \mathbf{L}_{\mathrm{em}}^{\dagger} &= (x - x^3 + \varepsilon)\frac{\partial}{\partial x} - \frac{1}{\tau}\varepsilon\frac{\partial}{\partial \varepsilon} + \frac{D}{\tau^2}\frac{\partial^2}{\partial \varepsilon^2}\end{aligned} \tag{6.14}$$

which is solved by the adjoint eigenfunctions (H_n denotes the Hermite polynomial of order n)

$$\psi_{0n}^{\dagger}(x, \varepsilon) = H_n\left(\frac{\varepsilon}{\sqrt{2D/\tau}}\right) \tag{6.15}$$

with the corresponding eigenvalues

$$\lambda_{0n} = n\frac{1}{\tau} \tag{6.16}$$

It is important to note that for large correlation times these eigenvalues become small and that the corresponding relaxation modes can therefore influence even the large time behavior of dynamical quantities such as correlation functions!

The symmetry of the pair process $(x(t), \varepsilon(t))$ allows us to construct two isospectral Fokker–Planck systems. An equivalent pair process to Eqs. (6.4) and (6.6) is given by

$$\begin{aligned}\dot{x} &= x - x^3 - \varepsilon\\ \dot{\varepsilon} &= -\frac{1}{\tau}\varepsilon + \frac{\sqrt{D}}{\tau}\xi(t)\end{aligned} \tag{6.17}$$

The spectrum of the corresponding Fokker–Planck operator

$$\tilde{\mathbf{L}}_{\mathrm{em}}(x, \varepsilon) = -\frac{\partial}{\partial x}(x - x^3 - \varepsilon) + \frac{1}{\tau}\frac{\partial}{\partial \varepsilon}\varepsilon + \frac{D}{\tau^2}\frac{\partial^2}{\partial \varepsilon^2} \tag{6.18}$$

is therefore isospectral with the Fokker–Planck operator $\mathbf{L}_{\mathrm{em}}(x, \varepsilon)$ [164]. The eigenfunctions of $\mathbf{L}_{\mathrm{em}}(x, \varepsilon)$ follow from the eigenfunctions of $\tilde{\mathbf{L}}_{\mathrm{em}}(x, \varepsilon)$ by the substitution $\varepsilon \to -\varepsilon$. Another isospectral Fokker–Planck operator can be constructed by inverting the potential [164]. This is a general property of a colored noise driven *overdamped* system. In order to show this we convert the eigenvalue problem Eq. (6.11) [here we use a general force field $h(x)$ instead of $x - x^3$] according to

$$\psi_\lambda(x, \varepsilon) = \exp\left(\frac{H(x)}{2D} - \frac{\tau}{2D}\varepsilon^2\right)\hat{\psi}_\lambda(x, \varepsilon) \tag{6.19}$$

where the (negative of the) potential is given by

$$H(x) = \int^x h(y)\, dy \tag{6.20}$$

into the eigenvalue problem for $\hat{\psi}_\lambda(x, \varepsilon)$

$$\hat{\mathbf{L}}_{\mathrm{em}}\hat{\psi}_\lambda(x, \varepsilon) = -\lambda\hat{\psi}_\lambda(x, \varepsilon) \tag{6.21}$$

with

$$\hat{\mathbf{L}}_{\mathrm{em}} = -\hat{\mathbf{a}}_x\mathbf{a}_x - \hat{\mathbf{a}}_x^2 - \hat{\mathbf{a}}_\varepsilon\mathbf{a}_\varepsilon + \tfrac{1}{2}\hat{\mathbf{a}}_x(\mathbf{a}_\varepsilon + \hat{\mathbf{a}}_\varepsilon) \tag{6.22}$$

and the operators are defined as

$$\begin{aligned}
\hat{\mathbf{a}}_x &= -\sqrt{D}\frac{\partial}{\partial x} - \frac{h(x)}{2\sqrt{D}} \\
\mathbf{a}_x &= \sqrt{D}\frac{\partial}{\partial x} - \frac{h(x)}{2\sqrt{D}} \\
\hat{\mathbf{a}}_\varepsilon &= -\frac{\sqrt{D}}{\tau}\frac{\partial}{\partial \varepsilon} + \frac{\varepsilon}{2\sqrt{D}} \\
\mathbf{a}_\varepsilon &= \frac{\sqrt{D}}{\tau}\frac{\partial}{\partial \varepsilon} + \frac{\varepsilon}{2\sqrt{D}}
\end{aligned} \tag{6.23}$$

The eigenvalue problem with the inverted potential, that is, $H(x) \to -H(x)$ can be treated analogous, yielding the converted Fokker–Planck operator

$$\tilde{\hat{\mathbf{L}}}_{\mathrm{em}} = -\tilde{\hat{\mathbf{a}}}_x \tilde{\mathbf{a}}_x - \tilde{\hat{\mathbf{a}}}_x^2 - \tilde{\hat{\mathbf{a}}}_\varepsilon \tilde{\mathbf{a}}_\varepsilon + \tfrac{1}{2}\tilde{\hat{\mathbf{a}}}_x(\tilde{\mathbf{a}}_\varepsilon + \tilde{\hat{\mathbf{a}}}_\varepsilon) \tag{6.24}$$

where the operators with the tilde are given by

$$\begin{aligned} \tilde{\hat{\mathbf{a}}}_x &= -\mathbf{a}_x \\ \tilde{\mathbf{a}}_x &= -\hat{\mathbf{a}}_x \\ \tilde{\hat{\mathbf{a}}}_\varepsilon &= -\mathbf{a}_x \\ \tilde{\mathbf{a}}_\varepsilon &= -\hat{\mathbf{a}}_x \end{aligned} \tag{6.25}$$

In view of the isospectral property with respect to inversion of ε, we have also performed the inversion $\varepsilon \to -\varepsilon$. From the equations above, we can establish the following operator relation

$$\mathbf{a}_x \hat{\mathbf{L}}^\dagger(x, \varepsilon) = \tilde{\hat{\mathbf{L}}}(x, -\varepsilon)\mathbf{a}_x \tag{6.26}$$

If $\hat{\psi}^\dagger(x, \varepsilon)$ is an eigenfunction of $\hat{\mathbf{L}}^\dagger(x, \varepsilon)$, that is,

$$\hat{\mathbf{L}}^\dagger(x, \varepsilon)\tilde{\psi}^\dagger(x, \varepsilon) = -\lambda\hat{\psi}^\dagger(x, \varepsilon) \tag{6.27}$$

then we find by multiplying from the left with $\mathbf{a}_x$ and by using Eq. (6.26)

$$\tilde{\hat{\mathbf{L}}}(x, -\varepsilon)\mathbf{a}_x \hat{\psi}^\dagger(x, \varepsilon) = -\lambda \mathbf{a}_x \hat{\psi}^\dagger(x, \varepsilon) \tag{6.28}$$

that is, $\mathbf{a}_x \hat{\psi}^\dagger(x, \varepsilon)$ is an eigenfunction of $\tilde{\hat{\mathbf{L}}}(x, -\varepsilon)$ with the *same eigenvalue* λ. The isospectral property is thus proven.

2. *Application of the Matrix Continued Fraction Technique*

The two-variable Fokker–Planck equation in the extended phase space can be solved by using the matrix continued fraction (MCF) technique [5]. Since our Fokker–Planck operator $\mathbf{L}_{\mathrm{em}}(x, \varepsilon)$ has inversion symmetry, we make this technique more efficient [163] by expanding the even and odd eigenfunctions separately in complete sets of orthogonal functions with respect to both variables x and ε. Thus,

$$\psi^{(e)}(x,\varepsilon)=\rho_0(x)w_0(\varepsilon)\sum_{n,m=0}^{\infty}[c_{2n}^{2m}\varphi_{2m}(x)w_{2n}(\varepsilon)+c_{2n+1}^{2m+1}\varphi_{2m+1}(x)w_{2n+1}(\varepsilon)]$$
$$\psi^{(o)}(x,\varepsilon)=\rho_0(x)w_0(\varepsilon)\sum_{n,m=0}^{\infty}[c_{2n}^{2m+1}\varphi_{2m+1}(x)w_{2n}(\varepsilon)+c_{2n+1}^{2m}\varphi_{2m}(x)w_{2n+1}(\varepsilon)] \tag{6.29}$$

where the complete set $\{w_n(\varepsilon)\}$ is given by the eigenfunctions of the operator

$$\mathbf{L}_H=\frac{D}{\tau^2}\frac{\partial^2}{\partial\varepsilon^2}-\frac{1}{4D}\varepsilon^2+\frac{1}{2\tau} \tag{6.30}$$

that is,

$$w_n(\varepsilon)=\frac{1}{\sqrt{2^n n!\sqrt{2\pi D/\tau}}}H_n\left(\frac{\varepsilon}{\sqrt{2D/\tau}}\right)\exp\left(-\frac{\tau\varepsilon^2}{4D}\right) \tag{6.31}$$

and the complete set in x is given by the Hermite functions

$$\varphi_n(x)=\sqrt{\frac{\alpha}{n!2^n\sqrt{\pi}}}H_n(\alpha x)\exp\left(-\frac{1}{2}\alpha^2x^2\right) \tag{6.32}$$

The constant α is an adjustable positive parameter to optimize the speed of the convergence. The form function $\rho_0(x)$ is assumed to be symmetric and positive and should decay to zero for $x\to\pm\infty$. Inserting those expansions into the eigenvalue equation Eq. (6.11), we find a coupled system of algebraic equations for the expansion coefficients c_n^m, which can be arranged in a tridiagonal vector recurrence relation. We find for the even eigenfunctions [163]

$$\mathbf{0}=\left(\underline{\underline{A}}_0-\left(\frac{k}{\tau}-\lambda\right)\right)\mathbf{c}_k+\sqrt{\frac{kD}{\tau}}\,\underline{\underline{B}}_{0e}\mathbf{c}_{k-1}+\sqrt{\frac{(k+1)D}{\tau}}\,\underline{\underline{B}}_{0e}\mathbf{c}_{k+1} \tag{6.33}$$

for odd k, and

$$\mathbf{0}=\left(\underline{\underline{A}}_e-\left(\frac{k}{\tau}-\lambda\right)\right)\mathbf{c}_k+\sqrt{\frac{kD}{\tau}}\,\underline{\underline{B}}_{e0}\mathbf{c}_{k-1}+\sqrt{\frac{(k+1)D}{\tau}}\,\underline{\underline{B}}_{e0}\mathbf{c}_{k+1} \tag{6.34}$$

for even k. The components of the matrices $\underline{\underline{A}}_e$, $\underline{\underline{A}}_0$, $\underline{\underline{B}}_{e0}$, $\underline{\underline{B}}_{0e}$ are given by

$$\begin{aligned}
(\underline{\underline{A}}_0)^{i,j} &= \int_{-\infty}^{\infty} \rho_0^{-1}(x)\varphi_{2i+1}(x)\Big(-\frac{\partial}{\partial x}(x-x^3)\Big)\rho_0(x)\varphi_{2j+1}(x)\,dx = A^{2i+1,2j+1} \\
(\underline{\underline{A}}_e)^{i,j} &= \int_{-\infty}^{\infty} \rho_0^{-1}(x)\varphi_{2i}(x)\Big(-\frac{\partial}{\partial x}(x-x^3)\Big)\rho_0(x)\varphi_{2j}(x)\,dx = A^{2i,2j} \\
(\underline{\underline{B}}_{e0})^{i,j} &= \int_{-\infty}^{\infty} \rho_0^{-1}(x)\varphi_{2i}(x)\Big(-\frac{\partial}{\partial x}\Big)\rho_0(x)\varphi_{2j+1}(x)\,dx = B^{2i,2j+1} \\
(\underline{\underline{B}}_{0e})^{i,j} &= \int_{-\infty}^{\infty} \rho_0^{-1}(x)\varphi_{2i+1}(x)\Big(-\frac{\partial}{\partial x}\Big)\rho_0(x)\varphi_{2j}(x)\,dx = B^{2i+1,2j}
\end{aligned} \tag{6.35}$$

while the components of the vectors $\mathbf{c}_k$ read

$$(\mathbf{c}_k)^i = \begin{cases} c_k^{2i} & \text{for even } k \\ c_k^{2i+1} & \text{for odd } k \end{cases}$$

For odd eigenfunctions the conditions for k have to be interchanged. The tridiagonal vector recurrence relation Eqs. (6.33) and (6.34) can be solved for the eigenvalues by iterating a matrix continued fraction. For the form junction, the Gaussian, $\rho_0(x) = \exp(-cx^2)$, has been chosen, where the constant c has been adjusted to obtain a good convergence of the matrix continued fraction. The matrices $A^{m,n}$ and $B^{m,n}$ read for this Gaussian form function

$$\begin{aligned}
A^{m,n} = &-\frac{\beta^6}{2D}\{\sqrt{(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)}\}\delta_{n,m+6} \\
&+\left\{\frac{\beta^6}{2D}\sqrt{m(m-1)(m-2)(m-3)(m-4)(m-5)}\right\}\delta_{n,m-6} \\
&+\left\{\frac{1}{2}\beta^2 + 2c\beta^4\right\}\sqrt{(m+1)(m+2)(m+3)(m+4)}\delta_{n,m+4} \\
&+\left\{-\frac{1}{2}\beta^2 + 2c\beta^4\sqrt{(m(m-1)(m-2)(m-3)}\right\}\delta_{n,m-4} \\
&+\left\{-\frac{1}{2} + \beta^2(m+3+2c) + 4c\beta^2(2m+3)\right\}\sqrt{(m+1)(m+2)}\delta_{n,m+2} \\
&+\left\{\frac{1}{2} - \beta^2(m-2-2c) + 4c\beta^4(2m-3)\right\}\delta_{n,m-2}
\end{aligned}$$

$$+\left\{-\frac{1}{2}+3\beta^{2}\left(m+\frac{1}{2}+2c(2m+1)\right)+12c\beta^{4}m(m+1)+6c\beta^{4}\right\}\delta_{nm} \tag{6.36}$$

and

$$B^{n,m}=-\left\{\frac{1}{2\beta}+2c\beta\right\}\sqrt{m+1}\,\delta_{n,m+1}+\left\{\frac{1}{2\beta}-2c\beta\right\}\delta_{n,m-1} \tag{6.37}$$

with $\beta=\alpha/\sqrt{2}$.

