Collapse and revival oscillations as a probe for the tunneling amplitude in an ultracold Bose gas

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We present a theoretical study of the quantum corrections to the revival time due to finite tunneling in the collapse and revival of matter-wave interference after a quantum quench. We study hard-core bosons in a superlattice potential and the Bose-Hubbard model by means of exact numerical approaches and mean-field theory. We consider systems without and with a trapping potential present. We show that the quantum corrections to the revival time can be used to accurately determine the value of the hopping parameter in experiments with ultracold bosons in optical lattices.

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I. INTRODUCTION

Collapse and revival oscillations in a Bose-Einstein condensate loaded in an optical lattice were first experimentally observed in 2002 [1] and since then have been a subject of much theoretical and experimental interest [2]. This phenomenon is understood as an oscillation between an initial coherent state and a final noncoherent (collapsed) state in a lattice where, after a quench, the hopping parameter between sites is negligible. Very recently it has been argued that such collapse and revival oscillations can be used as a very sensitive probe for effective three-body (and higher) interactions [3] by studying the time evolution of the visibility of the interference pattern. This has been investigated theoretically in Ref. [4]. It was assumed both in experiment and in theory [3,4] that the initial state is a coherent state and that after the quench the tunneling amplitude is negligible, that is, that the systems were in the atomic limit. Following these assumptions, the time-evolving state is a product of coherent states localized at each lattice site and one deals with an effective one-site problem.

Our goal in this article is to go beyond the previous analysis and present a full many-body study of collapse and revival phenomena in one-, two-, and three-dimensional cubic lattices. We study hard-core bosons in the presence of a superlattice and the Bose-Hubbard model. For both cases we consider systems without and with a trapping potential present. For the hard-core case we use exact numerical approaches in one and two dimensions and compare them with the predictions of a Gutzwiller mean-field theory. We show that the latter is qualitatively and quantitatively correct when determining the revival time for small hopping amplitudes. Building on that, we present an analysis for the Bose-Hubbard model that is solely based on the Gutzwiller mean-field approach. We also provide an analytical solution for the homogeneous hard-core case.

A previous study [5] considered the effect of a finite hopping on the damping of the collapse and revival oscillations in a lattice without a confining potential. The quantitative results presented there applied to an initial coherent state as in the articles mentioned previously. In contrast, we study the dynamics starting from initial states that are the exact many-body ground state in some cases and the appropriate Gutzwiller ansatz in the other cases.

We find the functional form of those corrections to the revival time in the atomic limit and show that if one knows the values of the superlattice potential for the hard-core case or the on-site interaction \( U \) for the Bose-Hubbard model, such corrections can be used to accurately determine the tunneling amplitudes in experiments in optical lattices. In the atomic limit, for the Bose-Hubbard model, the revival time \( t_{\text{rev}} \) is

\[ t_{\text{rev}} = \frac{2\pi\hbar}{U} \quad [6,7]. \]

Our general strategy is to numerically calculate the deviations from this value for \( 0 < J/U < 1 \). Given that in experiments with ultracold gases in optical lattices the hopping parameter \( J \) is exponentially sensitive to the lattice depth, while the on-site repulsion \( U \) is only power-law dependent on the lattice depth, this method of determining \( J \) by means of the revival time may be more accurate than the approaches followed so far. Our results are also relevant to cases in which the tunneling is small but one is still interested in estimating its value.

The article is organized as follows. In Sec. II, we study the collapse and revival in the hard-core case in the presence of a superlattice potential. This is done using numerically exact methods in one and two dimensions. In Sec. III, we introduce the time-dependent mean-field approach to the hard-core boson problem and compare its results with numerically exact ones in order to quantify the predictive power of the mean-field approximation for the revival time. In addition, we make some general theoretical statements and provide an analytical solution for the case without the trapping potential. Section IV is devoted to analyzing the Bose-Hubbard model in three-dimensional systems of soft-core bosons with and without confining potentials present. Finally, we present our conclusions in Sec. V.

II. NUMERICALLY EXACT RESULTS FOR HARD-CORE BOSONS IN A SUPERLATTICE

We first introduce the model Hamiltonian considered to study hard-core bosons in a superlattice. We also discuss its relation to the interaction quench in the soft-core case that is studied in Sec. IV.

