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Zufallsmatrix–Modell für Quanten-Brownsche Bewegung

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Random–matrix model for quantum Brownian motion

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Zusammenfassung

Motiviert von dem Wunsch (i) die Modellabhängigkeit der Beschreibung der Quanten–Brownschen Bewegung zu untersuchen und (ii) den üblichen Caldeira–Leggett Zugang zu erweitern, führen wir ein Bandzufallsmatrix–Modell für die System–Bad Wechselwirkung ein. Mit Hilfe dieses Modells leiten wir Markovsche Mastergleichungen für die Zeitentwicklung eindimensionaler Quantensysteme, die schwach mit dem Wärmebad gekoppelt sind, her. Wir betrachten zwei Formen der Zufallsmatrix–Kopplung. Insbesondere diskutieren wir eine genäherte Form, symmetrisch sowohl in den System- als auch in den Badvariablen. Wir finden, daß diese Form mit der "rotating wave" Näherung zusammenhängt. Wir studieren ausführlich zwei einfache Systeme — den gedämpften harmonischen Oszillator und das Zweiniveau–System. Unsere Ergebnisse stimmen vollkommen mit den Resultaten des Caldeira–Leggett Modells und mit den Agarwal Gleichungen überein. Dies beweist die Modellunabhängigkeit dieser Markovschen Mastergleichungen. Die Mittelung über das Zufallsmatrix–Ensemble wird mit Hilfe einer diagrammatischen Störungsentwicklung und mit der Supersymmetrie–Methode ausgeführt. Die letztere Methode dient der Erweiterung unseres Zufallsmatrix–Modells auf Situationen, in denen die Umgebung stark durch das System beeinflußt wird.

Abstract

Motivated by the desire to (i) investigate the dependence of the description of quantum Brownian motion on model assumptions and (ii) go beyond the standard Caldeira–Leggett approach, we use a random band–matrix model for the system–bath interaction to derive Markovian master equations for the time evolution of one–dimensional quantum system weakly coupled to a heat bath. We consider two forms of the random–matrix coupling. In particular, we discuss an approximated form, symmetric in both the variables of the system and the bath, which is found to be related to the rotating wave approximation. We study in detail two simple systems, the damped harmonic oscillator and the dissipative two–level system. Our results are in complete agreement with those of earlier models, like the Caldeira–Leggett model and with the Agarwal equations. This proves the universality of these Markovian master equations. The average over the random–matrix ensemble is performed by employing a diagrammatic perturbation expansion as well as the supersymmetry technique. The latter method is suitable for further generalization of our random–matrix model to situations where the environment is strongly influenced by the system.

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Chapter 1

Introduction

The description of the interaction of an open system with its environment is an important problem both in classical and in quantum physics. If the environment is modeled as a heat bath, the interaction will lead to relaxation and dissipation processes in the system, and to an irreversible approach toward equilibrium [Cof85, Wei93]. A typical example of an open system, taken from classical mechanics, is a heavy particle (the system) in suspension in a viscous fluid (the surrounding medium). Due to the incessant bombardment by the molecules of the liquid, the particle moves in a complex, highly irregular way [Ein56]. Brownian motion, as this irregular movement is now called, is perhaps the simplest dissipative process and, since its first systematic observation in 1827, it has become the archetype of all open systems (an account of its historical development can be found in Ref. [Bru76]). Brownian motion has been intensely investigated, both in classical [Wax54, Cof85] and in quantum mechanics [Gra88, Wei93] and the classical theory is quite well understood [Wax54, Cof85]. One may start either from a Fokker-Planck equation which describes the relaxation to equilibrium of the phase–space density of the Brownian particle, or, equivalently, from a Langevin equation for its position coordinate [1]. On the other hand, the situation is not so clear at the quantum level. The reason for this is that for most complex systems, a detailed understanding of the microscopic origin of damping is not available and the corresponding Hamiltonian is simply unknown [Wei93]. One has then to resort to model Hamiltonians to describe the system–environment interaction. Starting from such a Hamiltonian, one may for instance derive a master equation for the system which reduces in the appropriate limit to the classical Fokker–Planck equation [2]. During the last decades, various models of this type have been introduced in different branches of physics and chemistry. Notable examples are the Redfield theory in nuclear magnetic resonance [Red57, Red65], the Oppenheim–Romero-Rochin model in condensed–phase chemical physics [Opp87], the phase–space approach of Agarwal in quantum optics [Aga73], and the influence functional method used by Caldeira and Leggett in condensed–matter physics [Cal83a]. The Markovian master equations obtained in these approaches have recently been compared in Ref. [Koh97].

