The ring of Brownian motion: the good, the bad, and the simply silly

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Brownian motion

Gypsum crystals in a closterium moniliferum

source: IWF Wissen und Medien gGmbH, 1949
Why you should not do Brownian motion

- You know nothing about the subject
- Many very good people worked on it (Einstein, Langevin, Smoluchowski, Ornstein, Uhlenbeck, Wiener, Onsager, Stratonovich, ...)
- You don’t have your own pet theory yet
Why you should do
Brownian motion

- You know nothing about the subject
- Many very good people worked on it
- You still can do your own pet theory
Robert Brown (1773-1858)

1827 – irregular motion of granules of pollen in liquids

- Brown, Phil. Mag. 4, 161 (1928)
To see clearly how one can deceive one’s mind on this point if one is not careful, one has only to place a drop of alcohol in the focal point of a microscope and introduce a little finely ground charcoal therein, and one will see these corpuscles in a confused, continuous and violent motion, as if they were animalcules which move rapidly around.
Mean squared displacement

\[ \langle x^2(t) \rangle \propto t^\alpha \]

Brownian movement \( \alpha = 1 \)  
Lévy-Brownian movement \( \alpha = \frac{4}{3} \)

Theory of Brownian motion

W. Sutherland (1858-1911)  A. Einstein (1879-1955)  M. Smoluchowski (1872-1917)

\[ D = \frac{RT}{6\pi \eta aC} \]

Source: www.theage.com.au

\[ \langle x^2(t) \rangle = 2Dt \]

Source: wikipedia.org

\[ D = \frac{RT}{N} \frac{1}{6\pi kP} \]

Source: wikipedia.org

\[ D = \frac{32 mc^2}{243 \pi \mu R} \]

Source: wikipedia.org

Brownian noise

- Information Sciences & Engineering & Geophysics
- Finance & Linguistics
- Quantum & Relativity
- High Energy & Astrophysics
- Biology & Medicine
- Chemistry & Chemical Physics
- Soft Matter & Condensed Matter
Quantum-Mechanics

= Brownian Motion ?
= Stochastic Mechanics ?

(E. Nelson; 1966, 1986)

\[ p(x, t) = |\Psi(x, t)|^2 \]

\[ \Psi(x, t) = |\Psi(x, t)| e^{iS(x, t)} \]

\[ \dot{p}(x, t) = -\frac{\hbar}{m} \nabla \left[ (\nabla \ln |\Psi(x, t)| + \nabla S(x, t)) p(x, t) \right] + \frac{\hbar^2}{2m} \nabla^2 p(x, t) \]

\[ f_1 \geq 0, \ f_2 \geq 0 : \quad \frac{1}{2} \langle f_1(t_1) f_2(t_2) + f_2(t_2) f_1(t_1) \rangle \geq 0 \]

QM: NO!

Diffusion: space-time only

classical diffusion (Markovian)

\[ p(t, x|t_0, x_0) = \left[ \frac{1}{4\pi D(t-t_0)} \right]^{1/2} \exp\left[ -\frac{(x-x_0)^2}{4D(t-t_0)} \right]. \]

telegraph equation (non-Markovian)

\[ \tau_v \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial t} \rho = D \nabla^2 \rho, \]

alternative approach

\[
p(\tilde{x}|\tilde{x}_0) \propto \int_{a_-}^{a_+} \int_{\tilde{x}|\tilde{x}_0} da \, \exp\left( -\frac{a}{2D} \right)
\]

\[ PRD 75:043001 (2007) \]

J Masoliver & G H Weiss
Jüttner Gas

\[
f_{\text{Maxwell}}(\vec{p}) = \left[ \frac{\beta}{(2\pi m)} \right]^{d/2} \exp \left( -\frac{\beta p^2}{2m} \right)
\]

\[
f_{\text{Jüttner}}(\vec{p}) = Z_d^{-1} \exp \left[ -\beta_J \left( m^2 c^4 + p^2 c^2 \right)^{1/2} \right]
\]

\[\langle \vec{p} \cdot \vec{v} \rangle = dk_B T = d/\beta_J\]

statistical relativistic temperature

\[T = T = (k_B/\beta_J)^{-1}\]

Measuring temperature in Lorentz invariant way

Lorentz invariant equipartition theorem

\[ k_B \mathcal{T} = m \gamma(u)^3 \langle \gamma(v') (v' + u)^2 \rangle_{t'} \]

with \( u = -\langle v' \rangle_{t'} \)
“Temperature” problem in RTD?