A. Motivation and methods

The hard-core boson Hamiltonian on a superlattice with period 2 in the presence of a harmonic confining potential can
be written as
\[ \hat{H}_{\text{HCB}} = -J \sum_{\langle ij \rangle} (\hat{b}_i^\dagger \hat{b}_j + \text{H.c.}) + A \sum_i (-1)^i \hat{n}_i + V \sum_i r_i^2 \hat{n}_i, \]
(1)
where the hard-core creation and annihilation operators at site \( i \) are denoted by \( \hat{b}_i^\dagger \) and \( \hat{b}_i \), respectively, and the local density operator by \( \hat{n}_i = \hat{b}_i^\dagger \hat{b}_i \). Hard-core boson operators satisfy standard commutation relations for bosons. However, on the same site, they also satisfy the constraint \( \hat{b}_i^\dagger \hat{b}_i = 0 \), which precludes multiple occupancy of the lattice sites. The other parameters in Eq. (1) are the hopping constant \( J \) between nearest-neighbor sites \( \langle ij \rangle \), the strength of the superlattice potential \( A \), and the curvature of the harmonic trap \( V \). The distance from site \( i \) to the center of the trap \( r_i \) is measured in units of the lattice constant \( a \), which we set to unity. In what follows, we also denote the total number of lattice sites by \( L \) and the total number of particles by \( N \).

The hard-core model on a superlattice is particularly suitable to study collapse and revival phenomena because in one dimension (1D) it can be exactly solved by means of the Jordan-Wigner mapping to noninteracting fermions [8]. In equilibrium, these systems were studied in Ref. [9], where they were shown to exhibit ground-state phases that are similar to those of the Bose-Hubbard model. The nonequilibrium dynamics of hard-core bosons in a superlattice potential was studied in Ref. [10], where collapse and revival oscillations of the momentum distribution function were observed.

A quench of the superlattice potential \( A \) in Eq. (1) has a similar effect to a quench of \( U \) in the Bose-Hubbard model. From a simple band-structure calculation it follows that in 1D \( A \) opens a gap \( \Gamma = 2A \) in the hard-core boson energy spectrum [9,10],
\[ \epsilon_{\pm}(k) = \pm \sqrt{4J^2 \cos^2(ka) + A^2}, \]
(2)
where “+” (“−”) denotes the upper (lower) band. In two (2D) and three (3D) dimensions, hard-core bosons cannot be mapped to noninteracting fermions. The phase diagrams for the ground state of such systems were studied in detail in Refs. [11,12] by various numerical and analytical approaches and were shown to be qualitatively similar to the phase diagrams of the Bose-Hubbard model in 2D and 3D [13–15].

As already noted, in the atomic limit of the Bose-Hubbard model, the revival time after the interaction quench is given by \( t_{\text{rev}} = 2\pi / U \) (we set \( \hbar = 1 \) henceforth); similarly, for hard-core bosons in a superlattice potential, it follows that \( t_{\text{rev}} = \pi / A \) [10].

Hence, in this section we take advantage of the fact that the nonequilibrium dynamics of hard-core bosons in 1D can be exactly solved for large system sizes by means of the approach presented in Refs. [16], which makes use of the Jordan-Wigner transformation to noninteracting fermions [8]. In 2D, due to the reduced Hilbert space (when compared to soft-core bosons), one can perform full diagonalization calculations for small, but meaningful, periodic systems. All our exact results for 2D hard-core systems were obtained in \( 4 \times 4 \) lattices with periodic boundary conditions.

The two preceding approaches allow us to make exact predictions for the quantum corrections due to finite hopping amplitudes to the revival time \( t_{\text{rev}} \),
\[ \Delta t_{\text{rev}} = t_{\text{rev}}^{\text{atom}} - t_{\text{rev}}, \]
(3)
which in turn will help us gauge the accuracy of the mean-field approach that we use later for studying the Bose-Hubbard model.

In the latest experimental and theoretical studies [3,4], the main observable under consideration was the visibility of the interference pattern. Here, instead, we focus our attention on the time evolution of the \( n_{k=0} \) momentum occupation number,
\[ n_{k=0} = \frac{1}{L} \sum_{ij} \langle \hat{b}_i^\dagger \hat{b}_j \rangle, \]
(4)
which is also measured in experiments.

### B. Results for hard-core bosons in a periodic potential in 1D and 2D

In all cases for hard-core bosons presented here we consider the following quench. A system is prepared in a superfluid state with \( J_{\text{fin}} = 1 \) (which sets our energy scale) and no superlattice potential. At time \( t = 0 \) the superlattice potential \( A \) is quenched to a constant value \( A_{\text{fin}} = 1 \) and the hopping constant \( J \) is reduced to several values \( J_{\text{fin}} < 1 \) (in the remainder of the text, the notation \( A_{\text{fin}} \equiv A \) and \( J_{\text{fin}} \equiv J \) is used in all unambiguous cases).

In Fig. 1, we show the time evolution of \( n_{k=0} \) after this quench in (a) a chain and (b) a \( 4 \times 4 \) cluster, both at quarter filling. Results are presented for three different final values of \( J \), where the atomic limit (\( J = 0 \)) revival time can be clearly seen to correspond with the prediction \( t_{\text{rev}}^{\text{atom}} = \pi / A \). Two effects of finite final hopping are evident in those plots: first,
The quantum corrections due to finite hopping $\Delta t_{\text{rev}}$ versus $J$ for constant $A = 1$ are presented in a log-log plot in Fig. 2(a) for 1D and 2D systems. We find from those plots that the corrections follow a quadratic behavior for values of $J \ll 0.1$. This is depicted by a quadratic fit $f(x) = ax^2$ to data points with $J \ll 0.1$. Also evident from those plots is the very weak dependence of $\Delta t_{\text{rev}}$ on the density both in 1D and 2D. This turns out to be very convenient later when studying the harmonically trapped systems.

