An Introduction to Bosonization

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1. Why is $d = 1$ special for interacting electrons?
2. The bosonization method
3. Tomonaga-Luttinger model
4. Spin-charge separation
5. Summary

References

Timeline of bosonization in condensed-matter physics: (incomplete)

- Tomonaga (1950)
- Luttinger (1963)
- Mattis & Lieb (1965)
- Mattis (1974), Luther & Peschel (1974)
- Haldane (1981)

Parallel developments in particle physics:

- Coleman (1975), Mandelstam (1975), . . .

Reviews/Tutorials/Applications:

- Sólyom (1978)
- Stone (1994)
- Schönhammer & Meden (1996), Schönhammer (1997)
- von Delft & Schoeller (1998)
- Sénéchal (1999)
- Rao & Sen (2000)
- see http://www.arxiv.org/ . . .

1. Why is $d = 1$ special for interacting electrons?

cf. Sénéchal (1999)

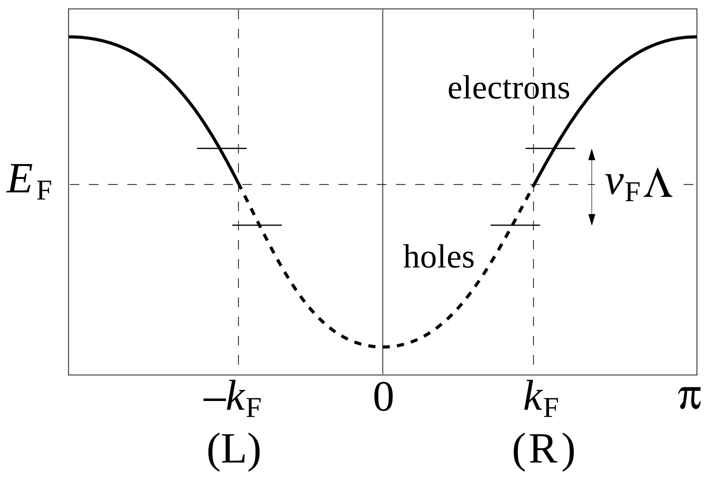
Electrons in a solid

General Hamiltonian: $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$

Kinetic energy (single energy band)

$$\hat{H}_0 = \sum_{\sigma=\uparrow,\downarrow} \sum_k \epsilon_k \, \hat{c}^+_{k\sigma} \hat{c}_{k\sigma}$$

$\underline{d = 1}$: Fermi surface consists of two Fermi points



Coulomb interaction:

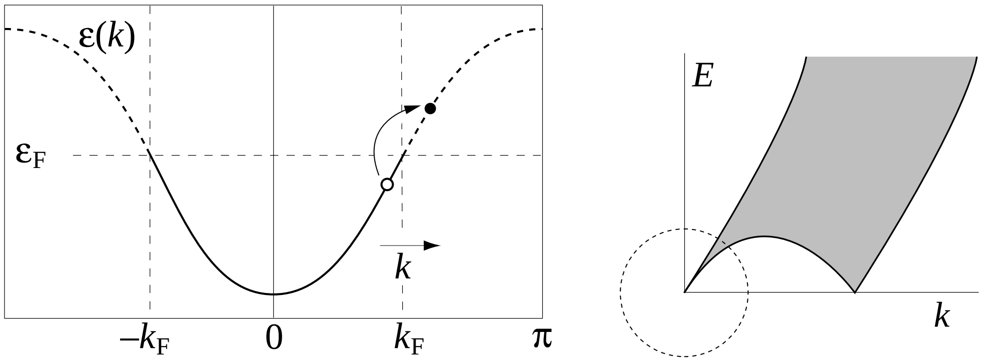
$$\hat{H}_{\text{int}} = \frac{1}{2L} \sum_{kk'q\sigma\sigma'} V^{\sigma\sigma'}_{kk'q} \; \hat{c}^+_{k+q\sigma} \hat{c}^+_{k'-q\sigma'} \hat{c}_{k'\sigma'} \hat{c}_{k\sigma}$$

⇒ creates two particle-hole pairs in Fermi sea

spectrum of $\hat{H}$: complicated many-body problem!