1907/08 moving bodies appear cooler

1923/1963.. hotter!

\[ T'(w) = T (1 - w^2)^{\alpha/2} \]
\[ \alpha = \begin{cases} 
+1 & \text{Planck, Einstein} \\
0 & \text{Landsberg, van Kampen} \\
-1 & \text{Ott} 
\end{cases} \]

\[ T = T' \text{ 1966-69} \]

1940s maybe ... not

Two prominent examples

- Stochastic Resonance
- Brownian Motors
Stochastic Resonance
( in a nutshell )

Weak signal

Noise source

System

Output signal
# papers in 2009: ≈ 460
> 85000 cites in total
Why are the ice-ages so periodic?

Milankowitch cycles:
Small changes in earth orbit eccentricity with 100k year periodicity

Changes are small! (<0.1% of solar constant)

What can amplify those small changes?

M. Milankowitch, Handbuch der Klimatologie I (1930)
- The 100ky cycles only bias the climate
- Fluctuations make climate switch
- small changes of conditions can have huge impact

Benzi, Sutera and Vulpiani (Tellus, 1981)
C. Nicolis and G. Nicoli (Tellus, 1981)
Noise-assisted synchronized hopping

\[ T_{\text{period}} \approx 2T_{\text{escape}} \]
Synchronization

\[ T_e \sim 2\Gamma^{-1} \]

SIGNAL

ESCAPE

\[ 2\pi/\omega_0 \]

\[ 0 \]

\[ 0.1 \]

\[ 0 \]

\[ 0.2 \]
Power spectral density

\[ \dot{X} = X - X^3 + \xi(t) \]

\[ \dot{X} = X - X^3 + A \sin(\Omega t) + \xi(t) \]

Spectrum

Frequency
Measuring SR

- Signal to noise ratio
- Spectral amplification
- Mutual information
- Cross-correlation: input ↔ output
- Peak area, (phase-) synchronization, ...

SR-reviews:
Amplification of small signals by noise


More noise, more signal!!
SR in colloidal systems

$T_{\text{period}} = 2 \text{ sec}$

SR - Ingredients

- Threshold system
- Weak (subthreshold) signal
- Noise

Anomalous amplification properties
Stochastic Resonance in Neurobiology

Input: currents at synapses

Processing: action potential if the sum of currents exceeds threshold

Output: electric pulses traveling down the axon

Basic idea: Signals below threshold can be detected in the presence of additional noise
SR in Visual Perception

Experiment:

SR and human posture control

Somatosensory function declines with age and in diabetic patients. Can additional noise help restore function?

• Parkinson
• Multiple sclerosis (MS)
• Stroke / skull-brain-trauma
• Cross-section paralysis
• Depression
• Pain
• …

SRT Zeptor Training - Powerslide Team
• Parkinson
• Multiple sclerosis (MS)
• Stroke / skull-brain-trauma
• Cross-section paralysis
• Depression
• Pain
• ...
SR trends

- Spatio – temporal SR
- Aperiodic SR
- Quantum SR
Motors $\rightarrow$ Brownian motors

Two heat reservoirs

One heat reservoir

Perpetuum mobile of the second kind?

NO !
Brownian motor
Temperature / Flashing Ratchet

- **LOW T**
- **HIGH T**
- **LOW T**

Diagram:
- Free diffusion
- Retrapping
- Return to original well
- Ratchet forward
Brownian motors - Characteristics

- Noise & AC-Input $\rightarrow$ DC-Output
- Non-equilibrium Noise $\rightarrow$ Directed Transport
- Current reversals

Applications:
- Novel pumps and traps for charged or neutral particles
- Brownian diodes & transistors
Ask not what physics can do for biology, ask what biology can do for physics
Drift Ratchet - Device

Source: F. Müller, MPI for microstructure physics, Halle
Drift Ratchet - Theory


Setup

Particle Separation
Drift Ratchet – Experiment

Quantum Demon?