In Fig. 2(b) we consider the damping of the oscillations, which can be characterized by the amplitude of the first revival $n_{k=0}^{\text{rev}} = n_{k=0}(t = t_{\text{rev}})$ subtracted from its value in the atomic limit: $\Delta n_{k=0}^{\text{rev}} = n_{k=0}^{\text{atom}} - n_{k=0}^{\text{rev}}$. For this quantity we find a quartic behavior, as illustrated by the fits in Fig. 2(b). Furthermore, we find a much stronger dependence on the density (note the 12 decades on the $y$ axis). The very fast reduction of the damping with decreasing $J$ makes it a less attractive tool for experimentally probing small values of $J$.

We find numerically the scaling of $t_{\text{rev}}$ with respect to the system parameters $J$ and $A$ to have the following functional form $t_{\text{rev}}(J, A) \equiv t_{\text{rev}}(J/A)/A$, whereas for the damping $n_{k=0}^{\text{rev}}(J, A) \equiv n_{k=0}(J/A)$; that is, the former depends on the value of $A$ and $J/A$ whereas the latter is only a function of the ratio $J/A$. In the atomic limit, the revival time scales with $A$: $t_{\text{rev}}^{\text{atom}}(J) = \pi/A$ and therefore the preceding scaling holds also true for $\Delta t_{\text{rev}}$: $\Delta t_{\text{rev}}(t, A) \equiv \Delta t_{\text{rev}}(t/A)/A$. In Sec. III we are able to analytically confirm this result for the mean-field approximation. On the other hand, in the atomic limit, $n_{k=0}^{\text{max}}(A)$ is independent of $A$ as the system exhibits perfect revivals so $\Delta n_{k=0}^{\text{rev}}$ is only a function of $J/A$.

C. Results for hard-core bosons in a trap in 1D

Experimental systems are in general different from the ones discussed in Sec. II B. This is because a confining potential is required for containing the gas of bosons. The confining potential in experiments is to a good approximation harmonic and generates an inhomogeneous density profile. Given the results shown in Fig. 2(a), where the revival time was shown to depend only weakly on the density, one would expect the outcome in the presence of a trap not to be strongly dependent on the confining potential and the total number of particles.

Up to small corrections, the preceding turns out to be the case for the changes induced in the revival time by a finite hopping. However, as shown in Fig. 3(a), if one quenches...
J and A keeping constant the trapping potential, then a very high damping rate can be seen even in the atomic limit. Hence, measurements at a constant curvature of the trap are not the best way to proceed in trapped systems. They mix the effects of the trapping potential and the finite hopping in the outcome. In fact, even the quadratic behavior obvious in the homogeneous case (Fig. 2) becomes obscured in the trap if the confining potential is kept the same from the initial state.

In previous work in equilibrium it has been argued that the correct way to define the thermodynamic limit for a trapped system is by keeping constant the so-called characteristic density \( \tilde{\rho} = N[\sqrt{dJ}]^2 \), where \( d \) is the dimensionality of the system (see, e.g., Ref. [17]). This is equivalent to what is done in homogeneous systems when one keeps constant the density \( N/L \). Since in Sec. II B all quenches were performed keeping constant \( N/L \), we have studied quenches in the trap in which the characteristic density is kept constant; that is, one needs to reduce the trapping potential by the same amount that the hopping parameter is reduced. We denote this scenario quench type (i). Another way of reproducing the homogeneous results that comes to mind is to remove the trapping potential concurrently with the superlattice quench and observe oscillations which then take place in a homogeneous potential. This scenario is denoted quench type (ii).

For both quench scenarios (i) and (ii), we find a quadratic behavior for \( \Delta_{rev} \), which is similar to what was shown in Fig. 2 for homogeneous systems. Interestingly, this behavior is, as depicted in Fig. 4(a), practically independent of the quench type and the characteristic density of the initial state. We note that for \( \tilde{\rho} = 3.0 \) the initial state has an insulating \( (n = 1) \) domain in the center of the trap, while for the other characteristic densities the system is purely superfluid. In 1D, the insulator appears in the center of the trap when \( \tilde{\rho} \sim 2.6–2.7 \) [16]. The independence of the asymptotic behavior of \( \Delta_{rev} \) on the initial state suggests that by measuring the correction to the revival time due to the finite value of \( J \) in experiments, it is possible to determine \( J \) if one knows \( A \). The same can be said for systems without a trap.