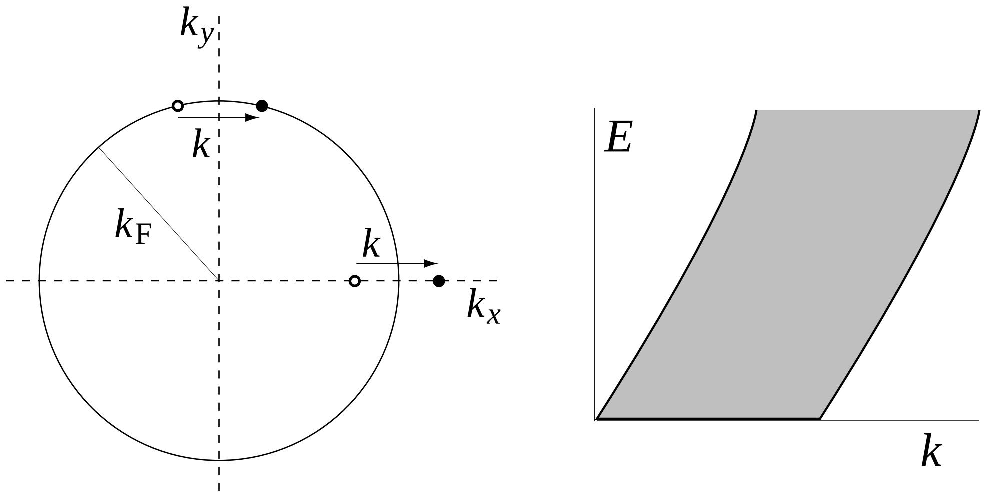
Propagation of particle-hole pairs

$\underline{d = 1}$:



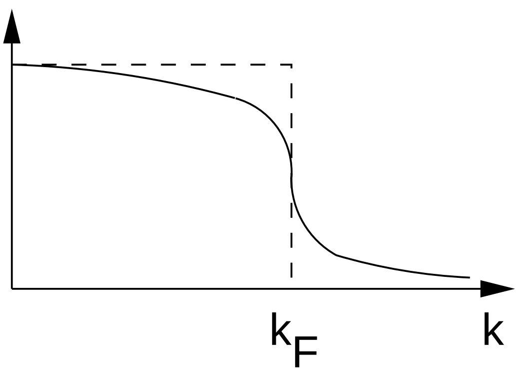
⇒ particle/hole have approximately same velocity v_F
⇒ coherent propagation almost like a single particle

$\underline{d = 2}$:

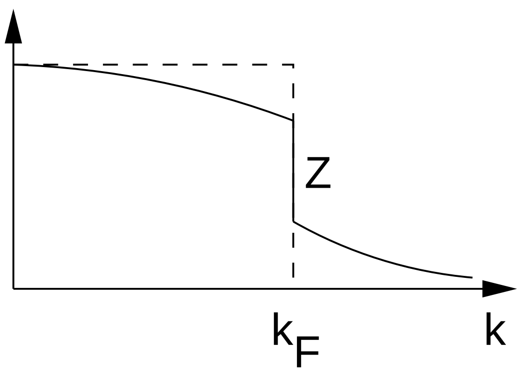


⇒ particle-hole pair with small relative k can be created at various E
⇒ incoherent motion more likely

Momentum distribution:



Luttinger liquid ($d = 1$)



Fermi liquid ($d \geq 2$)

2. The bosonization method

Fermionic field operators

Let us consider a standard fermion on the line $x = -\frac{L}{2} \dots \frac{L}{2}$:

$$\{\hat{\psi}(x), \hat{\psi}^+(x')\} = \delta(x - x') \qquad \text{if } x, x' \in [-\tfrac{L}{2}; \tfrac{L}{2}],$$
$$\{\hat{\psi}(x), \hat{\psi}(x')\} = 0,$$

with boundary conditions (periodic: $\delta = 0$, antiperiodic: $\delta = 1$)

$$\hat{\psi}(x + \tfrac{L}{2}) = e^{\pi i \delta}\, \hat{\psi}(x - \tfrac{L}{2}), \qquad \delta \in [0; 2).$$

We expand in Fourier components

$$\hat{\psi}(x) = \frac{1}{\sqrt{L}} \sum_k e^{-ikx} \hat{c}_k,$$

where $k = \frac{2\pi}{L}(n_k - \frac{\delta}{2})$, with $n_k \in \mathbb{Z}$, and

$$\{\hat{c}_k, \hat{c}^+_{k'}\} = \delta_{kk'},$$
$$\{\hat{c}_k, \hat{c}_{k'}\} = 0.$$