3. *Stationary Probability Density in the Extended Phase Space*

For large times, the probability distribution $W(x, \varepsilon, t)$ approaches the time-independent stationary density $W_{st}(x, \varepsilon)$. This is the stationary joint probability density, that is, the probability of finding x in the interval $[x, x+dx]$ and the auxiliary variable ε in the interval $[\varepsilon, \varepsilon+d\varepsilon]$. The expansion of the (even) stationary probability density $W_{st}(x, \varepsilon)$ in the complete sets (see Section VI.A.2) with respect to x and ε reads

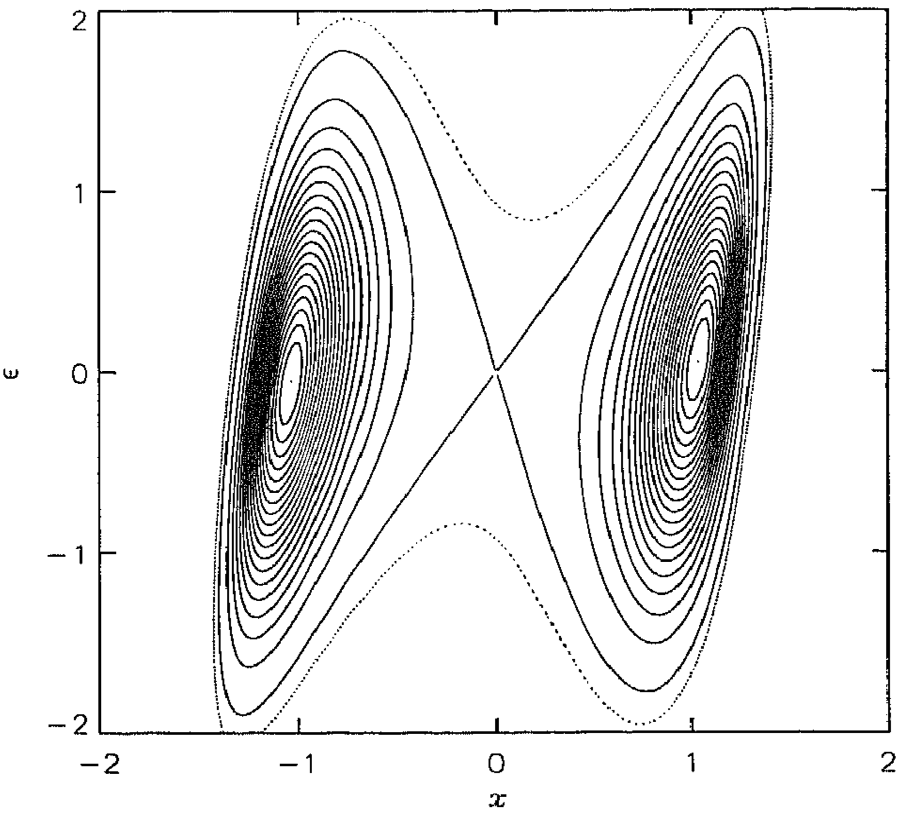
$$W_{st}(x,\varepsilon)=\rho_0(x)w_0(\varepsilon)\sum_{n,m=0}^{\infty}\left[d_{2n}^{2m}\varphi_{2m}(x)w_{2n}(\varepsilon)+d_{2n+1}^{2m+1}\varphi_{2m+1}(x)w_{2n+1}(\varepsilon)\right] \tag{6.38}$$

where the expansion coefficients have been determined by using the matrix continued fraction method [163]. In Fig. 6.2, $W_{st}(x, \varepsilon)$ is shown by using altitude charts. For increasing correlation times τ, the distribution exhibits a scewing [147, 163, 165], that is, the ridge of the distribution lies on a curved manifold. This manifold is approximately given by

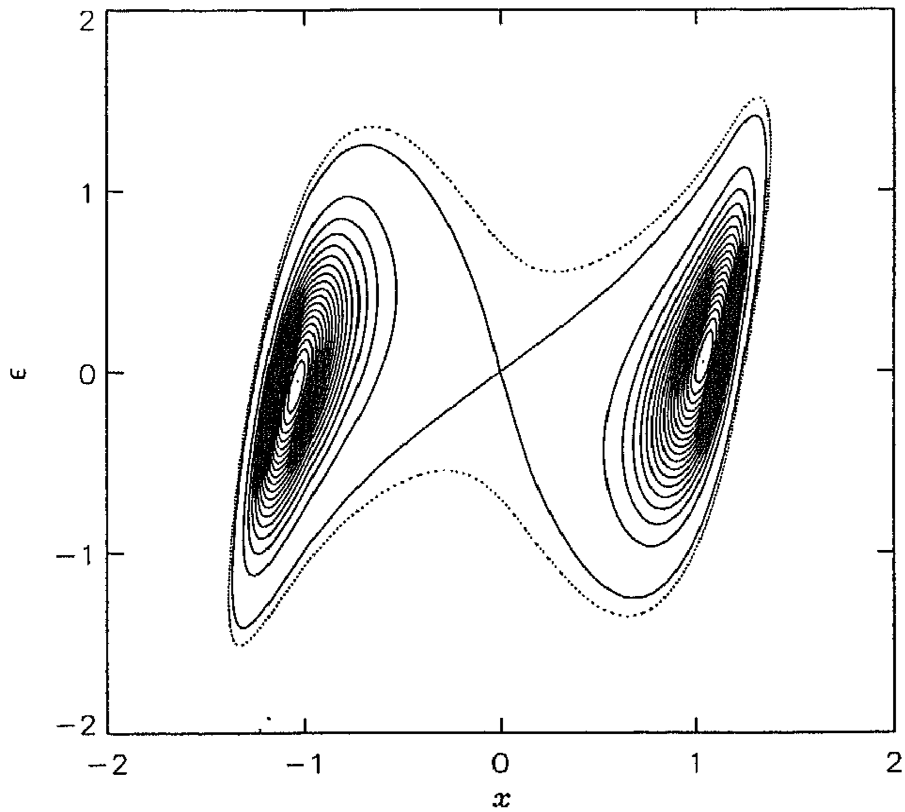
$$\varepsilon(x)=x^3-x \tag{6.39}$$

At a critical value of the correlation time $\tau_c(D)$, the character of the probability distribution at the origin changes from a saddle point to a minimum. One should note, however, that this crater is very shallow, cf. Fig. 6.2(d), that is, it does not significantly affect transport properties such as escape rates. One should bear in mind, however, that the whole region around the origin is very flat [166].

For small correlation times τ, one can derive an approximate expression for the stationary probability density [69]. Expanding the potential



(a)



(b)

Figure 6.2. The colour lines of the stationary probability density in the extended phase space [Eq. (6.38)] are shown for $\tau = 0.2(a)$, $\tau = 0.5(b)$, $\tau = 1(c)$, and $\tau = 3.333(d)$ for $D = 0,$ 1.

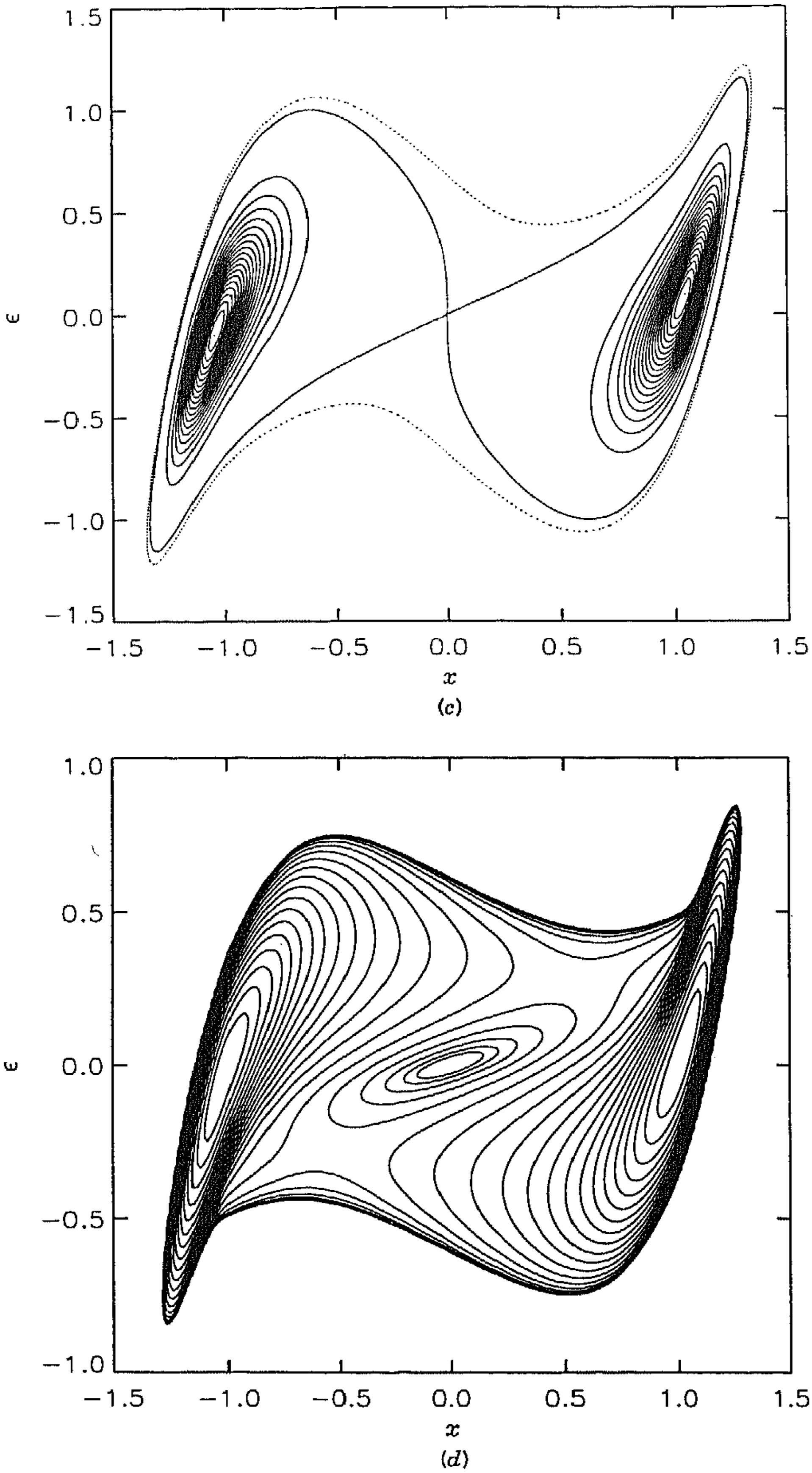


Figure 6.2. (*Continued*)

$\phi(x, \varepsilon)$, defined by

$$W_{st}(x, \varepsilon) = \exp -\left(\frac{\phi(x, \varepsilon)}{D}\right) \tag{6.40}$$

obeying the nonlinear partial differential equation with $h(x) = x - x^3$

$$[h(x) + \varepsilon]\frac{\partial\phi}{\partial x} + \frac{1}{\tau}\frac{\partial\phi}{\partial\varepsilon}\left(-\varepsilon + \frac{1}{\tau}\frac{\partial\phi}{\partial\varepsilon}\right) - D\left(h'(x) + \frac{1}{\tau^2}\frac{\partial^2\phi}{\partial\varepsilon^2}\right) = 0 \tag{6.41}$$

in a power series in τ, that is,

$$\phi(x, \varepsilon) = \phi^{(0)}(x, \varepsilon) + \tau\phi^{(1)}(x, \varepsilon) + \tau^2\phi^{(2)}(x, \varepsilon) + \tau^3\phi^{(3)}(x, \varepsilon) + O(\tau^4) \tag{6.42}$$

we find by equating all terms of equal power in τ

$$\begin{aligned}\phi(x, \varepsilon) = {} & \frac{\tau}{2}(1 - \tau h')(\varepsilon + h)^2 - \int^x \left\{h(y)[1 - \tau h'(y)] + \frac{1}{2}\tau^2 h(y)h''(y)\right\} dy \\ & + \frac{1}{2}\tau^3\varepsilon^2 h''\left[\frac{1}{2}h(x) + \frac{1}{3}\varepsilon\right] + \frac{3}{2}D\tau h'(x)\end{aligned} \tag{6.43}$$

By inserting the corresponding probability distribution into the Fokker–Planck equation one confirms that the errors are only of the orders $\tau^{n\geq 3}$ and $D\tau^{n\geq 2}$. Neglecting τ^3 terms, one finds in leading order

$$\begin{aligned}\phi(x, \varepsilon) = {} & \frac{\tau}{2}(1 - \tau h'(x))(\varepsilon + h(x))^2 - \int^x \Big(h(y)(1 - \tau h'(y)) \\ & + \frac{1}{2}\tau^2 h(y)h''(y)\Big)\, dy + \frac{3}{2}D\tau h'(x)\end{aligned} \tag{6.44}$$

The agreement with numerical solutions is very good even for correlation times up to $\tau = 0.5$.