On the other hand, as shown in Fig. 4(b), \( \Delta n_{rev}^0 \) reveals a strong dependence on the quench type and the initial density. For scenario (i), we obtain a quadratic behavior for pure superfluid initial states (\( \tilde{\rho} = 1.0 \) and \( \tilde{\rho} = 2.0 \)), while for the one having a Mott-insulating domain (\( \tilde{\rho} = 3.0 \)) a constant damping rate is always present for \( J < 1 \). For quench type (ii), the damping behaves completely differently and shows the quartic behavior observed in the homogeneous case. With respect to the aim to simulate the homogeneous case, this result indicates that the fact that the density profile remains unchanged during the time evolution under scenario (i) is less important than the fact that the potential is homogeneous after quench (ii).

![Figure 4](image-url) (Color online) Quantum corrections to (a) the revival time \( \Delta_{rev} \) and (b) the revival amplitude \( (\Delta n_{rev}^0) \) vs \( J \), both for the quench scenarios (i) keeping the characteristic density constant and (ii) turning off the trap. Results are presented for three different initial values of the characteristic density \( \tilde{\rho} = 1.0, \tilde{\rho} = 2.0, \) and \( \tilde{\rho} = 3.0 \), which correspond to curvatures \( V_{ini} = 4.44 \times 10^{-5} \), \( V_{ini} = 1.78 \times 10^{-4} \), and \( V_{ini} = 4 \times 10^{-4} \), respectively; the system sizes are \( L = 500, L = 400, L = 300 \), with \( N = 150 \) in all cases. Note that \( \tilde{\rho} = 3.0 \) already has a Mott-insulating region in the center of the trap, while \( \tilde{\rho} = 1.0 \) and \( \tilde{\rho} = 2.0 \) are entirely superfluid. The quadratic and quartic fits in (a) and (b) were done for points with \( J \leq 0.01 \).

### III. MEAN-FIELD APPROACH

Within the mean-field approximation, we can extend our analysis to consider the more experimentally relevant Bose-Hubbard model:

\[
\hat{H}_{SCB} = -J \sum_{\langle ij \rangle} (\hat{a}_i^\dagger \hat{a}_j + \text{H.c.}) + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) + \sum_i \hat{n}_i V_r r_i^2, \tag{5}
\]

where \( [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \) and \( [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \), as usual for bosons. The on-site interaction energy is denoted by \( U \).

The mean-field theory that we employ is based on the restriction of the wave function to the Gutzwiller-type product state,

\[
|\Psi_{MF}\rangle = \prod_{i=1}^{L} \sum_{n=1}^{n_c} \alpha_{in} |n\rangle_i, \tag{6}
\]
where \( n_c \to \infty \) for thermodynamic systems, \(|n_i|\), denotes a single-site Fock state for lattice site \( i \) and the complex coefficients \( \alpha_{in} \) allow for a time dependence. For all numerical calculations, a finite cutoff \( n_c \) is taken.

The mean-field ground state in equilibrium is found by minimization of the energy expectation value,

\[
\langle \Psi_{MF} | \hat{H}_{SCB} - \mu \hat{N} | \Psi_{MF} \rangle, \tag{7}
\]

where \( \mu \) is the chemical potential and \( \hat{N} \) counts the total number of particles. Hence, from here on we work on the grand-canonical ensemble. To find the time-evolution of the mean-field approximated system, we employ the time-dependent canonical ensemble. To find the time-evolution of the mean-field ground state in equilibrium is found by minimizing the energy expectation value, \( \langle \Psi_{MF} | i \partial_t - \hat{H}_{SCB} + \mu \hat{N} | \Psi_{MF} \rangle \), and yields the following set of differential equations:

\[
i \dot{\alpha}_{in} = -J \sum_{(j)} (\sqrt{n + 1} \alpha_{i(j+1)} \Phi_j^* + \sqrt{n} \alpha_{i(n-1)} \Phi_j) + \alpha_{in} \left[ \frac{U}{2} (n-1) + V r_i^2 - \mu \right]. \tag{9}
\]

Here \( \Phi_j = \langle a_j | = \sum_{n=1}^{\infty} \sqrt{n} \alpha_{i(j-1)}^* \alpha_{j(n)} \), \( \alpha_{i(i-1)} = \alpha_{i(n_c+1)} = 0 \), and \( \sum_{(j)} \) denotes summation over all \( j \) that are nearest neighbors of \( i \). This is a set of \( L \times n_c \) equations. The time evolution described by Eq. (9) preserves normalization and yields the following set of differential equations:

\[
i \dot{\alpha}_{in} = -J \sum_{(j)} (\sqrt{n + 1} \alpha_{i(j+1)} \Phi_j^* + \sqrt{n} \alpha_{i(n-1)} \Phi_j) + \alpha_{in} \left[ \frac{U}{2} (n-1) + V r_i^2 - \mu \right]. \tag{10}
\]

