Statistical mechanics for prethermalization in nearly integrable systems

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Quench of an isolated quantum system

- Sudden change of a parameter

Hamiltonian = \[
\begin{cases}
    H_0 & \text{for } t < 0 \\
    H_0 + g H_1 & \text{for } t \geq 0
\end{cases}
\]

- Time evolution of the state with \( H = H_0 + g H_1 \)

\[
|\psi(t = 0)\rangle = |\psi_0\rangle
\]
\[
|\psi(t \geq 0)\rangle = \exp(-iHt)|\psi_0\rangle = \sum_n \exp(-iE_n t) |n\rangle \langle n|\psi_0\rangle
\]

- Thermalization of isolated system?

\[
\langle A \rangle_t = \langle \psi(t)|A|\psi(t)\rangle \xrightarrow{t \to \infty} \langle A \rangle_{\text{therm}} = \frac{\text{Tr} A e^{-\beta H}}{\text{Tr} e^{-\beta H}}
\]

with effective \( \beta \) from \( \langle H \rangle_{\text{therm}} = E = \langle \psi_0|H|\psi_0\rangle \)
Fundamental postulate of statistical mechanics:

All accessible microstates are equally probable

⇒ Gibbs ensemble \( \rho_{\text{Gibbs}} \propto e^{-\beta H} \)

Integrable systems: No thermalization

- Many (\( \propto L \)) constants of motion \( I_\alpha \)
  
  \([ H, I_\alpha ] = 0 \) ⇒ \( \langle I_\alpha \rangle_t = \langle I_\alpha \rangle_0 = \text{const} \)
- Much fewer accessible states ⇒ \( \rho_{\text{Gibbs}} \) not appropriate

Nonthermal steady state: Statistical description

- Generalized Gibbs ensemble \( \rho_{\text{GGE}} \)

\[
\rho_{\text{GGE}} \propto \exp \left( - \sum_\alpha \lambda_\alpha I_\alpha \right) \quad \text{with} \quad \langle I_\alpha \rangle_{\text{GGE}} = \langle I_\alpha \rangle_0
\]

⇒ often correct, but not always
Relaxation in an integrable system

**Integrable 1/r Hubbard chain:** Quench from \( U = 0 \) to \( U = 0.5 \)

\[
d(t) = \langle n_{i\uparrow} n_{i\downarrow} \rangle_t
\]

Nonthermal steady state: described by GGE

\[
\lim_{t \to \infty} d(t) = d_{\text{GGE}} \neq d_{\text{therm}}
\]
Thermalization in nonintegrable systems

- **Nonintegrable systems:**
  
  Thermalization possible

- **Fermionic Hubbard model:**

  \[ H = \sum_{k \sigma} \epsilon_k a_{k\sigma}^\dagger a_{k\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \]

  (in \( d=\infty \) / noneq-DMFT)

  Eckstein, Kollar, Werner, PRL 103 ('09); PRB 81 ('10)

**Quench from \( U=0 \) to \( U=3.3 \):**

- Blue line: equilibrium

  momentum distribution for \( U=3.3 \) and \( T_{\text{eff}}=0.84 \)

momentum distribution thermalizes
Thermalization in the Hubbard model

Interaction quench from $0$ to $U$:

Eckstein, Kollar, Werner, PRL 103 ('09)
Vicinity of an integrable point: Prethermalization

Near integrable point:

▶ $H_0 = \sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha$

⇒ integrable

▶ Quench to $H_0 + g H_1$ with $|g| \ll 1$

⇒ long-lived state for $\frac{\text{const}}{g} \ll t \ll \frac{\text{const}}{g^2}$

“Prethermalization”

Berges et al, PRL 93 ('04)
Moeckel & Kehrein, PRL 100 ('08)
Moeckel & Kehrein, Ann. Phys. 324 ('09)

Interaction quench from 0 to $U$:

Fermi surface discontinuity

Eckstein, Kollar, Werner, PRL 103 ('09)
see also Uhrig, PRA 80 ('09)
Second order perturbation theory

The “Factor 2” Theorem:  

- Quench from $H_0 = \sum_{\alpha} \epsilon_{\alpha} a_{\alpha}^\dagger a_{\alpha}$ to $H = H_0 + g H_1$

- Observable $A$ with $[H_0, A] = 0$ and $\langle A \rangle_{t=0} = 0$

- Prethermalization plateau for $\frac{\text{const}}{g} \ll T \ll \frac{\text{const}}{g^2}$

$$\lim_{T \to \infty} \int_0^T \frac{dt}{T} \langle A \rangle_t = \frac{2}{g^3} \langle \tilde{\psi}_0 | A | \tilde{\psi}_0 \rangle + O(g^3) =: A_{\text{pretherm}}$$

where $|\tilde{\psi}_0\rangle = \text{ground state of } H \text{ up to order } g^2$
Integrable vs. nonintegrable systems