4. *Eigenvalues and Eigenfunctions*

The eigenvalues of the Fokker–Planck operator in the extended-phase space describe the relaxation towards the stationary state,

$$W(x, \varepsilon, t) = \sum_{n,m=0}^{\infty} c_{nm}\psi_{nm}(x, \varepsilon)\exp(-\lambda_{nm}t) \tag{6.45}$$

Most important is the smallest nonvanishing eigenvalue. It describes the relaxation on the longest time scale. The eigenvalues have been com-

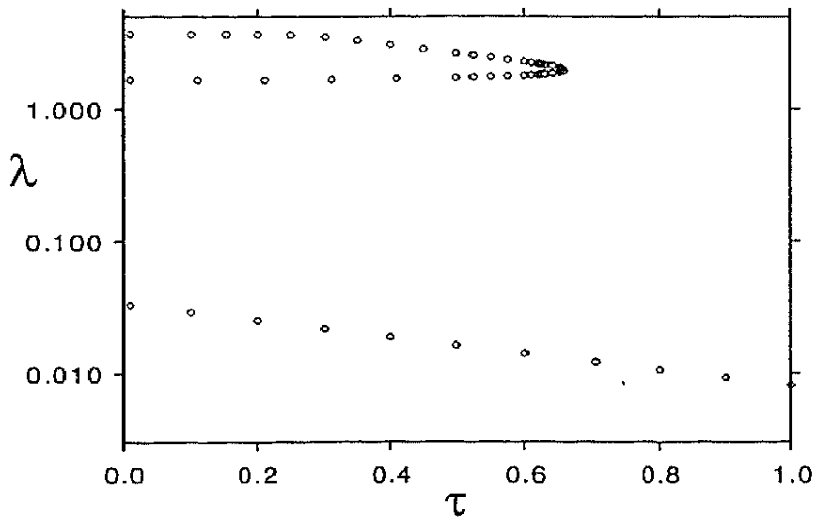


Figure 6.3. The first three branches of real valued eigenvalues, corresponding to odd eigenfunctions, are shown at $D = 0.1$ as a function of the correlation time of the noise. The intersection of the two branches of real values eigenvalues indicates the birth of a pair of complex conjugate eigenvalues.

puted in [154, 163] by applying the matrix continued fraction technique [5]. Our focus is on the dependence of the eigenvalues on the noise correlation time τ. In Fig. 6.3, this dependence is shown for several eigenvalues. The smallest nonvanishing eigenvalue decreases with increasing correlation time of the noise. It is also worthwhile to mention that at that critical value of τ, where the stationary probability distribution changes its shape from a saddle point to a minimum, none of the eigenvalues exhibits any characteristic behavior.

B. Stationary Probability Density

The stationary probability density $P_{st}(x)$ is obtained from the stationary joint probability density $P_{st}(x, \varepsilon)$ by tracing out the auxiliary variable ε,

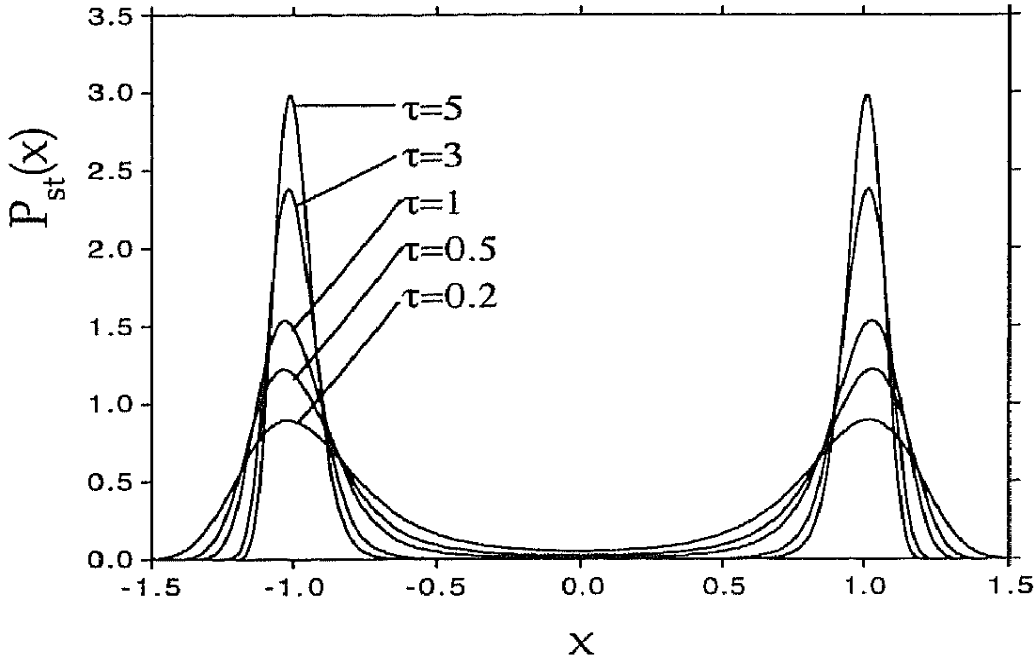
$$P_{st}(x) = \int_{-\infty}^{\infty} P_{st}(x, \varepsilon)\, d\varepsilon \tag{6.46}$$

In terms of the expansion coefficients d_m^n in Eq. (6.38) the symmetric stationary probability density reads

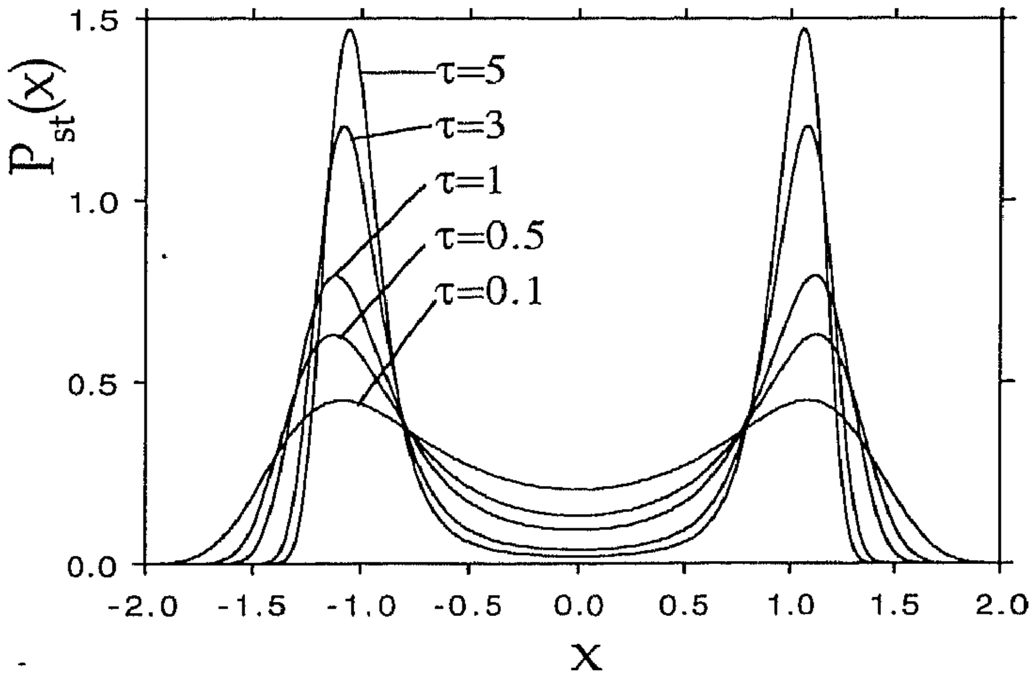
$$P_{st}(x) = \rho_0(x) \sum_{m=0}^{\infty} d_0^{2m} \varphi_{2m}(x) = P_{st}(-x) \tag{6.47}$$

In Fig. 6.4, $P_{st}(x)$ is plotted at $D = 0.1(a)$ and $D = 0.05(b)$. We observe [163],

1. That for increasing correlation times, the peaks become higher and



(a)



(b)

Figure 6.4. The numerically evaluated stationary probability density [Eq. (6.47)] is shown at $D = 0.1(a)$ and $D = 0.5(b)$ for various values of the correlation time τ of the noise.

more narrow and the probability density at $x = 0$ becomes smaller. This is in accordance with the reasoning put forward in Eq. (5.21) with the decoupling theory.

2. That starting at $\tau = 0$, the maxima for increasing τ shift to larger values of $|x|$ and shift back towards $|x| = 1$ for further increasing correlation times. The maximum shift increases with decreasing noise strength D; being in agreement with the UCNA prediction in Eq. (5.42).

The first observation can be explained by the decrease of the variance of the noise with increasing correlation times τ of the noise. The second observation can be explained qualitatively by changes of the stability of the oscillator equation in the variables $(x, v = \dot{x} = x - x^3 + \varepsilon)$.

Using the approximation schemes (introduced in Section V) we can derive approximate expressions for the stationary probability density. Within the small correlation time approximation, that is, for $\tau \to 0$, $\tau/D \to 0$, one finds with $V_0(x) = x^4/4 - x^2/2$ for the stationary probability density

$$P_{st}(x) = \frac{1}{Z}\left(1 - \tau|1 - 3x^2| - \frac{\tau}{2D}(x - x^3)^2\right)\exp\left\{-\frac{1}{D}V_0(x)\right\} \tag{6.48}$$

defined in the finite region of support

$$|x| < \sqrt{\frac{1+\tau}{3\tau}} \tag{6.49}$$

Using the unified colored noise approximation we find

$$P_{st}(x) = \frac{1}{Z}|1 - \tau(1 - 3x^2)|\exp\left\{-\frac{1}{D}V_0(x) - \frac{\tau}{2D}(x - x^3)^2\right\} \tag{6.50}$$

valid in the region of support, given by

$$1 - \tau(1 - 3x^2) > 0 \tag{6.51}$$

The decoupling approximation yields

$$P_{st}(x) = \frac{1}{Z}\exp\left\{\frac{-1}{\tilde{D}}V_0(x)\right\} \tag{6.52}$$

with

$$\tilde{D} = \frac{D}{1 + \tau(3\langle x^2\rangle - 1)} \tag{6.53}$$

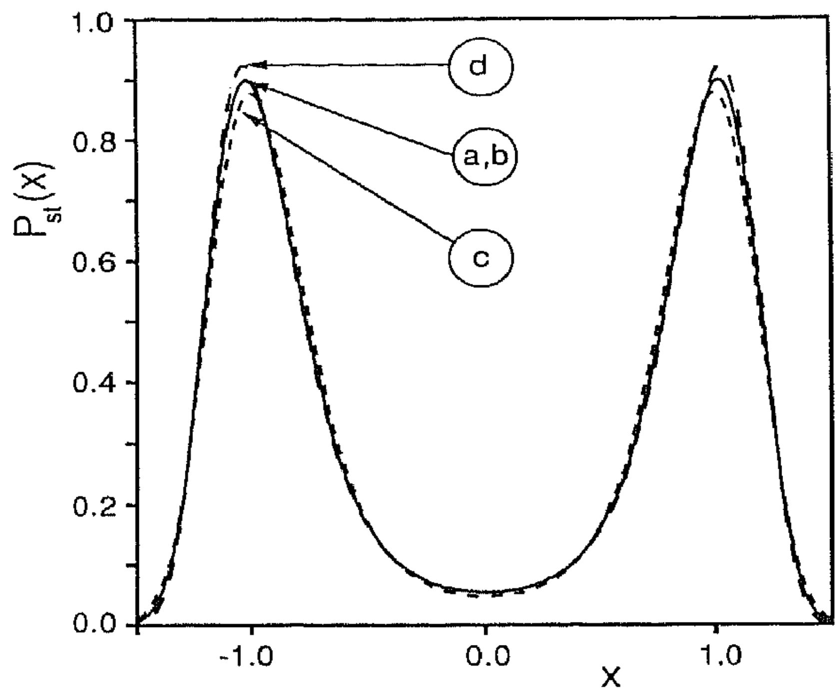


Figure 6.5. The approximate expressions, Eq. (6.48) (small correlation time approximation) (d), Eq. (6.50) (unified colored noise approximation) (b), and Eq. (6.52) (decoupling approximation) (c) are compared with the numerical solutions (a) at $\tau = 0.1$ and $D = 0, 1$.

In all expressions, Z is the respective normalization factor. In Fig. 6.5, the approximate expressions for the stationary probability densities are compared with numerical results obtained from the full numerical solutions at $D = 0.1$ and $\tau = 0.1$. The agreement is good for all approximations. In Fig. 6.6 we have compared the numerical result against the stationary densities obtained by using unified colored noise approximation and by using the decoupling ansatz for $\tau = 1$. The unified colored noise approximation breaks down locally at $x = 0$. The overall agreement, however, is still good. The decoupling approximation yields a distribution with infinite support, but the overall agreement is not very good.