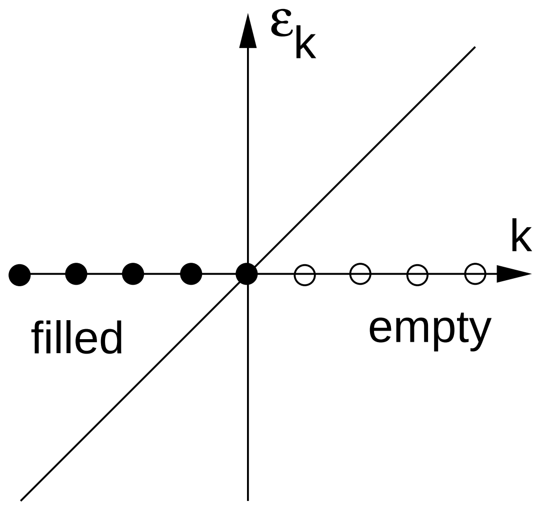
Note:

- L is large but finite ⇒ wavevector k is discrete
- no periodic lattice ⇒ number of modes n_k is infinite

Vacuum state and normal ordering

Consider a non-interacting system with energy dispersion ϵ_k.

ϵ_k is increasing with k
⇒ all states with (say) $k \le 0$ are occupied
⇒ infinite Dirac sea



We define a vacuum state $|0\rangle_0$

$$\hat{c}_k|0\rangle_0 = 0 \quad \text{for } k > 0\,, \qquad \hat{c}_k^+|0\rangle_0 = 0 \quad \text{for } k \le 0\,, \tag{1}$$

a corresponding normal ordering

$$ {}^*_*\hat{A}\hat{B}\hat{C}\ldots{}^*_* = \hat{A}\hat{B}\hat{C}\ldots - {}_0\langle 0|\hat{A}\hat{B}\hat{C}\ldots|0\rangle_0 \quad \text{for } \hat{A},\hat{B},\hat{C},\ldots \in \{\hat{c}_k, \hat{c}_k^+\}$$

and the number operator:

$$\hat{N} = \sum_k {}^*_*\hat{c}_k^+\hat{c}_k{}^*_* = \sum_{k>0}\hat{c}_k^+\hat{c}_k + \sum_{k\le 0}\underbrace{(-\hat{c}_k^+\hat{c}_k)}_{\hat{c}_k\hat{c}_k^+ - 1}$$

$$= \sum_k \left(\hat{c}_k^+\hat{c}_k - {}_0\langle 0|\hat{c}_k^+\hat{c}_k|0\rangle_0\right).$$

- $\hat{N}$ has eigenvalues $N \in \mathbb{Z}$
- we define $|N\rangle_0$ as $|0\rangle_0$ with lowest $|N|$ extra particles/holes
- Hilbert space $\mathcal{H}_N$ contains all states with $\hat{N}$-eigenvalue N
- Fock space $\mathcal{F}_c = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_{-1} \oplus \mathcal{H}_2 \oplus \mathcal{H}_{-2} \oplus \ldots$

Bosonic particle-hole excitations

Consider density operators $\hat{\rho}_q = \sum_k \hat{c}^+_{k+q}\hat{c}_k$

and their commutation relations:

$$[\hat{\rho}_q, \hat{\rho}_{q'}] = \sum_k \left(\hat{c}^+_{k+q}\hat{c}_{k+q'} - \hat{c}^+_{k+q-q'}\hat{c}_k\right) \quad \leftarrow \text{NOT separately convergent}$$

$$= \sum_k \left({}^*_*\hat{c}^+_{k+q}\hat{c}_{k+q'}{}^*_* - {}^*_*\hat{c}^+_{k+q-q'}\hat{c}_k{}^*_* \quad \leftarrow \text{separately convergent}\right.$$

$$\left. + {}_0\langle 0|\hat{c}^+_{k+q}\hat{c}_{k+q'}|0\rangle_0 - {}_0\langle 0|\hat{c}^+_{k+q-q'}\hat{c}_k|0\rangle_0\right)$$

$$= \delta_{qq'} \underbrace{\sum_k \Big(\Theta(k+q \le 0) - \Theta(k \le 0)\Big)}_{\propto\, q}$$

