Statistical mechanics for prethermalization in nearly integrable systems

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Quench of an isolated quantum system

- Sudden change of a parameter

$$\text{Hamiltonian} = \begin{cases} H_0 & \text{for } t < 0 \\ H_0 + g H_1 & \text{for } t \geq 0 \end{cases}$$

- Time evolution of the state with $H = H_0 + g H_1$

$$|\psi(t = 0)\rangle = |\psi_0\rangle$$

$$|\psi(t \geq 0)\rangle = \exp(-iHt)|\psi_0\rangle = \sum_n \exp(-iE_n t)\, |n\rangle\langle n|\psi_0\rangle$$

- Thermalization of isolated system?

$$\langle A \rangle_t = \langle \psi(t)|A|\psi(t)\rangle \xrightarrow{t \to \infty} \langle A \rangle_{\text{therm}} = \frac{\operatorname{Tr} A\, e^{-\beta H}}{\operatorname{Tr} e^{-\beta H}} \quad \textbf{?}$$

with effective β from $\langle H \rangle_{\text{therm}} = E = \langle \psi_0|H|\psi_0\rangle$

Statistical mechanics / Integrable systems

- Fundamental postulate of statistical mechanics:

  All accessible microstates are equally probable

  $\Rightarrow$ Gibbs ensemble $\rho_{\text{Gibbs}} \propto e^{-\beta H}$

- Integrable systems: No thermalization

  Kinoshita et al, Nature **440** ('06)
  Rigol et al, PRL **98** ('07)
  Cazalilla, PRL **97** ('06)
  Eckstein & Kollar, PRL **100** ('08)
  ...

  - Many ($\propto L$) constants of motion $\mathcal{I}_\alpha$
  - $[H, \mathcal{I}_\alpha] = 0 \;\Rightarrow\; \langle \mathcal{I}_\alpha \rangle_t = \langle \mathcal{I}_\alpha \rangle_0 =$ const
  - Much fewer accessible states $\Rightarrow$ ρ_{Gibbs} not appropriate

- Nonthermal steady state: Statistical description

  - Generalized Gibbs ensemble ρ_{GGE}

    Rigol et al, PRB **74** ('06)

$$\rho_{\text{GGE}} \propto \exp\left(-\sum_\alpha \lambda_\alpha \mathcal{I}_\alpha\right) \quad \text{with} \quad \langle \mathcal{I}_\alpha \rangle_{\text{GGE}} = \langle \mathcal{I}_\alpha \rangle_0$$

$\Rightarrow$ often correct, but not always

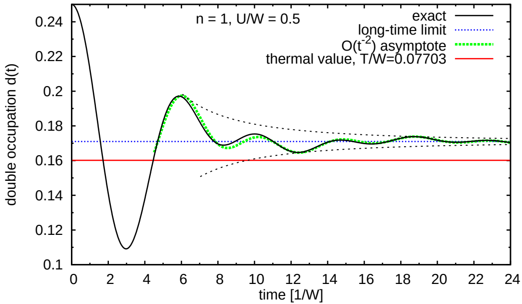
Gangardt & Pustilnik, PRA **77** ('08)
Barthel & Schollwöck, PRL **100** ('08)
Kollar & Eckstein, PRA **78** ('08)

Relaxation in an integrable system

Integrable $1/r$ Hubbard chain: Quench from $U = 0$ to $U = 0.5$

Kollar & Eckstein, PRA **78** ('08)

$$d(t) = \langle n_{i\uparrow} n_{i\downarrow} \rangle_t$$



Nonthermal steady state: described by GGE

$$\lim_{t \to \infty} d(t) = d_{\mathrm{GGE}} \neq d_{\mathrm{therm}}$$

Thermalization in nonintegrable systems

- Nonintegrable systems:

  Thermalization possible

Kollath et al, PRL **98** ('07)
Rigol et al, Nature **452** ('08)
Barmettler et al, PRL **102** ('09); NJP **12** ('10)
Rigol, PRL **103** ('10)

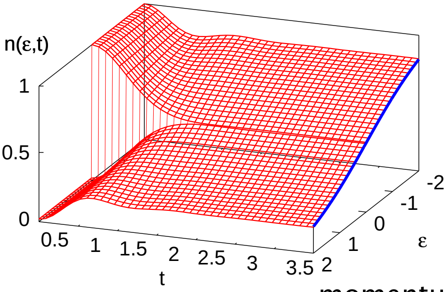
- Fermionic Hubbard model:

$$H = \sum_{\boldsymbol{k}\sigma} \epsilon_{\boldsymbol{k}}\, a^\dagger_{\boldsymbol{k}\sigma} a_{\boldsymbol{k}\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

(in $d = \infty$ / noneq-DMFT)

Eckstein, Kollar, Werner, PRL **103** ('09); PRB **81** ('10)

Quench from $U=0$ to $U=3.3$:



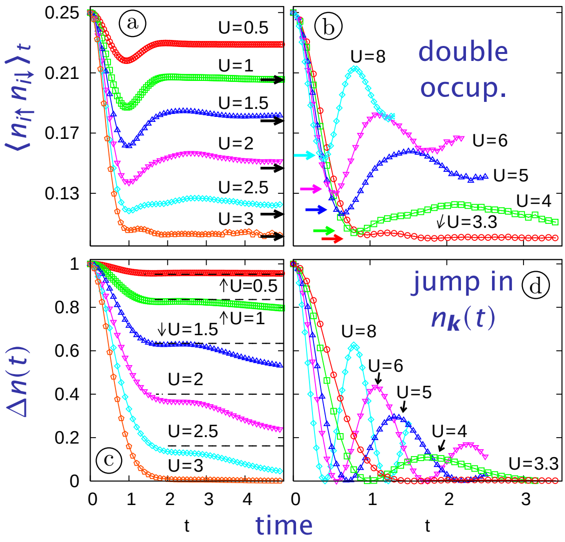
blue line: equilibrium momentum distribution for $U=3.3$ and $T_{\mathrm{eff}}=0.84$

momentum distribution thermalizes

Thermalization in the Hubbard model

Interaction quench from 0 to U:

Eckstein, Kollar, Werner, PRL **103** ('09)



Vicinity of an integrable point: Prethermalization

Near integrable point:

- $H_0 = \sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha$

  $\Rightarrow$ integrable

- Quench to

  $H_0 + g\,H_1$ with $|g| \ll 1$

  $\Rightarrow$ long-lived state for

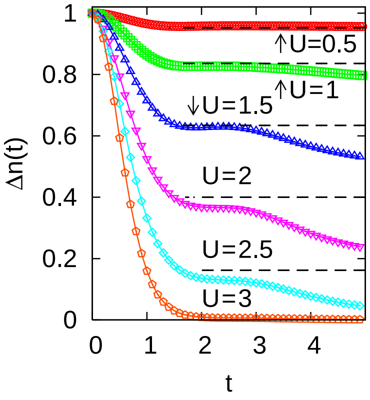
  $$\frac{\text{const}}{g} \ll t \ll \frac{\text{const}}{g^2}$$

"Prethermalization"

Berges et al, PRL **93** ('04)
Moeckel & Kehrein, PRL **100** ('08)
Moeckel & Kehrein, Ann. Phys. **324** ('09)

Interaction quench from 0 to U:

Fermi surface discontinuity



Eckstein, Kollar, Werner, PRL **103** ('09)
see also Uhrig, PRA **80** ('09)

Second order perturbation theory

The "Factor 2" Theorem:

Moeckel & Kehrein, Ann. Phys. **324** ('09)

- Quench from $H_0 = \sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha$ to $H = H_0 + g H_1$
- Observable A with $[H_0, A] = 0$ and $\langle A \rangle_{t=0} = 0$
- Prethermalization plateau for $\frac{\text{const}}{g} \ll T \ll \frac{\text{const}}{g^2}$

$$\lim_{T\to\infty} \int_0^T \frac{dt}{T} \langle A \rangle_t = \underbrace{2\,\langle \tilde{\psi}_0 | A | \tilde{\psi}_0 \rangle}_{=:\, A_{\text{pretherm}}} + O(g^3)$$

where $|\tilde{\psi}_0\rangle$ = ground state of H up to order g^2

Integrable vs. nonintegrable systems

Integrable systems vs. Nearly integrable systems

(Q1) Relaxation to nonthermal (quasi-)steady state

Nonthermal steady state $\overset{?}{\longleftrightarrow}$ Prethermalization plateau

(Q2) Statistical description of (quasi-)steady state

Generalized Gibbs ensemble

$\rho_{\mathrm{GGE}} \propto \exp\left(-\sum_\alpha \lambda_\alpha I_\alpha\right)$ $\overset{?}{\longleftrightarrow}$ $\rho_{\widetilde{\mathrm{GGE}}} \propto \exp\left(-\sum_\alpha \lambda_\alpha \tilde{I}_\alpha\right)$

GGE with exact constants of motion I_α

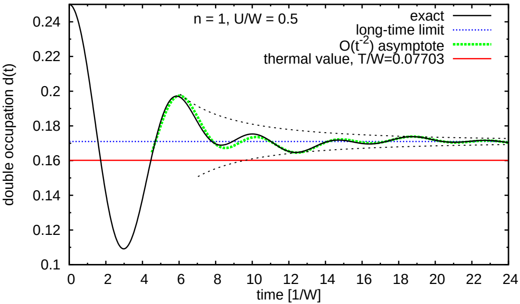
GGE with approximate constants of motion $\tilde{I}_\alpha$?