C. Colored Noise Induced Escape Rates and Mean First-Passage Times

A central, but in recent years also very controversial problem [134–138, 144, 154–158, 161–176, 193] is the dependence of the escape rates and the mean first-passage times on the noise correlation time τ. To describe a decay process out of a region Ω in phase space by a *escape rate*, the decay of the population in this region has to be *exponential* on its longest time scale, that is, Ω has to be a basin of attraction. The system has escaped when it has crossed the basin boundary of the basin of attraction. To uniquely identify a basin of attraction and a basin boundary we have to consider the stochastic dynamics in the extended-phase space. In 1-D x

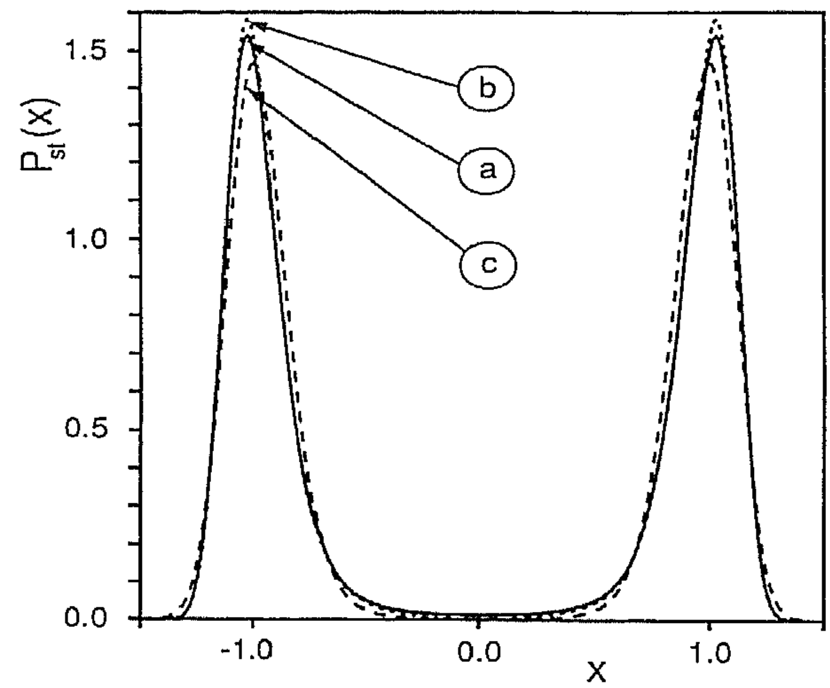


Figure 6.6. The approximate expressions, Eq. (6.50) (unified colored noise approximation) (b) and Eq. (6.52) (decoupling approximation) (c) are compared with the numerical stationary probabilities (a) at $\tau = 1$ and $D = 0, 1$.

space, a certain value of x cannot be considered to be in one or the other basin of attraction, since due to the memory of the noise, the time evolution depends on the prehistory of the process. The attractors of the 2-D system of equations

$$\begin{aligned} \dot{x} &= x - x^3 + \varepsilon \\ \dot{\varepsilon} &= -\frac{1}{\tau}\varepsilon \end{aligned} \tag{6.54}$$

are given by the points ($x_1 = -1$, $\varepsilon_1 = 0$) (stable node), ($x_2 = 1$, $\varepsilon_2 = 0$) (stable node), and ($x_3 = 0$, $\varepsilon_3 = 0$) (saddle point). The separatrix is obtained as the solution of the differential equation [158]

$$\frac{d\varepsilon}{dx} = -\frac{1}{\tau}\frac{\varepsilon}{x - x^3 + \varepsilon} \tag{6.55}$$

with the initial condition

$$\varepsilon(0) = 0 \tag{6.56}$$

Full analytical solutions of Eq. (6.55) are not known to the authors. Near the saddle point ($x=0$, $\varepsilon=0$), the solution is found to be [158]

$$\varepsilon(x) = -\left(1+\frac{1}{\tau}\right)x \tag{6.57}$$

while for large correlation times τ the solution is

$$\varepsilon(x) = \begin{cases} x^3 - x & \text{for } |x| < 1/\sqrt{3} \\ \frac{2}{3}\frac{1}{\sqrt{3}} & \text{for } x \leq -1/\sqrt{3} \\ -\frac{2}{3}\frac{1}{\sqrt{3}} & \text{for } x \geq 1/\sqrt{3} \end{cases} \tag{6.58}$$

In Fig. 6.7, we show some numerically obtained trajectories, cf. Eq. (6.54)], and the separatrix for $\tau=0.1$, $\tau=1$, $\tau=10$, and $\tau=50$. We note that the asymptotic separatrix for large τ is approached only at extremely large values of τ ($\tau=50$ is certainly not sufficient). The noise induced escape across the separatrix from the left to the right well takes place at positive values of ε. For small correlation times, the actual escape takes place at large values of $|\varepsilon|$, since the noise acts only in the ε direction. For increasing τ the separatrix bends over and the escape takes place across the separatrix at smaller values of $|\varepsilon|$. For large τ, the trajectories avoid a region around the origin. This unstable region is also responsible for the formation of the crater of the stationary joint probability density $W_{st}(x, \varepsilon)$ in the extended phase space.

For weak noise, that is, $D \to 0$, the mean first-passage time to the separatrix T_s is related to the escape rate r_s by [123, 167]

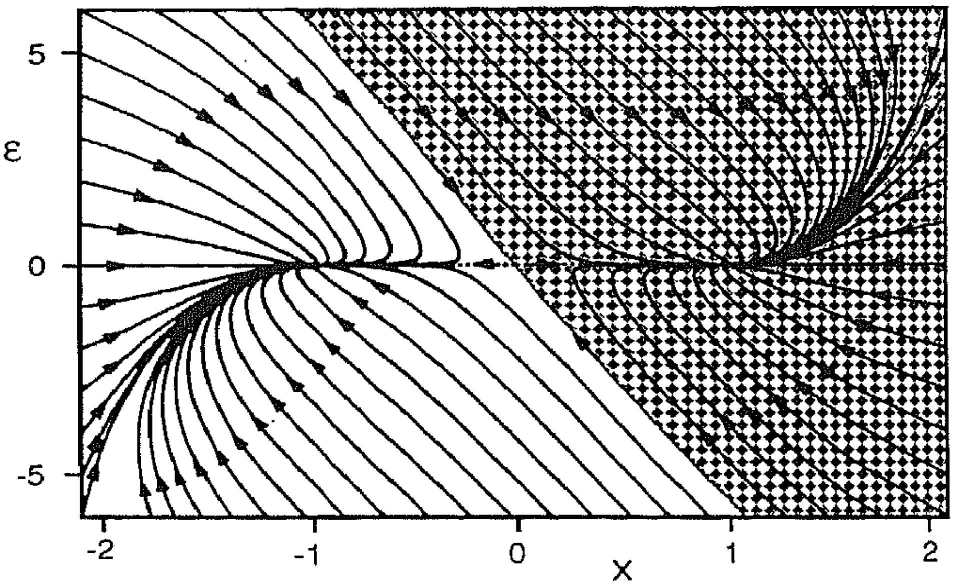
$$r_s = \frac{1}{2T_s} \tag{6.59}$$

The smallest nonvanishing eigenvalue λ_{min} is related to the escape rate r_s by

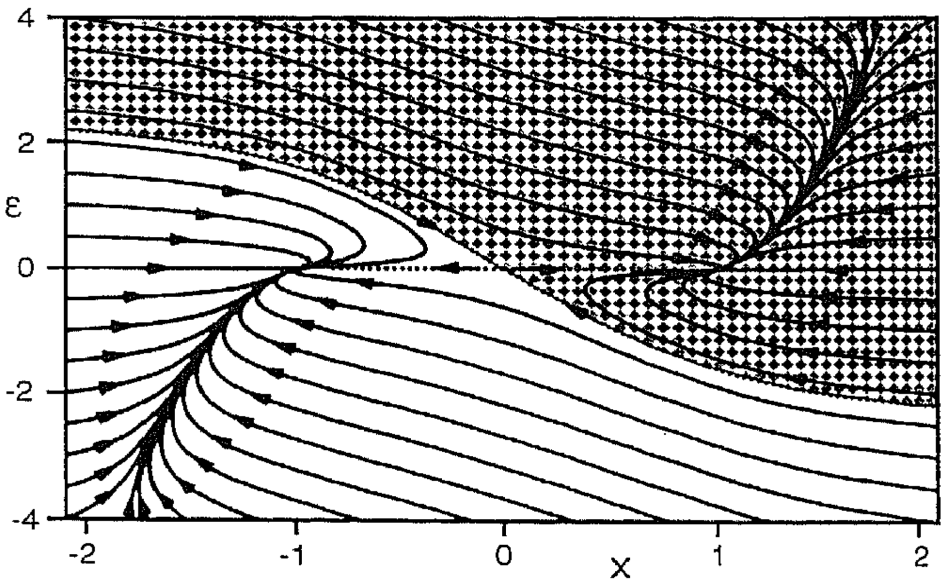
$$\lambda_{min} = 2r_s = \frac{1}{T_s} \tag{6.60}$$

The smallest nonvanishing eigenvalues obtained in [154], have been plotted as a function of the correlation time of the noise in Fig. 6.3. In Fig. 6.8, we depict the color induced rate suppression

$$\eta(\tau, D) = \frac{r_s(\tau)}{r_s(\tau = 0)} \tag{6.61}$$

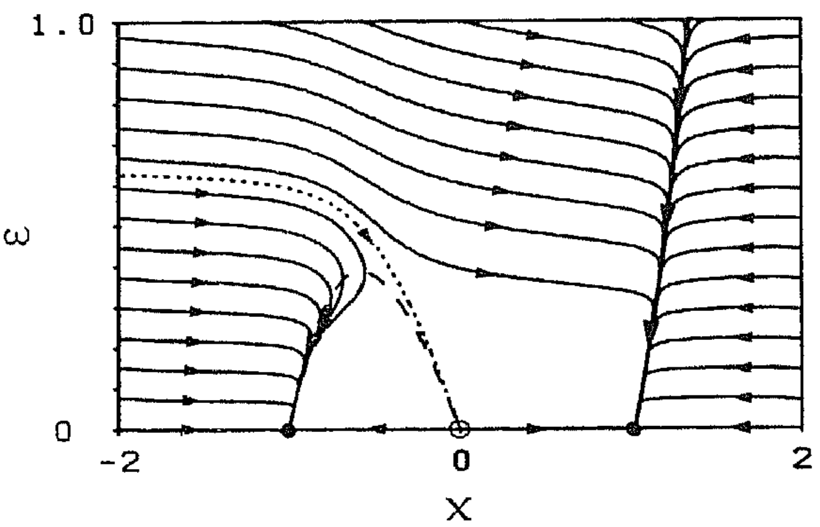


(a)

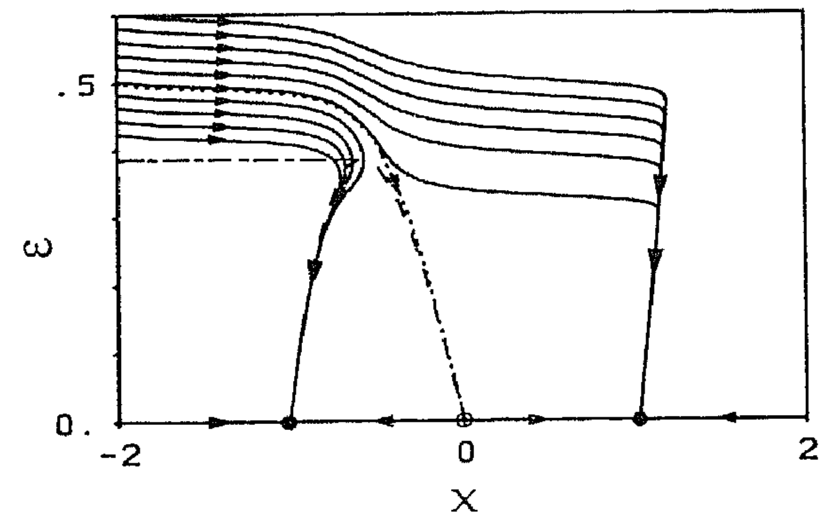


(b)

Figure 6.7. The deterministic trajectories, Eq. (6.54), are shown for $\tau = 0.1(a)$, $\tau = 1(b)$, $\tau = 10(c)$, and $\tau = 50(d)$. The dotted lines indicate the separatrix $\varepsilon(x)$ in (c) and (d) while in (a) and (b) the separatrix is the border between the hatched and nonhatched regions—the basins of attractions. The limiting result in Eq. (6.58) for $\varepsilon(x)$ is indicated by the dash-dotted line in (d).