- exact only if bottom of band extends to $k = -\infty$
- very different from $d \geq 2$ where $[\hat{\rho}_q, \hat{\rho}_{q'}] = 0$

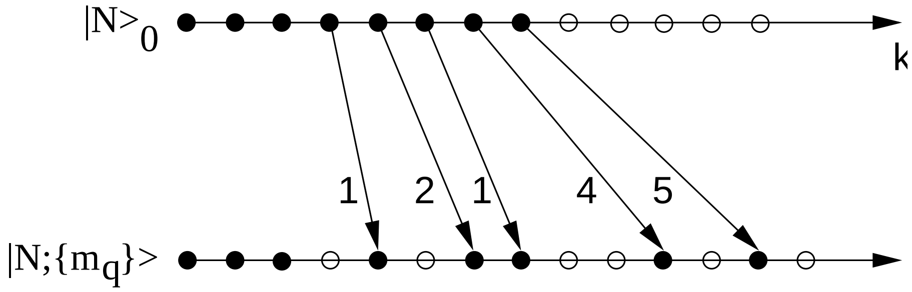
Define bosons: $\boxed{\hat{b}^+_q = \frac{i}{\sqrt{n_q}} \sum_k \hat{c}^+_{k+q}\hat{c}_k}$ for $q > 0$

- they obey $[\hat{b}_q, \hat{b}^+_{q'}] = \delta_{qq'}$ and $[\hat{b}_q, \hat{b}_{q'}] = 0$
  $\Rightarrow$ particle-hole excitations are independent bosons!
- $\hat{b}_q|N\rangle_0 = 0 \quad \Rightarrow$ in each $\mathcal{H}_N$ the state $|N\rangle_0$ is vacuum for $\hat{b}_q$
  $\Rightarrow$ boson normal-ordering = fermion normal-ordering
- bosons commute with $\hat{N}$

Reorganization of Hilbert space

Bosonic basis states: $|N;\{m_q\}\rangle = \prod_{q>0} \frac{(\hat{b}_q^+)^{m_q}}{\sqrt{m_q!}} |N\rangle_0$

Example:



Claim: Hilbert space $\mathcal{H}_N$ (all states with N fermions) can be obtained by acting with bosons on Dirac sea $|N\rangle_0$

Step 1: "Kronig identity"

$$\sum_{q>0} q \; {}^*_*\hat{b}_q^+ \hat{b}_q{}^*_* = \sum_k k \; {}^*_*\hat{c}_k^+ \hat{c}_k{}^*_* - \frac{2\pi}{L} \frac{\hat{N}(\hat{N}+1-\delta)}{2}$$

Note:

- $\hat{b}_q^+ \hat{b}_q$ contains 4 fermions, but only $\hat{N}^2$ survives
- can be verified via commutators with density operators
- purely algebraic identity, does not use bosonic commutators
- holds even for finite Fermi sea; works only for linear prefactor
- kinetic energy with linear dispersion is quadratic in bosons!

Equivalence of Fock spaces

Step 2: Completeness of bosonic states

- bosons are defined in terms of fermions $\Rightarrow \mathcal{F}_b \subseteq \mathcal{F}_c$
- calculate a sum of positive quantities using fermions and bosons

Grand partition function for non-interacting Hamiltonian: ($\delta = 1$)

$$\hat{h}_0 = \sum_k k \, {}^*_*\hat{c}_k^+ \hat{c}_k{}^*_* = \frac{\pi}{L}\hat{N}^2 + \sum_{q>0} q \, {}^*_*\hat{b}_q^+ \hat{b}_q{}^*_* \qquad \text{(Kronig)}$$

1) In fermionic basis:

$$\mathcal{Z}_c = \mathrm{Tr}_c[e^{-\beta \hat{h}_0}] = \prod_{n=1}^{\infty} (1 + w^{2n-1})^2, \qquad w = e^{-\beta\pi/L}$$