Integrable systems vs. Nearly integrable systems

(Q1) Relaxation to nonthermal (quasi-)steady state

Nonthermal steady state \(\leftrightarrow\) Prethermalization plateau

(Q2) Statistical description of (quasi-)steady state

Generalized Gibbs ensemble

\[
\rho_{\text{GGE}} \propto \exp\left(- \sum_{\alpha} \lambda_{\alpha} I_{\alpha}\right) \quad \leftrightarrow \quad \rho_{\widetilde{\text{GGE}}} \propto \exp\left(- \sum_{\alpha} \lambda_{\alpha} \tilde{I}_{\alpha}\right)
\]

GGE with exact constants of motion \(I_{\alpha}\) \quad GGE with approximate constants of motion \(\tilde{I}_{\alpha}\)
Q1: Nonthermal vs. prethermalized

Integrable $1/r$ Hubbard chain: Quench from 0 to $U$

\[ d(t) = \langle n_{i\uparrow} n_{i\downarrow} \rangle_t \]

Kollar & Eckstein, PRA 78 ('08)

\[ \lim_{t \to \infty} d(t) = \frac{n^2}{4} + \frac{n^2(2n - 3)}{6} U + O(U^2) = d_{\text{GGE}} \neq d_{\text{therm}} \]

Prediction for prethermalization plateau:

\[ d_{\text{pretherm}} = \frac{n^2}{4} + \frac{n^2(2n - 3)}{6} U + O(U^2) !! \]
**Q2: Approximate constants of motion**

**Before quench:**

- \( H_0 = \sum_\alpha \epsilon_\alpha I_\alpha \) with \( I_\alpha = a^\dagger_\alpha a_\alpha \) (bosons or fermions)
- \( [I_\alpha, I_\beta] = 0, \ [H_0, I_\alpha] = 0 \)
- Basis: \( I_\alpha |n\rangle = n_\alpha |n\rangle \)
- System in ground state \( |\psi_0\rangle \) of \( H_0 \)

**After quench:**

- \( H = H_0 + g H_1 \) with \( [H_0, H_1] \neq 0 \) and \( |g| \ll 1 \)
- \( H_1 \) can be expressed with \( a^\dagger_\alpha \) and \( a_\alpha \)
- \( |\tilde{\psi}_0\rangle = \) ground state of \( H \)
Construction of approximate constants of motion

- Canonical trafo: \( S = gS_1 + \frac{1}{2} g^2 S_2 + O(g^3) \)

\[
e^S H e^{-S} = H_0 + g(H_1 + [S_1, H_0]) + g^2 \left( \frac{1}{2} [S_2, H_0] + [S_1, H_1] + \frac{1}{2} [S_1, [S_1, H_0]] \right) + O(g^3)
\]

- With \( \tilde{I}_\alpha = e^{-S} I_\alpha e^S \) and \( |\tilde{n}\rangle = e^{-S} |n\rangle \):

\[
H = \sum_\alpha \epsilon_\alpha \tilde{I}_\alpha + \sum |\tilde{n}\rangle \langle \tilde{n}| (E_n^{(1)} + E_n^{(2)}) + O(g^3)
\]

= approx. const. of motion of \( H \)

- Generalized Gibbs ensemble:

\[
\rho_{\text{GGE}} = \exp \left( - \sum_\alpha \lambda_\alpha \tilde{I}_\alpha \right) \quad \text{with} \quad \langle \tilde{I}_\alpha \rangle_{\text{GGE}} = \langle \tilde{I}_\alpha \rangle_0
\]
Statistical prediction for pretherm. plateau

GGE prediction $\langle A \rangle_{GGE}$ vs. pretherm. plateau $\langle A \rangle_{pretherm}$:

- Simplest observable: $A = I_\alpha = a_\alpha^\dagger a_\alpha$
  
  $\Rightarrow \langle I_\alpha \rangle_{GGE} = \langle I_\alpha \rangle_{pretherm} + O(g^3)$

- General observable: $A = I_{\alpha_1} \cdots I_{\alpha_m}$
  
  $\Rightarrow \langle A \rangle_{GGE} - \langle A \rangle_{pretherm}$
  
  $= \langle I_{\alpha_1} \rangle_{\tilde{\psi}_0} \cdots \langle I_{\alpha_m} \rangle_{\tilde{\psi}_0} - \langle I_{\alpha_1} \cdots I_{\alpha_m} \rangle_{\tilde{\psi}_0} + O(g^3)$

Eckstein et al, EPJST 180, 217 ('10)
Kollar, Wolf, Eckstein, in preparation

similar to GGE criteria in Kollar & Eckstein, PRA 78 ('08)
’Dressed’ GGEs predict prethermalization in nearly integrable systems (e.g., for $n_{k\sigma}$ and $d$ in small-$U$ quenches)

Integrable / nearly integrable systems are connected, and described by one statistical theory

Nonthermal steady states in integrable systems evolve into prethermalization plateaus away from integrability

Statistical mechanics works, if done right