A. Analytical mean-field solution for hard-core bosons in a periodic potential

In the case of hard-core bosons, it is possible to reduce the number of equations considerably and employ a parametrization for the \( \alpha_{in} \) in Eq. (9) that preserves normalization and deals with real variables—this is due to the equivalence of hard-core bosons to \( s = \frac{1}{2} \) spins, which leads to the following ansatz for the Gutzwiller wave function [19]:

\[
|\Psi_{HCB} \rangle = \prod_{i=1}^{L} e^{\chi_i} \left( \sin \frac{\theta_i}{2} + \cos \frac{\theta_i}{2} e^{i\phi} a_i^d \right) |0\rangle. \tag{11}
\]

\[\chi_i = -\gamma A \sum_{k=0}^{n_k=0} \left( \sin \frac{\theta_i}{2} + \cos \frac{\theta_i}{2} e^{i\phi} a_i^d \right) |0\rangle \tag{12}\]

If there is no trap in the system \( (V = 0) \), it is possible to use translational invariance to simplify the equations (9), in which in the hard-core limit we again employ the superlattice quench introduced before. This leads to a formal replacement of \( U(n-1)/2 \) by \( A \) in (9). As all sites with the same potential must have the same properties and the system’s wave function is a product of single-site states, the \( L \)-site system can be reduced to an effective two-site problem (for two on-site potentials \( \mu_{1/2} = \mu \pm A \) independent of the dimension. Then

the ansatz Eq. (10) yields the following form of Eq. (9):

\[
\dot{\theta}_1 = -2dJ \sin \theta_2 \sin \phi, \tag{11a}
\]

\[
\dot{\theta}_2 = 2dJ \sin \theta_1 \sin \phi, \tag{11b}
\]

\[\dot{\phi} = 2A - 2dJ (\sin \theta_2 \cot \theta_1 - \sin \theta_1 \cot \theta_2) \cos \phi \tag{11c}\]

where \( \phi \equiv \phi_1 - \phi_2 \). Here it can be seen that dimensionality enters the equations only by a simple rescaling of the hopping parameter: \( J \to dJ \).

Also, the argument of translational invariance allows one to find a simple expression for \( n_{k=0} \):

\[
n_{k=0} = n + \frac{1}{2} (L - 1) \sin \theta_1 \sin \theta_2 \cos \phi. \tag{12}
\]

Figure 5 depicts the time evolution of \( n_{k=0} \) for the same systems for which the exact solution was presented in Fig. 2. One can clearly see that the mean-field and exact results show a similar shift of the frequency. However, the mean-field solutions exhibit no damping, and hence they are qualitatively incorrect for that quantity.

At this point it is instructive to extract the result for the atomic limit: for \( J = 0 \), Eq. (11) has the trivial solution \( \phi(t) = 2At + \phi(0) \) and \( \theta_{1/2} \) constant. Insertion of this result in Eq. (12) immediately reveals the revivals time \( t_{\text{rev}} = \pi / A \).

Obtaining the solution of the system of Eqs. (11) for finite \( J \) is possible by treating them like a classical system. Identification of Hamilton functions and the observation of their trajectories leads to an analytical expression for the period of \( n_{k=0} \):

\[
t_{\text{rev}} = \int_{u_1}^{u_2} du f(u), \text{ where } f(u) = \left[ d^2 J^2 (1 - u^2)^2 (1 - (2\gamma - u)^2) - 2 \gamma (1 - (2\gamma - u)^2)^{3/2} \right]^{-1/2}, \tag{13}\]

where \( \gamma = 2n - 1 \) and \( \gamma = 2n - 1 \) and \( \mathcal{H}_0 = -\gamma n (1-n) dJ - 2 \gamma A \) is the expectation value of the energy in the system before the

FIG. 5. (Color online) Plot of the time evolution of hard-core bosons in a periodic potential in the mean-field approximation for 1D (a) and 2D (b) after a superlattice quench \( A_{in} = 0 \to A_{in} = 1 \) for a system at quarter filling. This is to be compared with the exact results in Fig. 1, where exactly the same system parameters were used.

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quench. A closed expression for the preceding integral exists. However, it is cumbersome and does not provide any apparent
information on the functional form of $\Delta_{\text{rev}}$. However, $\Delta_{\text{rev}}$ is given by the analytic expression

$$
\Delta_{\text{rev}} = \frac{\sqrt{1 + 8d^2J^2} - 1}{\pi t_{\text{fin}}}
$$

for $\Delta_{\text{rev}}$ for a system of hard-core bosons in a periodic potential and a superlattice quench $A_{\text{fin}} = 0 \rightarrow A_{\text{fin}} = 1$. $t_{\text{fin}}$ is given by the analytic solution Eq. (13). Depicted are results for 1D, 2D, and 3D. The data points are numerical solutions of the set of Eqs. (9).