2) In bosonic basis: $|N; \{m_q\}\rangle$ has $\hat{h}_0$-eigenvalue $\pi N^2/L + \sum_{q>0} q m_q$

$$\mathcal{Z}_c = \sum_{N=-\infty}^{\infty} w^{N^2} \sum_{M=0}^{\infty} P(M) w^{2M} = \sum_{N=-\infty}^{\infty} w^{N^2} \Big/ \prod_{M=1}^{\infty} (1 - w^{2M})$$

where $P(M)$ = number of partitions of integer M

$\Rightarrow \mathcal{Z}_c = \mathcal{Z}_b$ due to an identity for the elliptic theta function ϑ_3

bosonic Fock space spans full fermionic Fock space

The bosonization identity

Commutator of bosons with fermions:

$$[\hat{b}_q, \hat{\psi}(x)] = \frac{i\, e^{iqx}}{\sqrt{n_q}}\, \hat{\psi}(x)$$

$$\Rightarrow \qquad \hat{b}_q \hat{\psi}(x)|N\rangle_0 = \frac{i\, e^{iqx}}{\sqrt{n_q}} \underbrace{\hat{\psi}(x)|N\rangle_0}_{\text{eigenstate of } \hat{b}_q!} \qquad \text{due to } \hat{b}_q|N\rangle_0 = 0$$

Coherent-state representation:

$$\hat{\psi}(x)|N\rangle_0 = \lambda_N(x)\, \exp\Bigg[\underbrace{\sum_{q>0} \frac{i\, e^{iqx}}{\sqrt{n_q}}\, \hat{b}_q^+}_{\equiv -i\hat{\varphi}^+(x)}\Bigg]\, \hat{F}\, |N\rangle_0$$

"Klein factor" $\hat{F}$ deletes one fermion (which is something bosons can't do):

$$[\hat{N}, \hat{F}] = -\hat{F}\,, \quad [\hat{N}, \hat{F}^+] = \hat{F}^+\,, \quad \hat{F}\hat{F}^+ = \hat{F}^+\hat{F} = 1\,, \quad \hat{F}^2 = \hat{F}^{+2} = 0$$

Now express any state $|N\rangle$ in terms of $\hat{b}_q^+$'s acting on $|N\rangle_0$, and commute these through $\hat{\varphi}^+(x)$:

$$\boxed{\hat{\psi}(x) = \sqrt{\frac{2\pi}{L}}\, e^{-\frac{2\pi}{L}(\hat{N}-\frac{\delta}{2})x}\, \hat{F}\, e^{-i\hat{\varphi}^+(x)}\, e^{-i\hat{\varphi}(x)}}$$

"bosonization identity"

- operator identity in Fock space, independent of Hamiltonian
- right-hand side is normal ordered

Bosonic fields

We use the bosonic field $\hat{\varphi}$

$$\hat{\varphi}(x) = -\sum_{q>0} \frac{e^{-iqx}}{\sqrt{n_q}}\,\hat{b}_q\, e^{-aq/2}\,, \qquad \text{regularized by } a \to 0^+$$

and the hermitian field $\hat{\phi}(x)$

$$\hat{\phi}(x) = \hat{\varphi}(x) + \hat{\varphi}^+(x)$$

with commutators (for $L \to \infty$)

$$[\hat{\varphi}(x), \hat{\varphi}^+(x)] = -\ln\left(\tfrac{2\pi}{L}(x - x' - ia)\right)$$

$$[\hat{\phi}(x), \hat{\phi}(x')] = -i\pi\,\mathrm{sgn}(x - x')$$

This leads to the expression

$$\sum_{q>0} q \,{}^*_*\hat{b}_q^+\hat{b}_q{}^*_* = \frac{1}{4\pi}\int dx\,{}^*_*(\partial_x\hat{\phi}(x))^2{}^*_*$$

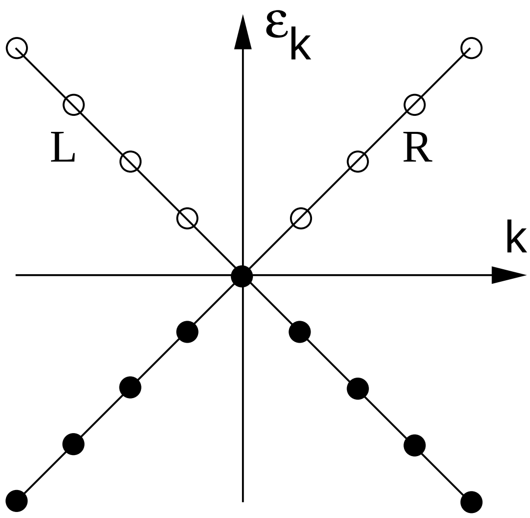
and allows fields in bosonization identity to be combined:

$$e^{-i\hat{\varphi}^+(x)}\, e^{-i\hat{\varphi}(x)} = \sqrt{\frac{L}{2\pi a}}\;\underbrace{e^{-i\hat{\phi}(x)}}_{\text{not normal ordered!}}$$

but this introduces a factors a^{-1} $\Rightarrow$ limit $a \to 0$ must be delayed