(c)



(d)

Figure 6.7. (*Continued*)

1. *Escape Rates for Weakly Colored Noise*

For small correlation times τ, the small correlation time approximation, in Eq. (5.6) valid on a large system time scale (i.e. small a), yields the 1-D eigenvalue equation

$$\begin{aligned} \mathbf{L}_{\mathrm{SRTA}}\psi_{\min} &= -\lambda_{\mathrm{SRTA}}\psi_{\min} \\ \mathbf{L}_{\mathrm{SRTA}} &= -\frac{\partial}{\partial x}(x-x^3)+\frac{\partial^2}{\partial x^2}[1+\tau(1-3x^2)] \end{aligned} \tag{6.62}$$

Accordingly, the approximation by Fox [106, 137] leads to the 1-D

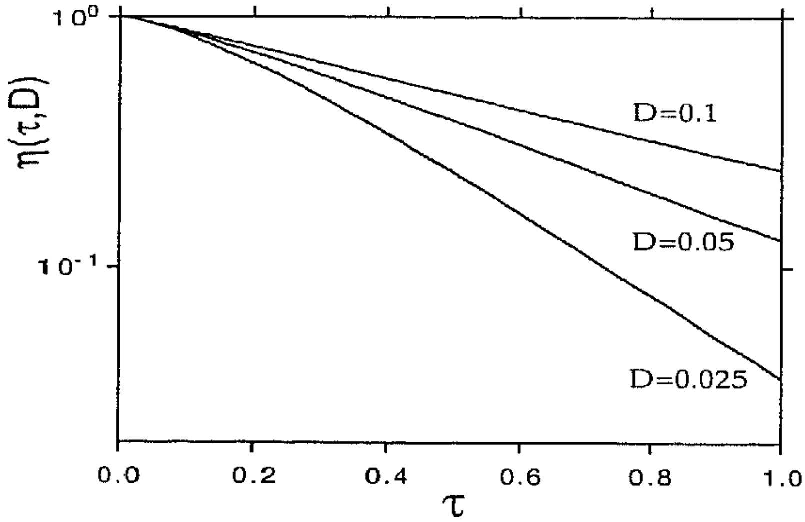


Figure 6.8. The noise color induced suppression of the escape rate $\eta(\tau, D)$ [Eq. (6.61)] is shown for $D=0.1$, $D=0.05$, and $D=0.025$.

eigenvalue equation

$$\begin{aligned} \mathbf{L}_{\text{Fox}}\psi_{\min} &= -\lambda_{\text{Fox}}\psi_{\min} \\ \mathbf{L}_{\text{Fox}} &= -\frac{\partial}{\partial x}(x-x^3) + D\frac{\partial^2}{\partial x^2}\frac{1}{1-\tau(1-3x^2)} \end{aligned} \tag{6.63}$$

The results of the numerical solutions of Eqs. (6.62) and (6.63) are compared in Fig. 6.9 with the smallest nonvanishing eigenvalue, obtained from the full 2-D Fokker–Planck equation.

Using the method of the effective small τ potential [168], one can find, within the small correlation time theory, an expression for the correlation time corrections of the escape rate valid for weakly colored noise. In the new variable

$$y = x - \tfrac{1}{2}\tau(x - x^3) \tag{6.64}$$

the diffusion coefficient of the Fokker–Planck operator in Eq. (6.62) is a constant, and the drift term corresponds to the potential

$$V_{\text{eff}}(y) = \tfrac{1}{4}x^4 - \tfrac{1}{2}(1 + 3D\tau)x^2 \tag{6.65}$$

which is again a potential of the quartic type, but with renormalized coefficients. Applying Kramers' formula for the escape rate valid for

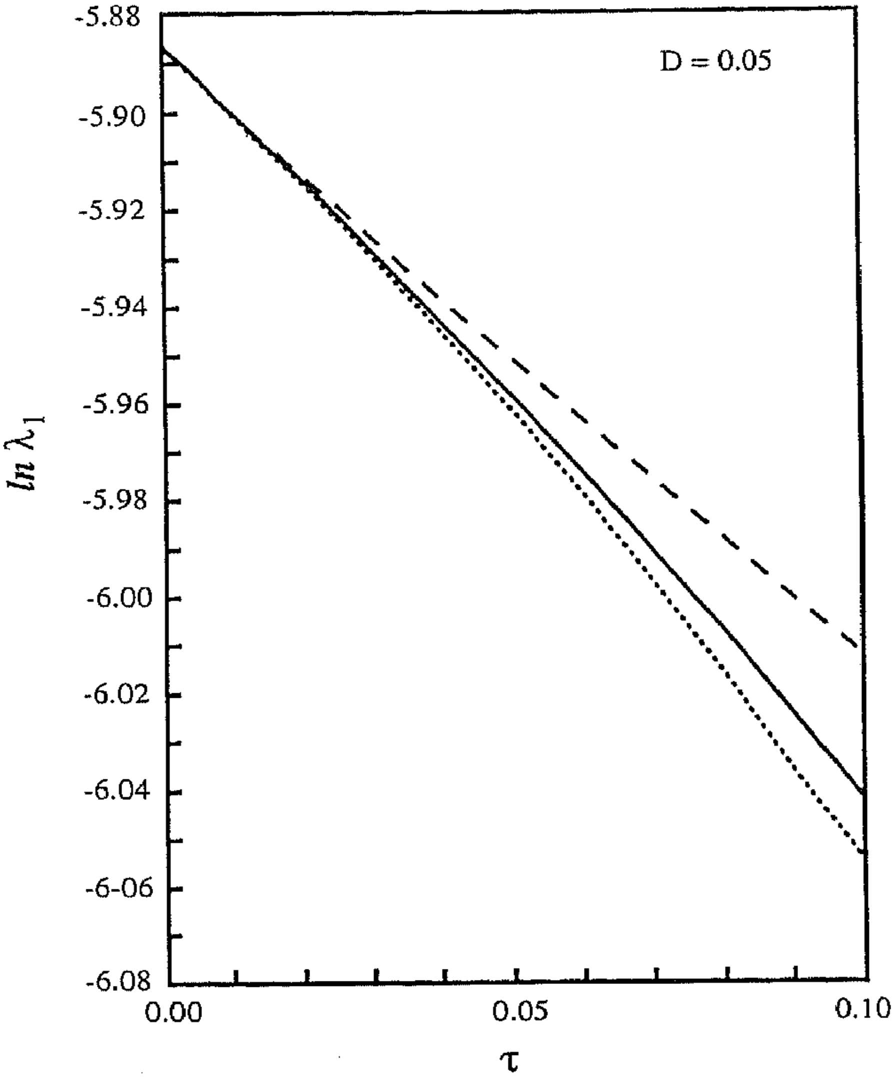


Figure 6.9. The smallest nonvanishing eigenvalues, obtained numerically from the approximate Fokker–Planck equations (6.62) (dotted) and (6.63) (dashed), are compared with the eigenvalues obtained from the full 2-D eigenvalue equation (6.11) (solid) at $D = 0.1$.

weak noise (for a recent review, see [123])

$$r_s = \frac{1}{2\pi}\sqrt{|V''_{\text{eff}}(x_{\text{min}})V''_{\text{eff}}(x_{\text{saddle}})|}\,\exp\left(-\frac{\Delta V}{D}\right) \tag{6.66}$$

where ΔV is the barrier height, x_{min} is the position of the local potential minimum, and x_{saddle} is the position of the barrier top. Inserting the expression for the effective potential, we find

$$r_s(\tau) = r_s(\tau = 0)[1 - \beta(D)\tau] \tag{6.67}$$

where

$$\beta(D) = \tfrac{3}{2} - 3D \tag{6.68}$$

This expression is asymptotically for $\tau \to 0$ exact. The dilemma, however, is that within the small correlation time theory for $D \to 0$, we also have to reduce τ according to the condition $\tau/D \to 0$.

In the limit $D \to 0$ and *small* but finite τ, corrections to the rate have been obtained by use of a variety of methods. The result reads [123, 134, 154, 156, 161, 169–171]

$$r_s(\tau) = r_s(\tau = 0)\{1 - \beta(D)\tau\} \exp -\left[\frac{\tau^2}{8D} + \frac{3\tau^4}{10D} + O\frac{\tau^6}{D}\right] \tag{6.69}$$

2. *Escape Rates for Strongly Colored Noise*

For large correlation times, corrections to the exponential part of the escape rates have been determined by using path integral techniques, adiabatic arguments, or by using the unified colored noise approximation. The result in leading order reads [154, 157]

$$r_s \propto \exp\left(-\frac{2\tau}{27D}\right) \tag{6.70}$$

Comparison with numerical solutions for the escape rate at finite values of τ, however, shows that this result is actually very asymptotic [154, 158, 166]. The dependence of the exponential part of the numerically evaluated rate in fact shows a dependence of the type in Eq. (6.70) but with a factor other than 2/27, that is $2/27 = 0.074\ldots \to \sim 0.1$ [154]. Luciani and Verga [156] derived a bridging formula, connecting the approximations at small τ and large τ, given for $\Delta V = \frac{1}{4}$

$$r_s = \frac{1}{\sqrt{2}\pi}(1 + 3\tau)^{-1/2} \exp\left[-\frac{1}{4D}\left\{\frac{1 + \frac{27}{16}\tau + \frac{1}{2}\tau^2}{1 + \frac{27}{16}\tau}\right\}\right] \tag{6.71}$$

3. *Mean First-Passage Times for Other Boundary Conditions*

So far, we have discussed escape rates and its connection to mean first-passage times to leave the basin of attraction. Since the concept of mean first-passage times is valid for more general regions in phase space, one can also ask for the mean first-passage time to leave the right or left infinite one-half plane, $x < 0$ or $x > 0$. This problem has been studied by Doering et al. [172] for weakly colored noise. They solved the Fokker–Planck equation in the extended-phase space $(x, z = \varepsilon/\langle \varepsilon^2 \rangle)$ with a

Gaussian source term $\delta(x-1)\rho_0(z)$ and an absorbing boundary at $x=0$ for $z>0$. Those boundary conditions yield a current-carrying stationary solution from which they can derive an expression for the mean first-passage time. In order to obtain a solution for weakly colored noise, they expand the stationary current carrying solution in functions of the orders of $\sqrt{\tau}$. As a result one finds

$$T_{x=0}=\frac{\pi}{\sqrt{2}}\exp\left(\frac{1}{4D}\right)\left[1+\sqrt{\frac{2}{\pi}}\,\lambda_M\sqrt{\tau}+\frac{3}{2}\tau\right] \tag{6.72}$$

with the Milne extrapolation length λ_M, given in terms of the Riemannian ζ function, by $\lambda_M=-\zeta(1/2)=1.460354$. This $\sqrt{\tau}$ correction has been established by simulating the Langevin equations $\dot{x}=x-x^3+\varepsilon$, $\dot{\varepsilon}=-(1/\tau)\varepsilon+\sqrt{D}/\tau\xi(t)$. Those results have been contrasted with the escape rates over the separatrix in detail in [173–176].

D. Colored Noise Driven Systems with Inertia

Up to now, we have neglected the inertial effect completely, that is, we have assumed that the velocity relaxation takes place on a very fast time scale in comparison with other time scales. For finite inertia, we introduce another finite time scale $\tau_r=1/\gamma$ into the system. The normalized Langevin equations read

$$\begin{aligned}\dot{x}&=v\\ \dot{v}&=-\gamma v+f(x)+\varepsilon\\ \dot{\varepsilon}&=-\frac{1}{\tau}\varepsilon+\frac{\sqrt{D}}{\tau}\xi(t)\end{aligned} \tag{6.73}$$

with Gaussian white noise $\xi(t)$,

$$\begin{aligned}\langle\xi(t)\rangle&=0\\ \langle\xi(t)\xi(t')\rangle&=2\delta(t-t')\end{aligned} \tag{6.74}$$

The corresponding Fokker–Planck equation

$$\frac{\partial}{\partial t}P(x,v,\varepsilon,t)=\left[-\frac{\partial}{\partial x}v+\gamma\frac{\partial}{\partial v}(v-f(x)-\varepsilon)+\frac{1}{\tau}\frac{\partial}{\partial\varepsilon}\varepsilon+\frac{D}{\tau^2}\frac{\partial^2}{\partial\varepsilon^2}\right]\times P(x,v,\varepsilon,t) \tag{6.75}$$

can be solved analytically for a quadratic potential, $f(x)=-\omega_0^2x$, only.