The standard model for quantum dissipation (the Caldeira–Leggett model) consists of a quantum system coupled to an infinite set of harmonic oscillators. Caldeira and Leggett

[1] However there are still some open questions concerning the microscopic foundations of these equations (see e.g. Ref. [Maz78]).

[2] One may also describe the evolution of the system in terms of a quantum Langevin equation [For65, For87, For88].

have shown that it is always possible to treat the bath as an ensemble of independent oscillators provided the system–bath coupling is weak. These authors also assumed that the coupling is linear in both the position coordinate of the quantum system and the bath variables. For the quantum system, this choice follows from the requirement that, in the classical limit, the friction force in the Langevin equation should be linear in the velocity. For the bath, the choice was primarily made for computational convenience: To make the elimination of the bath coordinates tractable, the restriction to linearity is necessary. To the best of our knowledge, there is no compelling argument for choosing the interaction term linear in the bath coordinates.

It is expected, of course, that the relaxation process described by a master equation should be insensitive to the detailed form of the system–bath interaction. The first motivation of this work aims at proving this statement. We do so with the help of an alternative model for the interaction. We use an ensemble of random matrices. The ensemble encompasses all forms of system–bath interaction which are linear in the position coordinate of the quantum system, and which respect fundamental symmetries of the problem like time–reversal invariance. The ensemble is characterized by a few parameters which establish the relevant time scales. The Markovian master equations derived in this fashion are then valid for all possible forms of the interaction between quantum system and heat bath, except for a set of measure zero. We find that in the high–temperature limit, the Markovian master equations derived by Caldeira and Leggett and others are independent of both the specific structure of the bath and of the specific form of the system–bath interaction. This establishes the universality of the Markovian master equation for quantum Brownian motion.

A second motivation for our work relates to the use of random–matrix models in *closed* quantum systems with many degrees of freedom. In such systems, only few degrees of freedom usually command physical interest. We refer to such degrees of freedom (to the remainder) as to the collective (the remaining) degree(s) of freedom, respectively. A case in point is nuclear fission. Here, interest is focussed on the shape degree of freedom, and little attention is usually paid to the intrinsic degrees of freedom of the fissioning nucleus. In cases like this, the dynamical behavior of the remaining degrees of freedom depends, however, strongly on that of the collective degree of freedom and, therefore, cannot be modeled as a heat bath. The success of RMT in self–bound many–body quantum systems [Guh98] then suggests that we model the remaining degrees of freedom in terms of a suitable random–matrix model. We have in mind an ensemble of random matrices which depends parametrically on the collective degree(s) of freedom. Such an approach has been taken in the papers by Bulgac et al. [Bul96]. Before investigating the consequences of such an idea, it is necessary to study the limiting case where the environment can indeed be modeled as a heat bath, and to ask whether in this case, the heat bath can be replaced by a suitable random–matrix model. Our proof of universality answers this question in the affirmative.