The Hamiltonian

Consider two branches:
one right-moving fermion, $\Psi_R(x) = \psi_1(x)$,
one left-moving fermion $\Psi_L(x) = \psi_2(-x)$



$$\hat{H}_0 = v_F^0 \int dx \, {}^*_*\hat{\Psi}_R^+ i\frac{\partial}{\partial x}\hat{\Psi}_R - \hat{\Psi}_L^+ i\frac{\partial}{\partial x}\hat{\Psi}_L{}^*_*$$
$$= v_F^0 \sum_{\eta=1,2} \sum_k k \, {}^*_*\hat{c}^+_{k\eta}\hat{c}_{k\eta}{}^*_*$$
$$= v_F^0 \sum_{q>0,\eta} q \, {}^*_*\hat{b}^+_{q\eta}\hat{b}_{q\eta}{}^*_* + \frac{v_F^0\pi}{L}\sum_\eta \hat{N}_\eta^2$$

⇒ kinetic energy is quadratic in bosons (as before)

Coulomb interaction acts on the densities $\hat{\rho}_{L,R}(x) = \hat{\Psi}^+_{L,R}(x)\hat{\Psi}_{L,R}(x)$:

$$\hat{H}_{\text{int}} = \iint V_2(x-x')\, {}^*_*\hat{\rho}_L(x)\hat{\rho}_R(x') + \hat{\rho}_R(x)\hat{\rho}_L(x'){}^*_*\, dx\, dx'$$
$$+ \iint V_4(x-x')\, {}^*_*\hat{\rho}_L(x)\hat{\rho}_L(x') + \hat{\rho}_R(x)\hat{\rho}_R(x'){}^*_*\, dx\, dx'$$

⇒ interaction is quartic in fermions, quadratic in bosons

$\hat{H}_{TL} = \hat{H}_0 + \hat{H}_{\text{int}}$ is quadratic in bosons
and can be diagonalized

Diagonalization of $\hat{H}_{\mathrm{TL}}$

Bosonic form of $\hat{H}_{\mathrm{TL}}$:

$$\hat{H}_{\mathrm{TL}} = \sum_{q>0,\eta} \left(v_F^0 q + \frac{g_4(q)\,q}{2\pi}\right) {}^*_*\hat{b}^+_{q\eta}\hat{b}_{q\eta}{}^*_*$$

$$+ \sum_{q>0} \frac{g_2(q)\,q}{2\pi}\, {}^*_*\hat{b}^+_{q1}\hat{b}^+_{q2} + \hat{b}_{q1}\hat{b}_{q2}\,{}^*_*$$

$$+ \tfrac{2\pi}{L}\left[\left(v_F^0 + \frac{g_4(0)}{2\pi}\right)\frac{\hat{N}_1^2 + \hat{N}_1^2}{2} + \frac{g_2(0)}{2\pi}\hat{N}_1\hat{N}_2\right]$$

Couplings:

$g_{2,4}(q)$ = Fourier transforms of $V_{2,4}(x-x')$

$\approx g_{2,4}(0)$ for $q < q_c$, else ≈ 0

Bogoljubov transformation: $\hat{B}_{q\eta} = \dfrac{1+K}{2\sqrt{K}}\,\hat{b}_{q\eta} + \dfrac{K-1}{2\sqrt{K}}\,\hat{b}^+_{q\bar{\eta}}$

$$\boxed{\hat{H}_{\mathrm{TL}} = \tfrac{\pi}{2L}\left[v_N\hat{\mathcal{N}}^2 + v_J\hat{\mathcal{J}}^2\right] + \sum_{q>0,\eta} v_F q\, {}^*_*\hat{B}^+_{q\eta}\hat{B}_{q\eta}{}^*_*}$$ free bosons!

with $\hat{\mathcal{N}} = \hat{N}_1 + \hat{N}_2$, $\quad \hat{\mathcal{J}} = \hat{N}_1 - \hat{N}_2$

$$v_N = v_F^0 + \frac{g_4 + g_2}{2\pi}, \quad v_J = v_F^0 + \frac{g_4 - g_2}{2\pi},$$

$$v_F = \sqrt{v_N v_J}, \quad K = \sqrt{v_J / v_N}$$

$\hat{H}_{\mathrm{TL}}$ is completely solvable!