For the stationary distribution one then obtains

$$P_{st}(x, v, \varepsilon) = \frac{1}{Z} \exp\left(-\frac{\phi(x, v, \varepsilon)}{D}\right) \tag{6.76}$$

with the potential

$$\begin{aligned}\phi(x, v, \varepsilon) = {} & \tfrac{1}{2}\gamma\omega_0^2(1 + \tau^2\omega_0^2)x^2 + \gamma^2\tau^2\omega_0^2 xv + \tfrac{1}{2}\gamma[(1 + \gamma\tau)^2 + \tau^2\omega_0^2]v^2 \\ & + \tfrac{1}{2}\tau(1 + \gamma\tau)\varepsilon^2 - \gamma\tau(1 + \gamma\tau)\varepsilon v - \gamma\tau^2\omega_0^2 x\varepsilon \end{aligned} \tag{6.77}$$

The quadratic form for the potential can be diagonalized by introducing the new variable q instead of ε [111]

$$q = -\gamma v + \varepsilon - \frac{\gamma\tau\omega_0^2 x}{1 + \gamma\tau} \tag{6.78}$$

The potential then factorizes, that is,

$$\phi(x, v, q) = \tfrac{1}{2}\tau(1 + \gamma\tau)q^2 + \tfrac{1}{2}(1 + \gamma\tau + \tau^2\omega_0^2)\gamma v^2 + \tfrac{1}{2}\gamma\omega_0^2\left(1 + \frac{\tau^2\omega_0^2}{1 + \gamma\tau}\right)x^2 \tag{6.79}$$

This factorization for the parabolic potential implies for a general force field $f(x)$ the introduction of the new variable [111]

$$q = -\gamma v + \varepsilon - \frac{\gamma\tau f(x)}{1 + \gamma\tau} \tag{6.80}$$

The system of Langevin equations then reads [111]

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= q + \frac{f(x)}{1 + \gamma\tau} \\ \dot{q} &= -\left(\gamma + \frac{1}{\tau}\right)q + \frac{\gamma}{\tau}\left[\frac{\tau^2}{1 + \gamma\tau} f'(x) - 1\right]v + \frac{\sqrt{D}}{\tau}\,\xi(t) \end{aligned} \tag{6.81}$$

1. *Small Correlation Time Approximation*

Starting from the Fokker–Planck equation in Eq. (6.75),

$$\frac{\partial}{\partial t} P = (\mathbf{A} + \varepsilon\mathbf{B} + \mathbf{L}_\varepsilon)P \tag{6.82}$$

where

$$\mathbf{A} = -\frac{\partial}{\partial x} v + \gamma \frac{\partial}{\partial v} v - f(x) \frac{\partial}{\partial v}$$

$$\mathbf{B} = -\frac{\partial}{\partial v} \tag{6.83}$$

$$\mathbf{L}_\varepsilon = \frac{1}{\tau} \frac{\partial}{\partial \varepsilon} \varepsilon + \frac{D}{\tau^2} \frac{\partial^2}{\partial \varepsilon^2}$$

we apply the same small τ-expansion technique as presented in Appendix A1 of [5] (generalized to a three dimensional (3-D) Fokker–Planck equation). As the result one finds for large times and $\gamma \ll 1/\tau$ up to order $O(\tau^2)$ the 2-D Fokker–Planck approach

$$\frac{\partial}{\partial t} P(x, v, t) = \mathbf{L}_{\text{SCTA}} P(x, v, t) \tag{6.84}$$

where

$$\mathbf{L}_{\text{SCTA}} = \mathbf{A} + D\mathbf{B}^2 + \tau D \mathbf{B}[\mathbf{A}, \mathbf{B}] + D\tau^2 \mathbf{B}[\mathbf{A}, [\mathbf{A}, \mathbf{B}]]$$
$$+ D^2\tau^2(\mathbf{B}[[\mathbf{B}, \mathbf{A}], \mathbf{B}]\mathbf{B} + \tfrac{1}{2}\mathbf{B}^2[[\mathbf{B}, \mathbf{A}], \mathbf{B}]) \tag{6.85}$$

with $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$. The same result can also be obtained by extending the functional technique to higher dimensions [146, 177]. Inserting the expressions for the operators $\mathbf{A}$ and $\mathbf{B}$, we obtain the Fokker–Planck type operator (see Appendix of [111])

$$\mathbf{L}_{\text{SCTA}} = -\frac{\partial}{\partial x} v + \gamma \frac{\partial}{\partial v} v - f(x) \frac{\partial}{\partial v} + D(1 - \gamma\tau + \tau^2 f'(x) + \tau^2\gamma^2) \frac{\partial^2}{\partial v^2}$$
$$+ \tau D(1 - \gamma\tau) \frac{\partial^2}{\partial x\, \partial v} + O(\tau^3) \tag{6.86}$$

The stationary distribution can be obtained analytically up to order τ and reads

$$P_{\text{st}}^{(1)}(x, v) = \frac{1}{Z} \exp\left[-\frac{\gamma}{D} U(x) - \frac{\gamma}{2D(1-\gamma\tau)} v^2\right] \tag{6.87}$$

where

$$U(x) = -\int^x f(y)\, dy \tag{6.88}$$

The distribution in x and v factorize within this approximation. The shape of the distribution function in x only is in first order of τ, independent of the correlation time of the noise. For a harmonic potential $U(x) = \frac{1}{2}\omega_0^2 x^2$, the stationary probability density can be computed and obtained up to second order τ^2, yielding for the variance $\sigma_{xx} = \langle x^2 \rangle_{st} - \langle x \rangle_{st}^2 = \langle x^2 \rangle_{st}$

$$\sigma_{xx} = \frac{D}{\omega_0^2 \gamma}(1 - \omega_0^2 \tau^2) \tag{6.89}$$

which agrees with the exact solution up to order τ^2.

2. *Unified Colored Noise Approximation*

The starting point for the application of the unified colored noise approximation is the set of equations in Eq. (6.81). Performing a time scale transformation $t \to t/\sqrt{\tau}$, we observe that we can adiabatically eliminate q for [111]

$$\frac{\gamma\sqrt{D\tau}}{(\sqrt{1+\gamma\tau})^3} \ll 1 \tag{6.90}$$

and

$$t > \frac{\tau}{1+\gamma\tau} \tag{6.91}$$

The Fokker–Planck equation for this approach reads [111]

$$\frac{\partial}{\partial t} P(x, v, t) = \mathbf{L}_{\mathrm{UCNA}} P(x, v, t) \tag{6.92}$$

where

$$\begin{aligned} \mathbf{L}_{\mathrm{UCNA}} = & -\frac{\partial}{\partial x} v + \left[\frac{\gamma}{1+\gamma\tau}\left(1 - \frac{\tau^2}{1+\gamma\tau} f(x)\right)\right] \frac{\partial}{\partial v} v \\ & - \frac{f(x)}{1+\gamma\tau}\frac{\partial}{\partial v} + \frac{D}{(1+\gamma\tau)^2}\frac{\partial^2}{\partial v^2} \end{aligned} \tag{6.93}$$

The stationary probability density can only be obtained analytically for the harmonic potential $U(x) = \frac{1}{2}\omega_0^2 x^2$. In this case, it precisely agrees with the exact stationary probability density Eqs. (6.76) and (6.77). The unified colored noise approximation—as a truly Markovian approximation—also correctly describes the dynamical quantities, such as correlation functions within its range of validity Eqs. (6.90) and (6.91). In contrast to the small correlation time approximation, the high friction limit can be carried through by another adiabatic approximation, yielding the (Stratonovich) Langevin equation

$$\dot{x} = \frac{f(x)}{\gamma - \tau f'(x)} + \frac{\sqrt{D}}{\gamma - \tau f'(x)} \xi(t) \tag{6.94}$$

which is up to a time scale $(1/\gamma)$ identical with the standard unified colored noise approximation for overdamped systems, discussed in Section V.C. This higher dimensional UCNA in Eqs. (6.92) and (6.93) has recently been applied to study colored noise driven bistability by Schimansky-Geier and Zülicke [178], and for modeling the dynamics in dye lasers by Cao et al. [179].

3. *Decoupling Approximation*

The probability density $P(x, v, t)$ for the non-Markovian stochastic process, described by the Langevin equation, Eq. (6.73), obeys the integrodifferential equation [146]

$$\begin{aligned}
\frac{\partial}{\partial t} P(x, v, t) &= \frac{\partial}{\partial t} \langle \delta(x(t) - x)\delta(v(t) - x) \rangle \\
&= -v \frac{\partial}{\partial x} P(x, v, t) - f(x) \frac{\partial}{\partial v} P(x, v, t) + \gamma \frac{\partial}{\partial v} (vP(x, v, t)) \\
&\quad + \frac{\partial^2}{\partial x\, \partial v} \left[\frac{D}{\tau} \int_0^t \exp - \left(\frac{t-s}{\tau}\right) \left\langle \delta(x(t) - x)\delta(v(t) - v) \frac{\delta x(t)}{\delta \xi(s)} \right\rangle \right] \\
&\quad + \frac{\partial^2}{\partial v^2} \left[\frac{D}{\tau} \int_0^t \exp - \left(\frac{t-s}{\tau}\right) \left\langle \delta(x(t) - x)\delta(v(t) - v) \frac{\delta v(t)}{\delta \xi(s)} \right\rangle \right]
\end{aligned} \tag{6.95}$$

where the functional derivatives $\delta(x(t))/\delta(\xi(s))$ and $\delta(v(t))/\delta(\xi(s))$ obey the coupled integrodifferential equations [146]

$$\begin{aligned}
\frac{\delta x(t)}{\delta \xi(s)} &= \theta(t-s)\left[\int_s^t \frac{\delta v(r)}{\delta \xi(s)}\, dr \right] \\
\frac{\delta v(t)}{\delta \xi(s)} &= \theta(t-s)\left[1 - \int_s^t f(x(r)) \frac{\delta x(r)}{\delta \xi(s)}\, dr - \gamma \int_s^t \frac{\delta v(r)}{\delta \xi(s)}\, dr \right]
\end{aligned} \tag{6.96}$$

Factorization of the probability density and the functional derivatives yields a closed equation of the Fokker–Planck type for the probability density $P(x, v, t)$, that is [146],

$$\frac{\partial}{\partial t} P(x, v, t) = \mathbf{L}_{\mathrm{Dec}} P(x, v, t)$$
$$\mathbf{L}_{\mathrm{Dec}} = -v \frac{\partial}{\partial x} + \frac{\partial}{\partial v} [\gamma v - f(x)] + \frac{D}{1 + \gamma\tau - \tau^2 \langle f'(x) \rangle} \frac{\partial^2}{\partial v^2}$$
$$+ \frac{D\tau}{1 + \gamma\tau - \tau^2 \langle f'(x) \rangle} \frac{\partial^2}{\partial x\, \partial v} \tag{6.97}$$

The stationary probability density factorizes in x and v, that is,

$$P_{\mathrm{st}}(x, v) = \frac{1}{Z} \exp\left(-\frac{v^2}{2\sigma_{vv}}\right) \exp\left(-\frac{U(x)}{\sigma_{xx}}\right) \tag{6.98}$$

with the potential $U(x)$ defined in Eq. (6.88) and the covariances

$$\sigma_{xx} = \frac{D}{\gamma} \frac{1}{1 - \tau^2 [\langle f'(x) \rangle / (1 + \gamma\tau)]}$$
$$\sigma_{vv} = \frac{D}{\gamma} \frac{1}{1 + \gamma\tau - \tau^2 \langle f'(x) \rangle} \tag{6.99}$$

Tracing out the velocity, we obtain the equation of motion for the probability density in the position x only [146]

$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} \tilde{f}(x) P(x, t) + \frac{\tilde{D}(1 + (1/\gamma\tau))}{1 + (1/\gamma\tau) - \tau \langle \tilde{f}'(x) \rangle} \frac{\partial^2}{\partial x^2} P(x, t) \tag{6.100}$$

where

$$\tilde{f}(x) = \frac{f(x)}{\gamma}$$
$$\tilde{D} = \frac{D}{\gamma^2} \tag{6.101}$$

The prediction of the decoupling theory has been tested by using analogue simulations [147]. Although the agreement of the stationary joint probability density $P_{\mathrm{st}}(x, v)$ is not very good (incorrect symmetry due to factorization), the agreement of the reduced density with the analogue simulation result is excellent. The mean values, occurring in the

Fokker–Planck approach have to be determined self-consistently. Since the decoupling theory is not restricted to small correlation times, one can also perform the overdamped limit $\gamma\tau \gg 1$,

$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} \tilde{f}(x) P(x, t) + \frac{\tilde{D}}{1 - \tau\langle \tilde{f}'(x) \rangle} \frac{\partial^2}{\partial x^2} P(x, t) \quad (6.102)$$

This equation agrees precisely with the standard decoupling theory for overdamped systems.

VII. MULTIPLICATIVE COLORED NOISE AND PHOTON STATISTICS OF DYE LASERS

The study of dynamical systems in optical sciences is attracting rapidly growing interest. In particular, the fields of optical bistability and chaos in optical systems have become the main focus of interest for many researchers. Here, we restrict ourselves to the influence of noise in nonlinear optical system. Experiments with dye lasers strongly emphasize the role played by noise sources with a finite correlation time. For dye lasers strongly correlated noise enters via the pumping mechanism and it crucially impacts their photon statistics.

Some time ago it was found that the behavior of a single mode dye laser was not very well described by the usual single mode laser theory [180–182]. Short and co-workers [10, 182] suggested taking into account fluctuations of the pump parameter, to describe the dye laser close to threshold. Graham and co-workers [183, 184] and Schenzle and Brand [185] developed a stochastic model for the field of the single mode laser, which incorporates δ correlated pump noise. Some intensity correlation functions compared very nicely with the results of this one-fit parameter model. It turned out, however, that it is not possible to explain experimental data at different working points of the laser with one value of the fit parameter [10]. Short et al. [10] concluded from their experiments that the pump noise should be slower than the fluctuations of the intensity. Dixit and Sahni [11] and Schenzle and Graham [12] first discussed the impact of colored pump fluctuations for the photon statistics of dye lasers. While the stochastic equations have been simulated in [10], Schenzle and Graham [12] used a small correlation time expansion. Numerical studies of the dye laser model for arbitrary correlation times have been put forward in [186, 187].