A measurement → Increase information → Reduction of entropy

Quantum Brownian Motors

\[ i\hbar \dot{\rho} = \left[ H_S + H_{\text{INT}} + H_{\text{BATH}} \right] \rho \]

Hilbert space: SYSTEM @ BATH

SUPER BATH
Quantum-Langevin-equation

\[ m \ddot{x}(t) + \int_{-\infty}^{t} \gamma(t - t') \dot{x}(t') \, dt' + V'(x; t) = \xi(t) \]

\[ \frac{1}{2} \langle \xi(t) \xi(s) + \xi(s) \xi(t) \rangle_{\text{bath}} = \]

\[ \frac{m}{\pi} \int_{0}^{\infty} \Re \gamma(-i\omega + 0^+) \hbar \omega \coth \left( \frac{\hbar}{2k_B T} \right) \cos [\omega (t - s)] \, d\omega \]

And:

\[ [\xi(t), \xi(s)] = -i\hbar \cdots \neq 0 \]
Rocking Ratchet - Theory


Current enhancement due to tunneling

Tunneling induced current reversal

Finite current for $T \to 0$
Rocking QM Ratchet – Experiment

H. Linke, et al., SCIENCE 286, 2314 (1999)
Brownian motors:

EX(E/O)RCISING DEMONS

Field strength $E=106\ \text{V/cm}$

$\Omega=3\Delta$ corresponds to $4\mu\text{m}$ wavelength

Typical current: some nA
Quantum Gears: Driving Through Interactions

cold atoms (bosons, fermions)

\[ \hat{H} = H_R + H_B(t) + H_{RB}, \quad H_B(t + T) = H_B(t) \]

- targeting excitations
- experimentally feasible setup
- full many-body treatment
Generalizations of Brownian Motion
Brownian motion: Generalized Langevin-equation

Hamiltonian: \[ H_{\text{tot}} = H_{\text{sys}} + H_{\text{env}} + H_{\text{WW}} \]

\[ m\dddot{x}(t) + \int_{-\infty}^{t} \gamma(t - t') \dot{x}(t') \, dt' + V'(x; t) = \xi(t) \]

Asymptotically normal, anomalously fast, or anomalously slow – via fractional Brownian motion –

\[ \int_{0}^{\infty} \gamma(t) \, dt = \begin{cases} \text{const} & \Rightarrow \text{normal} \\ 0 & \Rightarrow \text{superfast} \\ \infty & \Rightarrow \text{superslow} \end{cases} \]

Connection to the fractional Fokker-Planck-equation
Fractional Fokker-Planck equation

Subdiffusion ($\alpha<1$):

$$\frac{\partial P(x, t)}{\partial t} = \left[ \frac{\partial}{\partial x} \frac{V'(x, t)}{\gamma_\alpha} + K_\alpha \frac{\partial^2}{\partial x^2} \right]_0 D_t^{1-\alpha} P(x, t)$$

Fat tails in the distribution of the residence times

Riemann-Liouville Operator

Superdiffusion ($\alpha>1$):

$$\frac{\partial P(x, t)}{\partial t} = \left[ \frac{\partial}{\partial x} \frac{V'(x, t)}{\gamma_\alpha} + K_\alpha \frac{\partial^{2/\alpha}}{\partial |x|^{2/\alpha}} \right] P(x, t)$$

Fat tails in the distribution of the jump lengths

Riesz-derivative
Fractional Fokker-Planck equation

subdiffusive ($\alpha<1$)

\[
\frac{\partial P(x, t)}{\partial t} = \left[ \frac{\partial}{\partial x} \frac{V'(x, t)}{\eta_\alpha} + K_\alpha \frac{\partial^2}{\partial x^2} \right]_0 D_t^{1-\alpha} P(x, t)
\]

Riemann-Liouville Operator

\[
0 D_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t \frac{f(t')}{(t-t')^{1-\alpha}} dt'
\]
Noise – always bad?

Source: Agilent Technologies
Brownian motion

EQ. & NONEQ. STAT. MECHANICS
NUISANCE
MISUSE
Driven Many-Body Quantum Systems

- quantum machinery
- role of heat transport in nanodevices
What “is” thermodynamics?
✓ non-local description in terms of symmetry (breaking) parameters

“Good” starting point in relativistic thermodynamics?
✓ (non-)conserved tensor densities, Noether currents

Origin of different temperature transformation laws?
✓ choice of space-time hyperplanes
✓ definition of heat, formulation of 1st/2nd law

How should one define thermodynamic observables in special and general relativity?
✓ invariant manifolds, lightcone integrals

Observable consequence: temperature-induced apparent drift