Momentum distribution

Expectation values of fermion fields can be calculated with bosonization identity:

$$\langle \hat{\psi}_\nu^+(x)\hat{\psi}_\nu(0)\rangle = \frac{i}{2\pi}\frac{1}{x+i0^+}\left(\frac{(2/q_c)^2}{x^2+(2/q_c)^2}\right)^\gamma ,$$

$$\sim (1/x)^d \quad \text{for small } x\text{, or } K=1 \Rightarrow \text{as in free case}$$

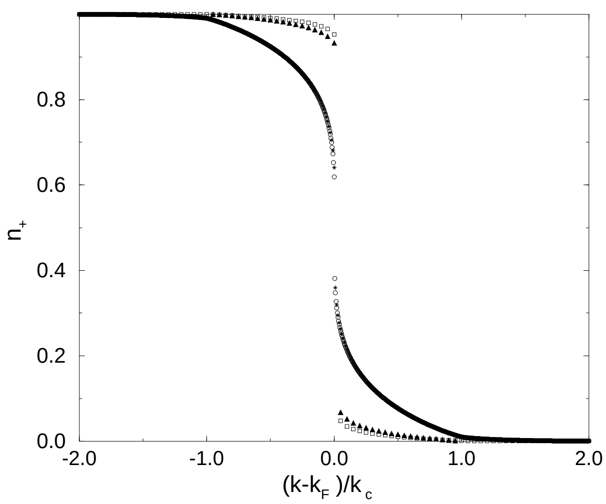
$$\sim (1/x)^{1+\gamma} \quad \text{for large } x \text{ and } K \neq 1$$

"anomalous dimension" $\gamma = \frac{1}{2}(K + \frac{1}{K} - 2)$

Momentum distribution:

$$n(k) = \langle \hat{n}_k^{\text{Phys}}\rangle = \int dx\, e^{-ik_F x}\, \langle \hat{\psi}_\nu^+(x)\hat{\psi}_\nu(0)\rangle$$

$$= n(k_F) - \text{const} \cdot \text{sgn}(k-k_F) \cdot |k-k_F|^\gamma$$



(Schönhammer 1997)

- continuous momentum distribution
- divergent derivative at k_F
- characteristic for Luttinger liquids

Field theory of the free system

Let us consider the hermitian fields

$$\bar{\phi}_\eta(x) = \hat{\varphi}_\eta(x) + \hat{\varphi}_\eta^+(x) + \frac{2\pi x}{L}\hat{N}_\eta$$

and the combinations

$$\hat{\Phi}_0(x) = \frac{1}{\sqrt{4\pi}}[\bar{\phi}_1(x) - \bar{\phi}_2(x)]$$
$$\hat{\Theta}_0(x) = \frac{1}{\sqrt{4\pi}}[\bar{\phi}_1(x) + \bar{\phi}_2(x)]$$

with commutators

$$[\hat{\Phi}_0(x), \hat{\Phi}_0(x')] = [\hat{\Theta}_0(x), \hat{\Theta}_0(x')] = 0$$
$$[\hat{\Phi}_0(x), \underbrace{\partial_{x'}\hat{\Theta}_0(x')}_{\hat{\Pi}_0(x)}] = i\delta(x - x') \qquad \Rightarrow \text{ canonically conjugate}$$

Then the non-interacting TL Hamiltonian is

$$\hat{H}_0 = \frac{v_N^0 \hat{\mathcal{N}}^2 + v_J^0 \hat{\mathcal{J}}^2}{2\pi^{-1}L} + \sum_{q>0,\eta} v_F^0 \, {}^*_* \hat{b}^+_{q\eta}\hat{b}_{q\eta} {}^*_*$$
$$= \frac{v_F^0}{2}\int dx \, {}^*_* \hat{\Pi}_0(x)^2 + (\partial_x\hat{\Phi}_0(x))^2 \, {}^*_*$$