To describe the behavior of the laser close to the threshold correctly, one has to take into account both pump and quantum fluctuations [188]. For adiabatically eliminated inversion and polarization, the equation of motion for the complex field amplitude E reads [149, 188, 189]

$$\dot{E} = a_0 E - AE|E|^2 + p(t)E + q(t) \tag{7.1}$$

where

$$\begin{aligned} E &= E_1 + iE_2 \\ p &= p_1 + ip_2 \\ q &= q_1 + iq_2 \end{aligned} \tag{7.2}$$

In Eq. (7.1) a_0 denotes the pump parameter; A denotes the saturation parameter, which limits the stationary field amplitude to a finite value; q denotes the quantum fluctuations due to spontaneous emission processes, being important at low-field intensities; and $p(t)$ denotes the pump fluctuations. Both, quantum and pump fluctuations are assumed to be Gaussian distributed with zero mean. The correlation functions are given by

$$\begin{aligned} \langle q_i(t) q_j(t') \rangle &= 2\tilde{q}\delta(t-t')\delta_{ij} \\ \langle p_i(t) p_j(t') \rangle &= \frac{\tilde{D}}{\tilde{\tau}} \exp\left[-\frac{1}{\tilde{\tau}}\left|t-t'\right|\right]\delta_{ij} \end{aligned} \tag{7.3}$$

Some typical sets of parameters $\tilde{D}$, $\tilde{q}$ have been obtained by Zhou et al. [13] and Roy et al. [190] by comparing simulations of Eqs. (7.2) and (7.3) with experimental data. A typical set of parameters is $a_0 = 0.7 \times 10^6\,\mathrm{s}^{-1}$, $A = 0.114 \times 10^6\,\mathrm{s}^{-1}$, $\tilde{q} = 2 \times 10^{-3}\,\mathrm{s}^{-1}$, $\tilde{D} = 4.9 \times 10^3\,\mathrm{s}^{-1}$, and $\tilde{\tau} = 5.0 \times 10^{-7}\,\mathrm{s}$.

Above threshold (i.e. $a_0 > 0$) we transform the equation of motion for the complex field amplitude into a set of equations for intensity I and phase ϕ. The equation of motion for the intensity reads in suitable normalized form, that is, with $t \to a_0 t$, $I \to A/a_0 I$

$$\begin{aligned} \dot{I} &= 2(I - I^2) + 2I\varepsilon + \frac{Q}{2} + \sqrt{QI}\,q(t) \\ \dot{\varepsilon} &= -\frac{1}{\tau}\varepsilon + \frac{\sqrt{D}}{\tau}\xi(t) \end{aligned} \tag{7.4}$$

where

$$\begin{aligned} \tau &= a_0 \tilde{\tau} \\ D &= \frac{\tilde{D}}{a_0} \\ Q &= 4 \frac{\tilde{q}}{a_0^2} A \end{aligned} \tag{7.5}$$

Equation (7.4) already takes advantage of the embedding property introduced in the last section. Inserting the experimental values for the parameters in Eq. (7.5), we obtain the values for the normalized parameters $D = 7 \times 10^{-3}$, $Q = 1.86 \times 10^{-9}$, $\tau = 0.35$. It is interesting to note that the quantum fluctuations are six orders of magnitude smaller than the pump fluctuations, and that the noise correlation time τ is not much smaller than the typical system time scale, which is in our normalized units of $\tau_s = 1$. Other experimental sets yield even much larger values of τ in the order 10. This makes it clear that in order to understand the dye laser one needs theories for dynamical systems driven by colored noise that are *nonperturbative* in the correlation time of the noise. If we neglect quantum fluctuations that can be safely done if we are interested only in the stationary properties of the laser light intensity and those properties that are not too close to threshold, the equation of motion for the field intensity reads

$$\begin{aligned} \dot{I} &= 2(a - I)I + 2I\varepsilon \\ \dot{\varepsilon} &= -\frac{1}{\tau}\varepsilon + \frac{\sqrt{D}}{\tau}\xi(t) \end{aligned} \tag{7.6}$$

with the δ correlated Gaussian stochastic process $\xi(t)$. In Eq. (7.6) we have made use of a different scaling as compared to Eq. (7.4) by invoking the dimensionless pump parameter a, i.e. $a_0 \rightarrow a \cdot a_0$. This gives the possibility to vary the working point of the laser. For $a < 0$, that is, below threshold, the laser is off. In Eq. (7.6) this is reflected in a vanishing stationary mean value of the intensity, that is, $\langle I_{\mathrm{st}} \rangle = 0$. For $a > 0$, that is, above threshold, the laser is on. Accordingly, the stationary mean value of the intensity is given by $\langle I_{\mathrm{st}} \rangle = a$. Note that the mean value agrees identically with the behavior of the noiseless system $\dot{I} = 2(a - I)I$; the noise has not shifted the bifurcation point. Some quantities of interest are the stationary probability distribution $P_{st}(I)$, since it directly relates to the photon counting statistics [191, 192], and the correlation function

$$\phi_I(t) = \frac{\langle (I(t) - \langle I \rangle_{st})(I(0) - \langle I \rangle_{st}) \rangle}{\langle I^2 \rangle_{st} - \langle I \rangle_{st}^2} \tag{7.7}$$

which relates, via the Wiener-Khintchine theorem, to the fluctuation induced line width. The line width is characterized by the effective eigenvalue λ_{eff}, or equivalently by the inverse of the relaxation time

$$T = \int_0^\infty \phi_I(t)\, dt = \frac{1}{\lambda_{eff}} \tag{7.8}$$

A. The White Noise Limit

In the white noise limit, the (Stratonovich) Fokker–Planck equation for the intensity reads

$$\frac{\partial}{\partial t} P(I, t) = \mathbf{L}_0 P(I, t) \tag{7.9}$$

where

$$\mathbf{L}_0 = -2 \frac{\partial}{\partial I}(a - I)I + 4D \frac{\partial}{\partial I} I \frac{\partial}{\partial I} I \tag{7.10}$$

Above threshold, the stationary distribution function is given by [6]

$$P_{st}(I) = P_{st}^{\tau=0}(I) = \frac{1}{Z}(4DI)^{\kappa_0} \exp\left(-\frac{I}{2D}\right) \tag{7.11}$$

with the normalization constant (Γ denotes the Gamma function)

$$Z = \frac{1}{4D}(8D^2)^{\kappa_0+1}\Gamma(\kappa_0 + 1) \tag{7.12}$$

and

$$\kappa_0 = \frac{a}{2D} - 1 \tag{7.13}$$

Below threshold, the stationary probability is given by the right-sided δ function $\delta_+(I)$. The stationary distribution function above threshold shows a noise induced transition at $D = a/2$ [6]. For $D < a/2$, the stationary distribution vanishes at $I = 0$, while for $D > a/2$, $P_{st}(I)$ has an integrable singularity at $I = 0$. This transition, however, does not show up in the stationary mean values

$$\langle I^n \rangle_{st} = \int_0^\infty I^n P_{st}(I) dI = (2D)^n \Gamma(\kappa_0 + n + 1)$$
$$= (2D)^n \Gamma\left(\frac{a}{2D} + n\right) \tag{7.14}$$

The effective eigenvalue, λ_{eff} for white noise can be obtained from Equation (7.15) [152], that is,

$$\lambda_{eff}^{-1} = \frac{1}{\langle I^2 \rangle - \langle I \rangle^2} \int_0^\infty \frac{f^2(x)}{D(x) P_{st}(x)} dx \tag{7.15}$$

with $D(x) = 4Dx^2$

$$f(x) = -\int_0^x (x' - \langle I \rangle_{st}) P_{st}(x') \, dx' \tag{7.16}$$

The integrals can be evaluated exactly, yielding the simple result [152]

$$\lambda_{eff} = 2a \tag{7.17}$$

that is, the bandwidth of the laser does not depend on the noise strength.

B. The Stationary Probability with Colored Noise

The two-variable Fokker–Planck equation

$$\frac{\partial}{\partial t} W(I, \varepsilon, t) = \mathbf{L}_{em} W(I, \varepsilon, t) \tag{7.18}$$

with the embedding Fokker–Planck operator

$$\begin{aligned}
\mathbf{L}_{em} &= \mathbf{A} + \varepsilon \mathbf{B} + \mathbf{L}_\varepsilon \\
\mathbf{A} &= -2 \frac{\partial}{\partial I} (a - I) I \\
\mathbf{B} &= -2 \frac{\partial}{\partial I} I \\
\mathbf{L}_\varepsilon &= \frac{1}{\tau} \frac{\partial}{\partial \varepsilon} \varepsilon + \frac{D}{\tau^2} \frac{\partial^2}{\partial \varepsilon^2}
\end{aligned} \tag{7.19}$$

Expanding the stationary solution of the Fokker–Planck equation, Eq. (7.18), into complete sets of orthogonal functions with respect to both variables I and ε,

$$W_{st}(I, \varepsilon) = \frac{\alpha}{\Gamma(1+\nu)} \psi_0(\varepsilon) \sum_{n,m=0}^{\infty} d_n^m (\alpha I)^\nu L_m^\nu(\alpha I) \exp(-\alpha I) \psi_n(\varepsilon) \tag{7.20}$$

with $L_m^\nu(x)$ being generalized Laguerre polynomials, α being a positive arbitrary scaling parameter, and $\psi_n(\varepsilon)$ being Hermite functions, that is,

$$\psi_n(\varepsilon) = \frac{1}{\sqrt{2^n n! \sqrt{2\pi D/\tau}}} H_n\left(\frac{\varepsilon}{\sqrt{2D/\tau}}\right) \exp\left(-\frac{\tau\varepsilon^2}{4D}\right) \tag{7.21}$$

Arranging the expansion coefficients in vectors,

$$\mathbf{c}_n = (c_n^0, c_n^1, c_n^3, \ldots) \tag{7.22}$$

we obtain the tridiagonal vector recurrence relation [186]

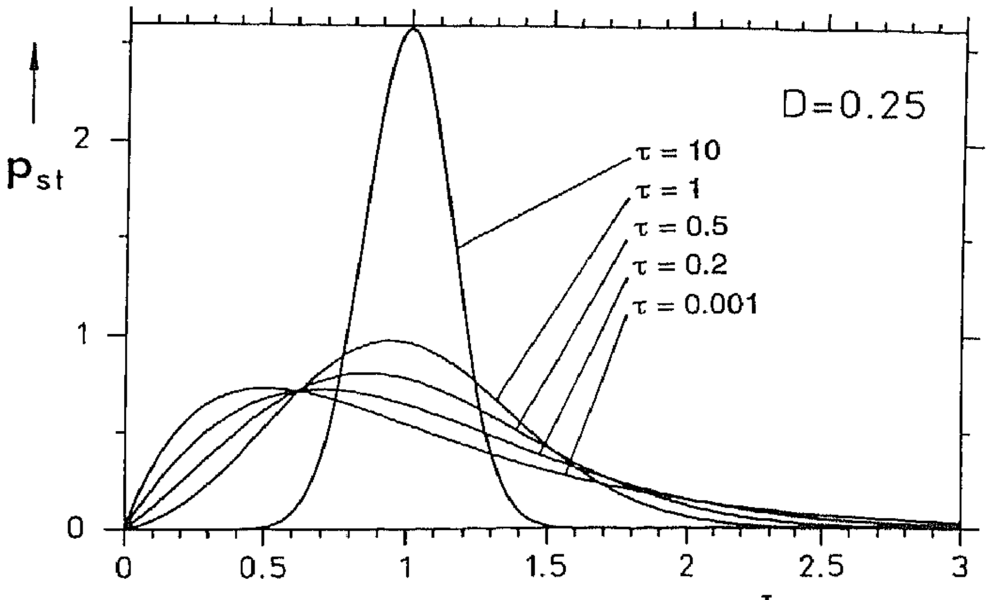
$$\sqrt{n\frac{D}{\tau}}\,\underline{\underline{B}}\mathbf{c}_{n-1} + \left(\underline{\underline{A}} - n\frac{1}{\tau}\underline{\underline{1}}\right)\mathbf{c}_n + \sqrt{(n+1)\frac{D}{\tau}}\,\underline{\underline{B}}\mathbf{c}_{n+1} = \mathbf{0} \tag{7.23}$$

which can be solved in terms of matrix continued fractions [186]. The matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are matrix representations of the operators **A** and **B**, respectively. They are given by

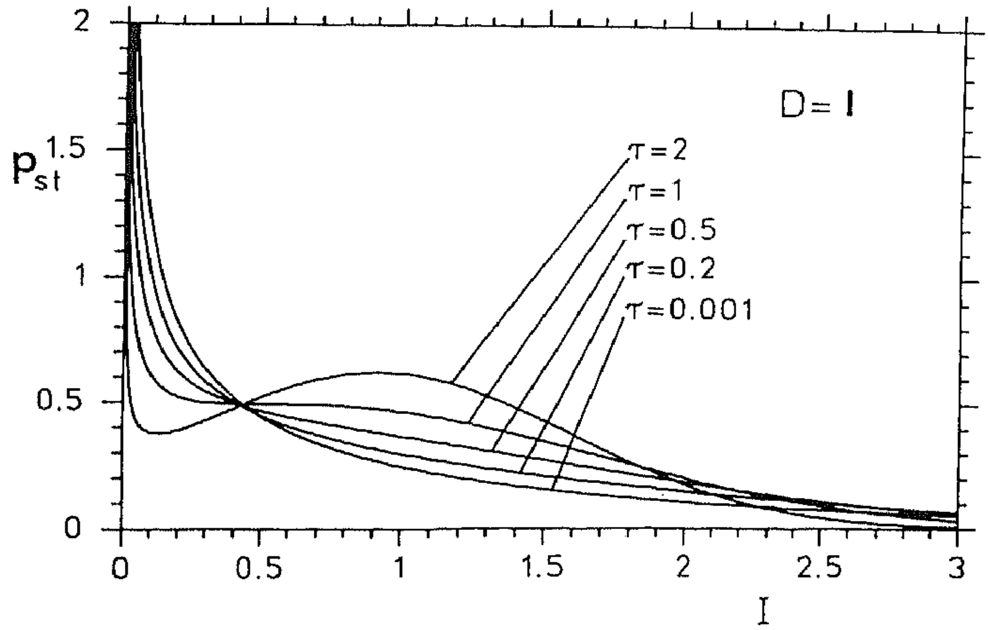
$$\begin{aligned} A^{mn} &= 2m\left(1 - \frac{3m+2\nu+1}{\alpha}\right)\delta_{m,n} + 2m\frac{m+1+\nu}{\alpha}\delta_{m,n-1} \\ &\quad + 2m\left(\frac{3m+\nu-1}{\alpha}\right)\delta_{m,n+1} - 2m\frac{m-1}{\alpha}\delta_{m,n+2} \\ B^{mn} &= 2m(\delta_{m,n} - \delta_{m,n+1}) \end{aligned} \tag{7.24}$$

The parameter ν has been chosen as $a/(2D) - 1$ [186] to match the dependency on I at low intensities. In Fig. 7.1, the numerically obtained stationary probabilities are shown for $D = 0.25(a)$ and $D = 1(b)$ for increasing correlation times of the noise. We observe crucial changes in the distribution function for increasing correlation times. The consequences for the photon-counting statistics are evident.