In Fig. 6, we plot the analytic solution for different dimensions at quarter filling. As mentioned before, dimensionality in the mean-field picture is captured by a simple rescaling of $J \rightarrow dJ$. For comparison, we depict numerical solutions of Eq. (9) as points in the plot. The latter are required for studying the inhomogeneous trapped hard-core boson case and soft-core bosons, for which no analytic solutions are available.

The analytic expression (13) allows us to confirm the numerical finding for the scaling relations of $t_{\text{rev}}$ and $n_{\text{rev}}$ with respect to the parameters $A$ and $J$. The proposed scaling for the revival time $t_{\text{rev}}(J,A) \equiv t_{\text{rev}}(A/J)A$ does obviously hold for the integrand $f(u)$ in Eq. (13). The fact that for the integration limits one has $u_{1/2}(J,A) \equiv u_{1/2}(A/J)$ proves the proposition for the mean-field case.

Furthermore, the evaluation of Eq. (12) leads to the expression

$$
n_{k=0} = n - \frac{1}{8dJ}(L - 1)(H_0 + 2A \cos \theta_1),
$$

where one can see, (i) that no damping of $n_{k=0}$ occurs within the mean-field approximation in the hard-core limit and (ii) that $n_{k=0}$ only depends on $J/A$ as found for the numerical solution.

We note also that for the case of half filling and $dJ = 1$, Eq. (13) yields $t_{\text{rev}} \rightarrow \infty$, reflecting the fact that the mean-field equations of motion do not predict any oscillations in this case—in contrast to the exact solution. With the absence of damping and the last observation, there are already two deficiencies of the mean-field approximation that we need to keep in mind for the analysis that follows—this makes the comparison to the exact solution an essential duty to ensure one has an idea of the limits of the validity of the mean-field results presented in Sec. IV.

B. Exact vs mean-field results

In equilibrium, a detailed comparison between the predictions of the mean-field theory introduced before and exact quantum Monte Carlo simulations for the ground state of hard-core bosons in the presence of a superlattice potential was presented in Refs. [11,12]. The Gutzwiller approach was found to correctly predict the two phases present in the ground state of this model, namely, a superfluid phase for all fillings but $n = 0,1/2$, and 1 and for $n = 1/2$ below a critical value of $A/J$ and a Mott insulator (a charge density wave) for $n = 1/2$ above a critical value of $A/J$. However, Gutzwiller mean-field theory was shown to provide a poor estimate of the critical value of $A/J$ for the superfluid-Mott-insulator transition. It overestimated it by around 100% in 2D and a 50% in 3D.

As shown in the previous section, after the quench in the periodic system, the mean-field solution does not exhibit any damping for the amplitude of the oscillation, whereas in the exact solution there obviously is damping. For this reason, we do not study the damping any further. In the remainder of the article we therefore focus on the predictions of mean-field theory for the corrections to the revival time.

In order to be more quantitative, we define the relative deviation of the mean-field approximation from the exact solution by

$$
\varepsilon(J) = \frac{\Delta_{\text{rev}}(J) - \Delta_{\text{rev}}^m(J)}{\Delta_{\text{rev}}(J)},
$$

where $\Delta_{\text{rev}}(J)$ and $\Delta_{\text{rev}}^m(J)$ are the corrections to the revival time due to a finite value of $J$ for the exact and mean-field solutions, respectively.

In Figs. 7(a) and 7(b), we plot $\varepsilon(J)$ in a linear-log scale vs $J$. That figure shows an almost constant error over two decades ($10^{-3} \leq J \leq 10^{-1}$). For $J \lesssim 10^{-3}$ rounding-off errors set in as one starts dealing with numbers $\sim 10^{-10}$; that is, the deviations seen in that region are not to be considered any further. The absolute value in the constant part of the deviation is around 16% for 1D (a) and 11% for 2D (b) for $n = 0.125$ and $n = 0.25$. At half filling, interestingly, the deviation is yet much smaller. In all cases, it is obvious that the mean-field approximation describes the 2D system better than the 1D case, even though the system size in 1D is one order of magnitude larger than in 2D and mean-field is expected to be more accurate as the system size is increased.

In Figs. 7(c) and 7(d), we present the same results as in Fig. 2 and compare them to the mean-field predictions, but this time in a linear scale. This scale emphasizes the differences between the mean-field results and the exact ones for values of $J$ close to $A = 1$. Once again, it is obvious that the mean-field approximation works better in 2D than in 1D and that it becomes a very good approximation of the exact results for $n = 0.125$ and $n = 0.25$. We further note the already-mentioned case of $dJ = 1$, which does not yield any revival in the mean-field approximation and therefore $\Delta_{\text{rev}} \rightarrow \infty$ as the figures suggest. Interestingly, there is a corresponding anomaly.
in the exact solution in 2D: there, in a neighborhood of $J = 0.5$, the exact solution does not produce a symmetric peak after the first revival oscillation, which makes it meaningless to determine a value for the revival time—therefore these data points are missing. Finally, Figs. 7(a) and 7(b) show that at half filling mean-field theory provides the most accurate results for the quadratic region of small values of $J$, whereas it provides the worse results for large values of $J$, as seen in (c) and (d).