⇒ free massless bosonic field in $1 + 1$ dimensions

Field theory of the interacting system

$\hat{H}_{\mathrm{TL}}$ has the same form as $\hat{H}_0$, only with renormalized parameters:

$$\hat{H}_{\mathrm{TL}} = \frac{v_N \hat{\mathcal{N}}^2 + v_J \hat{\mathcal{J}}^2}{2\pi^{-1} L} + \sum_{q>0,\eta} v_F \, {}^*_* \hat{B}^+_{q\eta} \hat{B}_{q\eta} {}^*_*$$

$\Rightarrow$ "refermionize" $\hat{H}_{\mathrm{TL}}$ by defining $\hat{\Pi}$ and $\hat{\Phi}$ in terms of $\hat{B}$ and $\hat{B}^+$:

$$\hat{H}_{\mathrm{TL}} = \frac{v_F}{2} \int dx \, {}^*_* \hat{\Pi}(x)^2 + (\partial_x \hat{\Phi}(x))^2 {}^*_*$$

They are related to $\hat{\Pi}_0$ and $\hat{\Phi}_0$ by

$$\hat{\Pi}(x) = \frac{1}{\sqrt{K}} \hat{\Pi}_0(x) \, , \qquad \hat{\Phi}(x) = \sqrt{K} \, \hat{\Phi}_0(x)$$

Useful for calculation of correlation functions:

"mass operator" $\quad \hat{M}(x) = \Psi_{\mathrm{R}}(x)^+ \Psi_{\mathrm{L}}(x) + \text{h.c.}$

$$\sim \frac{e^{i\sqrt{4\pi}\hat{\Phi}_0(x)} + e^{-i\sqrt{4\pi}\hat{\Phi}_0(x)}}{a} \quad \sim \; O(\hat{\Phi}_0^2)$$

Result:

$$\langle \hat{M}(x) \hat{M}(0) \rangle \sim \frac{a^{2(K-1)}}{x^{2K}}$$

$\Rightarrow$ $\hat{M}$ has scaling dimension K

4. Spin-charge separation

We now include the electron spin $\sigma = \uparrow, \downarrow$ in $\hat{H}_{\mathsf{TL}}$:

interaction: $V(x - x')\hat{\rho}(x)\hat{\rho}(x') \longrightarrow V_{\sigma\sigma'}(x - x')\hat{\rho}_\sigma(x)\hat{\rho}_{\sigma'}(x')$

Hamiltonian: $\hat{H}_{\mathsf{TL}} \longrightarrow \hat{H}_{\mathsf{TL}}^{\uparrow,\downarrow}$

$\Rightarrow \hat{H}_{\mathsf{TL}}^{\uparrow,\downarrow}$ separates into two independent parts:

- a "charge" part with symmetric combinations of ↑ and ↓
- a "spin" part with antisymmetric combinations of ↑ and ↓

spin-charge separation: $$\boxed{\hat{H}_{\mathsf{TL}}^{\uparrow,\downarrow} = \hat{H}_{\mathsf{TL},c} + \hat{H}_{\mathsf{TL},s}}$$

with interaction-dependent "charge" parameters v_c, K_c
and "spin" parameters v_s, K_s

Luttinger liquid phenomenology (Haldane 1981) for low T:

specific-heat coefficient γ: ($C = \gamma T$)
$$\frac{\gamma}{\gamma_0} = \frac{1}{2}\left(\frac{v_F^0}{v_c} + \frac{v_F^0}{v_s}\right)$$

compressibility κ:
$$\frac{\kappa}{\kappa_0} = \frac{v_F^0}{v_c} K_c$$

susceptibility χ:
$$\frac{\chi}{\chi_0} = \frac{v_F^0}{v_s} K_s$$

LL parameters can be obtained from thermodynamic properties

Summary

Bosonization of one-dimensional many-fermion systems:

- exact solution at Hamiltonian level
- calculation of correlation functions

Spin-charge separation:

- spin and charge excitations are independent
- occurs only in $d = 1$

Luttinger liquids:

- continuous momentum distribution with infinite slope
- non-universal $v_c, K_c, \ldots$ determine low-energy spectrum
- obtain parameters from low-T thermodynamics