Within the small correlation time approximation, one obtains the



(a)



(b)

Figure 7.1. The stationary probability density [Eq. (7.20)] is shown at $D = 0.25(a)$ and $D = 1(b)$ for various values of the correlation time τ of the noise, for $a = 1$. Here, the UCNA result in Eq. (7.29) coincides within line thickness with the numerical (MCF) result in Eq. (7.20).

Fokker–Planck type equation [186]

$$\frac{\partial}{\partial t} P(I, t) = \left(-2 \frac{\partial}{\partial I} I(a - I) + 4D \frac{\partial}{\partial I} I \frac{\partial}{\partial I} I\right) P(I, t) - 8D\tau \frac{\partial}{\partial I} I \frac{\partial}{\partial I} I^2 P(I, t) \tag{7.25}$$

The stationary probability density within small correlation time approximation is obtained from Eq. (7.25) without ad hoc exponentiation as

$$P_{st}(I) = P_{st}^{\tau=0}(I)\left\{1 - \frac{\tau}{2D}[(2D + a)(a - 2I) + I^2]\right\} \tag{7.26}$$

where $P_{st}^{\tau=0}$ is the stationary probability in the white noise limit Eq. (7.11). The agreement of Eq. (7.26) with the numerical solution is obviously nonuniform in the intensity. In order that the correction term for finite correlation term remains a *small* term, the ratio τ/D has to be small. How small it has to be is determined by the value of the intensity itself. In other words, the small correlation time approximation is only valid in the finite region of support where the resulting stationary probability density is positive. Using the unified colored noise approximation [144, 151, 179], we find a Fokker–Planck equation with a strictly positive diffusion coefficient

$$\frac{\partial}{\partial t} P(I, t) = -\frac{\partial}{\partial I}\left(\frac{2(a - I)I}{1 + 2\tau I} + \frac{4DI}{(1 + 2\tau I)^3}\right)P(I, t) + \frac{\partial^2}{\partial I^2}\frac{4DI^2}{(1 + 2\tau I)^2} P(I, t) \tag{7.27}$$

The condition of validity is given for this particular model by

$$\gamma(I, \tau) = \frac{1}{\sqrt{\tau}} + 2I\sqrt{\tau} \gg 1 \tag{7.28}$$

which is fulfilled for small and large correlation times of the noise on the whole intensity axis, except for $I = 0$. The stationary probability density is obtained within this approximation by

$$P_{st}(I) = \frac{1}{Z}(1 + 2\tau I)\exp\left[\frac{\tau}{2D}(2a - I)I\right]P_{st}^{\tau=0}(I) \tag{7.29}$$

where Z is a normalization constant and $P_{st}^{\tau=0}(I)$ is the stationary probability in the white noise limit. In Fig. 7.2, the numerically evaluated stationary probabilities are compared with those obtained by using the unified colored noise approximation at $D = 0.5$. The agreement is good for large-to-large correlation times of the noise. The largest deviations are in agreement with what one expects from Eq. (7.29) at small intensities I.

The fluctuational line width of the dye laser (i.e., the effective eigenvalue) is given by the inverse of the integral of the normalized intensity autocorrelation function over the complete time axis Eqs. (7.7) and (7.8). In the white noise limit, this quantity could be obtained exactly, yielding the simple expression $\lambda_{eff} = 2a$. Since the unified colored

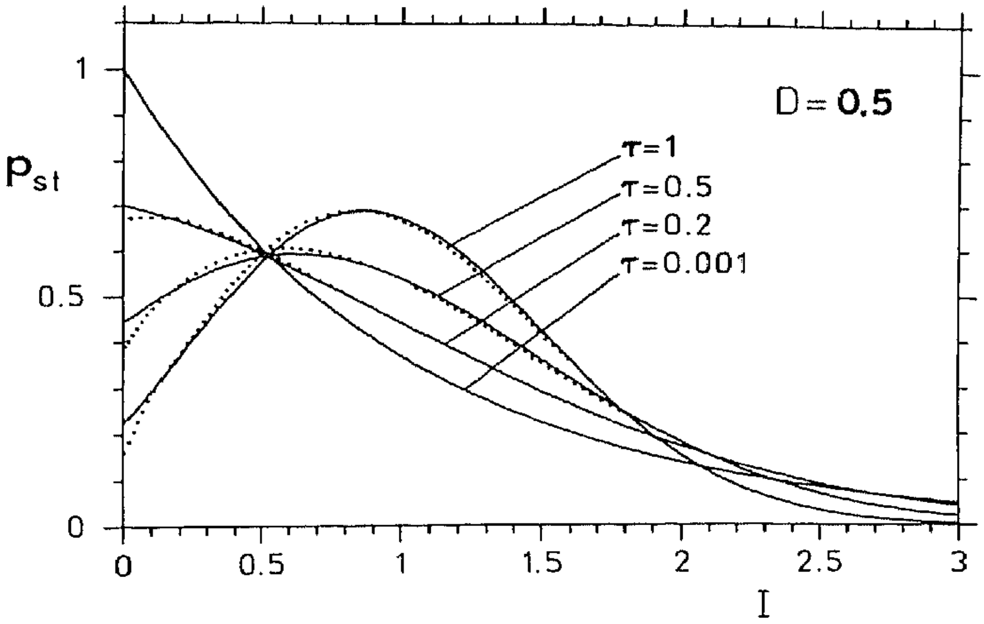


Figure 7.2. The approximate (solid line) expression [Eq. (7.29)] for the stationary probability density is compared with the full numerical (dotted line) result at $D = 0.5$, and $a = 1$.

noise approximation represents a true Markovian approximation of a non-Markovian process, we can use the closed expressions for the effective eigenvalue Eqs. (7.15) and (7.16) with $D(x) \to 4D_x^2/(1 + 2\tau x)^2$. The integrals, however, have to be evaluated numerically. The result is compared in Fig. 7.3 with the numerical results obtained in [144, 187].

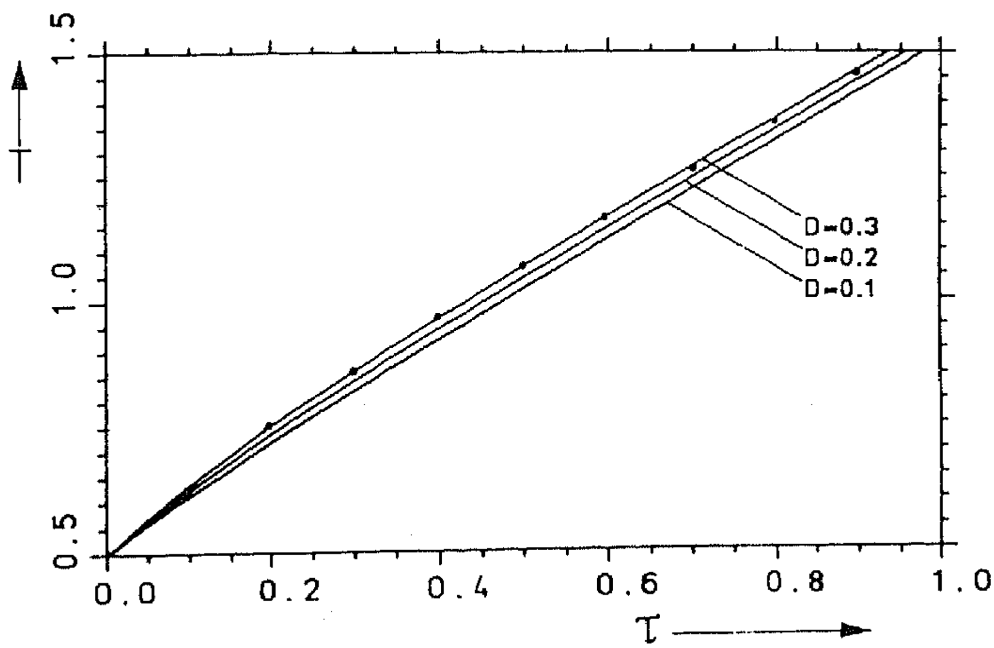


Figure 7.3. The inverse line width (i.e. the relaxation time) $\lambda_{\text{eff}}^{-1}$ [Eq. (7.15)] is depicted versus noise correlation time τ for different noise strengths D, and $a = 1$. The solid lines are the numerical exact results. The UCNA result [Eqs. (7.15, 7.27, 7.29)] for $D = 0.3$ is indicated by the full circles, being very close to the exact result.

The comparison is very favorable for the unified colored noise approximation. We should mention, also, that for the particular dye laser model we are using here, a weak noise expansion yields the analytical expression [187]

$$\lambda_{\text{eff}}^{-1} = \frac{1}{2a} + \tau + \frac{2D\tau(3a\tau + 1)}{(1 + a\tau)(1 + 2a\tau)(1 + 4a\tau)} \tag{7.30}$$

which agrees also well with numerical results.

VIII. SUMMARY AND OUTLOOK

We have reviewed current theories and applications of colored noise driven dynamical systems. Exact solutions are available for colored dichotomous noise driven dynamical systems (see Section IV) and for colored noise driven linear systems (see Section III.E, and Appendix A in [111]). For colored Gaussian noise driven nonlinear systems, one has to apply approximation schemes and/or to resort to numerical techniques (see Sections VI and VII).

For *small* correlation times τ of the noise, that is, when the noise correlation time is smaller than all system time scales, numerous approximate Fokker–Planck type equations have been derived in the literature (see Section V.A). Triggered by the need to describe experiments (photon statistics of dye lasers, electrohydrodynamic instabilities, analogue experiments, or turbulent transitions in liquid He II, see the various contributions in [194]), new theories valid for *intermediate-to-large* correlation times (decoupling approximation and unified colored noise approximation) have been developed (see Sections V.B and V.C, respectively).

In Section VI, the theoretical concepts introduced in Section V are applied to the problem of colored noise driven bistability. We focused on the noise color dependence of the escape rates and stationary correlation functions. The answers of the approximation schemes have been compared with precise numerical results. In Section VII, the impact of multiplicative colored noise for the photon-counting statistics and the fluctuational line width has been investigated.

The reader has certainly noted that throughout this chapter the emphasis has been placed on nonequilibrium systems described by stochastic differential equations driven by general multiplicative colored noise. Thus, we did not address the huge area of colored noise fluctuations within thermal equilibrium systems, being described by the epochmaking *generalized Langevin equation* [195–197] and obeying the fluctuation–dissipation theorem of the second kind with memory friction.

The vast research obtained for this thermal equilibrium colored noise has been extensively addressed in the literature, see, for example, [100, 123, 195–205].

Before we close our survey we would also like to mention other topics, related to colored noise driven dynamical systems that we did not cover in detail. One such topic is the influence of colored noise for transport quantities in periodic potentials [206–208], or the work on the relaxation times, see Eq. (5.24) [209, 210]. Another area, widely studied in recent years, refers to the *decay from unstable states* [211] when triggered by colored noise [212–218], where the decay time T obeys a characteristic scaling law (Suzuki's scaling [211]) of the form $T \propto \log(D) + B(\tau) + C$, where D denotes the noise strength. The influence of the noise color is accounted for by the function $B(\tau)$. Much of the progress on colored noise approximation schemes has originated from digital simulations. Likewise, the method of colored noise analogue simulations [194, 219, 220] has played a pioneering role in guiding the theoretical practitioners to improve upon their theoretical schemes.

Finally, throughout our survey we restricted the time evolution to continuous time. If on the other hand the dynamics is recorded by stroboscopic methods—commonly used in the study of chaotic dynamics—the dynamical flow is not governed by a stochastic differential equation but rather by a noisy map. Very recently, the study of correlated noise has been initiated for such discrete-time dynamical flows [221–223].

Undoubtedly, we shall witness more research work in future years aimed at completing, extending, and interpreting the present state of the art of colored noise driven systems in chemistry, physics, biology, and the engineering sciences.

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