Results for $\epsilon(J)$ in the 1D trapped case are presented in Fig. 8 for the two types of quenches introduced in the previous section: (i) the scenario of a constant characteristic density and (ii) switching off the trapping potential. The behavior is qualitatively the same as just discussed for the homogeneous case of Fig. 7(a). Quantitatively, we find an error $\sim 5\%$, which is in between the values for low and half filling in Fig. 7(a) and is similar for both quench types. Such an intermediate value is expected because the trap causes a density profile with different densities in different regions of the trap.

For the description of experiments, which are of more interest in 3D trapped geometries and very large system sizes, one can expect much smaller errors than the ones in Fig. 8. Therefore, we conclude that the shift of the revival time due to a finite hopping is correctly captured not only qualitatively but also quantitatively by the mean-field approximation described here. This is an interesting finding considering that, in contrast, the description of the evolution of the amplitude in terms of the mean-field approximation is incorrect even at the qualitative level.

IV. RESULTS FOR SOFT-CORE BOSONS

In Refs. [2,20] it has already been shown that ultracold bosonic gases in optical lattices can be well described by the bosonic Hubbard model (5). In light of the recent results in Refs. [3,4] mentioned in the Introduction, we note that the interaction constant $U$ and the hopping amplitude $J$ in the Hamiltonian (5) are the effective two-body interaction and one-body hopping, respectively, whose origin is not under discussion here. They may be obtained after multiorbital renormalization effects are taken into account. Multiorbital effects may also generate effective higher-body interactions that translate into additional frequencies during the collapse and revival of the matter-wave interference but are not considered here. These effects can be reduced by properly engineering the initial state. We focus on the effect that a finite effective hopping $J$ has on the first revival of the matter wave.

Collapse and revival oscillations like the ones observed experimentally in Refs. [1,3] have been reproduced in 1D bosonic [21] and fermionic [22] systems by means of numerically exact time-dependent renormalization group techniques. Here we use mean-field theory to study 3D bosonic systems theoretically. Following the results of the previous section, we expect the mean-field predictions for the tunneling-induced correction of the revival time of the value in the atomic limit $\Delta t_{\text{rev}}$ to be close to the exact results.

We proceed in the way introduced in the preceding section. For soft-core bosons, we solve the equations (9) for a cutoff of $n_\epsilon = 7$ numerically. This cutoff ensures convergence with respect to the energy expectation value and the momentum distribution and was successfully employed in Ref. [23] to study the superfluid–Mott-insulator transition. Initially, the system is prepared in the Gutzwiller mean-field ground state of the trapped system at an intermediate interaction $U_{\text{int}} = 6J_{\text{ini}}$ ($J_{\text{ini}} = 1$ sets our energy scale), which ensures the validity of a one-band model in experiments while the system is still far from the transition to the Mott insulator. Within the mean-field approximation $(U/J)_{\text{crit}} = 34.8$ (for six nearest neighbors) [13,24,25]. At time $t = 0$, the on-site interaction is doubled to $U_{\text{int}} = 12$ and we investigate the collapse and revivals for several values of $J_{\text{ini}} < 1$ (where, as in the section about hard-core bosons, the notation $U_{\text{int}} \equiv U$ and $J_{\text{int}} \equiv J$ is used in all unambiguous cases).

Figure 9 depicts the collapse and revival oscillations for the homogeneous case. These predictions, within mean-field theory, correspond to the solution of a single site problem. They clearly exhibit a different functional form when compared to the ones for hard-core bosons in a superlattice potential Fig. 2. Also, not only a shift in the revival time is present but...
Additionally, a damping of the amplitude can be observed. We do not consider this damping effect, as we cannot make any statement about the validity of the mean-field approximation for this quantity (see Sec. III). We note again that the revival time for the interaction quench in the atomic limit is given by $t_{\text{rev}} = 2\pi / U$.

In Fig. 10, we show $\Delta t_{\text{rev}}$ for several densities in the homogeneous case. We observe a linear relation emphasized by the fits for data points with $J \lesssim 0.01$ in the figure. Since soft-core bosons are not subject to particle-hole symmetry, the behavior with increasing density is different from the one observed for hard-core bosons in Fig. 2. The smallest deviation from the atomic limit is no longer reached at half filling; instead, for soft-core bosons we find that it appears for a density $n \sim 0.75$. Except for densities around this value, $\Delta t_{\text{rev}}$ is not strongly dependent on the density. With respect to the dependence of $\Delta t_{\text{rev}}$ on both parameters $J$ and $U$, we note that the same scaling as in the hard-core case (with $A$ and $U$ interchanged) holds true: $t_{\text{rev}}(J, U) \equiv t_{\text{rev}}(J/U) / U$.