Q1: Nonthermal vs. prethermalized

Integrable $1/r$ Hubbard chain: Quench from 0 to U

Kollar & Eckstein, PRA **78** ('08)

$$d(t) = \langle n_{i\uparrow} n_{i\downarrow} \rangle_t$$



$$\lim_{t \to \infty} d(t) = \frac{n^2}{4} + \frac{n^2(2n-3)}{6} U + O(U^2) \qquad = d_{\mathrm{GGE}} \neq d_{\mathrm{therm}}$$

Prediction for prethermalization plateau:

$$d_{\mathrm{pretherm}} = \frac{n^2}{4} + \frac{n^2(2n-3)}{6} U + O(U^2) \qquad \textbf{!!}$$

Q2: Approximate constants of motion

Before quench:

- $H_0 = \sum_\alpha \epsilon_\alpha I_\alpha$ with $I_\alpha = a_\alpha^\dagger a_\alpha$ (bosons or fermions)
- $[I_\alpha, I_\beta] = 0, \quad [H_0, I_\alpha] = 0$
- Basis: $I_\alpha |\boldsymbol{n}\rangle = n_\alpha |\boldsymbol{n}\rangle$
- System in ground state $|\psi_0\rangle$ of H_0

After quench:

- $H = H_0 + g H_1$ with $[H_0, H_1] \neq 0$ and $|g| \ll 1$
- H_1 can be expressed with $a_\alpha^\dagger$ and a_α
- $|\tilde{\psi}_0\rangle =$ ground state of H

Construction of approximate constants of motion

- Canonical trafo: $S = gS_1 + \frac{1}{2}g^2 S_2 + O(g^3)$

Harris & Lange, PR **157** (1963)

$$e^S H e^{-S} = H_0 + g(H_1 + [S_1, H_0]) + g^2(\tfrac{1}{2}[S_2, H_0] + [S_1, H_1] + \tfrac{1}{2}[S_1,[S_1,H_0]]) + O(g^3)$$

- With $\tilde{I}_\alpha = e^{-S} I_\alpha e^S$ and $|\tilde{\boldsymbol{n}}\rangle = e^{-S}|\boldsymbol{n}\rangle$:

$$H = \sum_\alpha \epsilon_\alpha \underbrace{\tilde{I}_\alpha}_{} + \sum_{\tilde{\boldsymbol{n}}} |\tilde{\boldsymbol{n}}\rangle\langle\tilde{\boldsymbol{n}}| \, (E_{\boldsymbol{n}}^{(1)} + E_{\boldsymbol{n}}^{(2)}) + O(g^3)$$

= approx. const. of motion of H

- Generalized Gibbs ensemble:

$$\rho_{\widetilde{\mathrm{GGE}}} = \exp\left(-\sum_\alpha \lambda_\alpha \tilde{I}_\alpha\right) \quad \text{with} \quad \langle\tilde{I}_\alpha\rangle_{\widetilde{\mathrm{GGE}}} \stackrel{!}{=} \langle\tilde{I}_\alpha\rangle_0$$

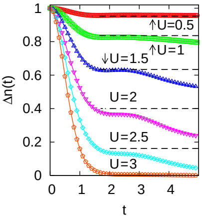
Statistical prediction for pretherm. plateau

GGE prediction $\langle A \rangle_{\widetilde{\mathrm{GGE}}}$ vs. pretherm. plateau $\langle A \rangle_{\mathrm{pretherm}}$:

Eckstein et al, EPJST **180**, 217 ('10)
Kollar, Wolf, Eckstein, in preparation

- Simplest observable: $A = I_\alpha = a_\alpha^\dagger a_\alpha$

  $\Rightarrow \quad \langle I_\alpha \rangle_{\widetilde{\mathrm{GGE}}} = \langle I_\alpha \rangle_{\mathrm{pretherm}} + O(g^3)$ **!!**



- General observable: $A = I_{\alpha_1} \cdots I_{\alpha_m}$

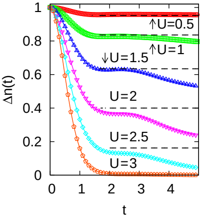
  $\Rightarrow \quad \langle A \rangle_{\widetilde{\mathrm{GGE}}} - \langle A \rangle_{\mathrm{pretherm}}$

  $$= \langle I_{\alpha_1} \rangle_{\tilde{\psi}_0} \cdots \langle I_{\alpha_m} \rangle_{\tilde{\psi}_0} - \langle I_{\alpha_1} \cdots I_{\alpha_m} \rangle_{\tilde{\psi}_0} + O(g^3)$$

similar to GGE criteria in Kollar & Eckstein, PRA **78** ('08)

Conclusion

- 'Dressed' GGEs predict prethermalization in nearly integrable systems (e.g., for $n_{\mathbf{k}\sigma}$ and d in small-U quenches)



- Integrable / nearly integrable systems are connected, and described by one statistical theory

- Nonthermal steady states in integrable systems
  evolve into
  prethermalization plateaus away from integrability

- Statistical mechanics works, if done right