The calculations for the trapped case are more demanding computationally. This is because translational invariance is broken and one has to deal with all the lattice sites in the system. The results reported here are obtained for a lattice with $L = 30 \times 30 \times 30 = 27000$ sites with $N = 1000$ up to $N = 11000$. As before, finite size effects for our observables of interest are extremely small. As a matter of fact, we found that it would be difficult to distinguish the results reported here from those of a $L = 10^3 = 1000$ system. Again, results are presented for the two quench scenarios analyzed in detail in Sec. II C for hard-core bosons.

Results for the time evolution of $n_{k=0}$ in the harmonic trap are shown in Fig. 11 for an initial state with characteristic density $\bar{\rho} = 39.8$ and the two different quench types: (i) keeping constant the characteristic density and (ii) turning of the trap. Here we observe an effect that is qualitatively different from the one seen in the hard-core limit (Fig. 3) and the homogeneous soft-core case, namely, the revival time in the case with finite hopping exceeds the atomic value [Fig. 11(a)]. This effect is only observed in the quench scenario (i) for high characteristic densities. For quench type (ii) the effect is not present for any density [Fig. 11(b)].

In Fig. 12, $\Delta t_{\text{rev}}$ is plotted for several values of the characteristic density of the initial state. In Fig. 12(a), results are shown for quench type (i). Note that we account for the fact that the revival time exceeds the atomic limit for high densities by plotting the absolute value of $\Delta t_{\text{rev}}$; in particular, $\bar{\rho} = 23.8$ and $\bar{\rho} = 39.8$ yield a negative value of $\Delta t_{\text{rev}}$. The strong dependence of the results on the characteristic density of the
uncertainty of the filling in the center of the trap would lead to a negative sign for $\tilde{\rho}$, due to the linearity of the relation and a prefactor $\sim 0.1$. This shows that the described effect is not small and one should be able to measure it in experiments.

Finally, we should mention that we also performed calculations for different values of the interaction constant $U$ before and after the quench. They all exhibited a similar qualitative behavior as depicted in Fig. 12. We therefore stress that our results do not depend on a particular value of $U$ but represent a general behavior that can be reproduced with experimentally relevant parameters. By comparing experimental results and calculations within mean-field theory, plus using the expected experimental values for $U$, one could then use experimentally measured values of $\Delta t_{\text{rev}}$ to determine the final value of $J$.

V. CONCLUSION

We have presented a detailed study of the dependence of collapse and revival properties of the matter-wave interference in lattice boson models on a finite tunneling amplitude after an interaction quench.

We first studied quenches of hard-core bosons on a superlattice potential. For those systems, we presented exact numerical results in 1D and 2D and compared them with the approximated mean-field solution. Both approaches exhibited the same functional form of the correction to the revival time produced by finite hopping parameters after the quench, with a leading behavior $\sim t^2 / A^3$. The mean-field results were also shown to have, as expected, smaller errors in 2D as compared to 1D. Since the largest errors for homogeneous 2D systems were $\sim 10\%$ and for trapped 1D systems $\sim 5\%$ (in contrast to $\sim 17\%$ for the 1D homogeneous case), we expect that in 3D trapped systems, mean-field theory should provide relatively accurate results for the corrections to the revival time.

We then studied interaction quenches in the Bose-Hubbard model in 3D. In this case, our analysis was solely based on the Gutzwiller mean-field theory. We showed that for soft-core bosons the corrections to the revival time in the atomic limit, due to finite values of $J$ after the quench, are $\sim J / U^2$. This is an effect that could be measured experimentally. Given the weak dependence of the correction on the initial density profile, provided the density in the center of the trap is greater than $n = 1.5$, we have proposed that the corrections to the revival time measured experimentally could be used to determine the actual value of $J$ after the quench. The only input one would need is the experimental value of $U$ and the mean-field predictions from calculations similar to the ones presented here.

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\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig12}
\caption{(Color online) Quantum corrections to the revival time ($\Delta t_{\text{rev}}$) vs $J$ for the scenarios (a) keeping the characteristic density constant and (b) turning of the trap; and an interaction quench $U_{\text{ini}} = 6 \rightarrow U_{\text{fin}} = 12$. Results are presented for five different initial values of the characteristic density---$\tilde{\rho} = 0.6$, $\tilde{\rho} = 3.1$, $\tilde{\rho} = 10.6$, $\tilde{\rho} = 23.8$, and $\tilde{\rho} = 39.8$---that correspond to the following values of the density in the middle of the trap: $n_{\text{center}} \sim 0.5$, $n_{\text{center}} \sim 1.0$, $n_{\text{center}} \sim 1.5$, $n_{\text{center}} \sim 2.0$, and $n_{\text{center}} \sim 2.5$. Note that in (a) $\Delta t_{\text{rev}}$ has a negative sign for $\tilde{\rho} = 23.8$ and $\tilde{\rho} = 39.8$; therefore, the absolute value is depicted.}
\end{figure}