The thesis is organized as follows. In the next chapter, we will consider two models for system plus bath: First, we will briefly review the Caldeira–Leggett model, where the system is linearly coupled to a set of harmonic oscillators, and then present our random–matrix model with emphasis on the statistical properties of the interaction, a random band–matrix. In both cases, an approximate form of the interaction Hamiltonian will be distinguished: The rotating wave approximation and the symmetrized random–matrix coupling, respectively. In chapter three, we shall evaluate the ensemble average of the evolution equations for the total system, namely, the Dyson equation for the time–evolution operator and the von Neumann equation for the density operator. The limit of weak coupling will be considered and averaging will be performed using a diagrammatic perturbation expansion. The fourth chapter is devoted to the derivation of the averaged Markovian master equation for the reduced density operator of the system. We will start from the foregoing evolution equations and consider the two forms of the random–matrix interaction separately. In chapter five, we apply our results to the damped harmonic oscillator and to the dissipative two–level system, respectively. We show that in the limit of large bandwidth we recover the Agarwal equations and in the high–temperature limit the Caldeira–Leggett master equation. Chapter six deals with a discussion of the symmetrized form of the random–matrix coupling. To this end, we will use the quantum Langevin equation approach to quantum relaxation. A comparison with the rotating wave approximation will be made and we will show that both approximations agree in the low temperature limit. In chapter seven, we employ the supersymmetry method to calculate the average over the random–matrix ensemble. This will provide us with a formalism which is more suitable for handling parameter dependent random matrices. This constitutes, therefore, a first step towards a generalization of our random–matrix model for quantum Brownian motion. In the last chapter we summarize our results.

Chapter 2

Description of the models

2.1 Introduction

This chapter is devoted to the presentation of two microscopic models for quantum Brownian motion, the Caldeira–Leggett model and our random–matrix model, respectively. We begin in Section 2.2 with a brief review of the standard oscillator bath model. We will consider both, the fully coupled Hamiltonian and the corresponding Hamiltonian in the rotating wave approximation. The Markovian master equations obtained within this model will then be applied to two simple systems, namely the damped harmonic oscillator and the dissipative two–level system. We will consider the classical limit of these evolution equations and discuss their correspondence with the classical Fokker–Planck and Langevin equations. In Section 2.3, we will present the random band–matrix model. As for the previous model, we shall distinguish two forms of the system–bath interaction: Form (I) which is not symmetric in the system–bath parameters and an approximated, symmetrized form (II). The latter will be shown to be related to the rotating wave approximation.

2.2 Oscillator bath models

We consider a simple quantum system interacting with a large environment, considered as a heat bath. We denote by S the system of interest and by B the heat bath. The system S could be, for instance, a particle of mass M moving in a potential $U(x)$ or a spin degree of freedom. We write the total Hamiltonian for system plus bath as

$$H = H_S + H_B + W \ , \tag{2.1}$$

where H_S and H_B are the unperturbed Hamiltonians of the system S and of the bath B, respectively, and W is the interaction Hamiltonian between the system and the bath.

When the coupling between S and B is weak, in the sense that any bath degree of freedom is only weakly perturbed by its interaction with the system, then it is appropriate to make the following simplifications: (i) one may just consider the linear response of the bath to the system and (ii) one is allowed to describe the former in the harmonic approximation (see Appendix C of Ref. [Cal83b]). In the limit of weak coupling, one may thus consider the following model in which the system is linearly coupled through its position operator to a collection of harmonic oscillators. This model is known as the "oscillator bath" model or the Caldeira–Leggett model [Cal83a]. The corresponding Hamiltonian may be written as

$$H = \frac{p^2}{2M} + U(x) + x\sum_i g_i x_i + \sum_i \left(\frac{p_i^2}{2m_i} + \frac{1}{2}m_i\omega_i^2 x_i^2\right) . \tag{2.2}$$

Here x and p (x_i and p_i) are the position and momentum operators of the system (of the bath oscillators, respectively). The g_i's are the coupling constants and the m_i's and ω_i's are the masses and frequencies of the bath oscillators. This model has been introduced by Ullersma [Ull66] to study the simplest model of a dissipative quantum system that one can envisage, the damped harmonic oscillator. In subsequent years, the model has been extended to other simple systems, including the damped free particle and the dissipative two–level system. The oscillator bath model has been analyzed by employing a number of different approaches, such as projection operator techniques [Aga71a, Zwa73], iterative methods [Car75], path integral formulations [Cal83a, Gra88, Hu92, Kar97], cumulant expansions [Mun96] and flow equations [Ker98]. It should be stressed that *weak coupling* between system and bath does not mean that the Hamiltonian (2.2) applies only for *weak damping* of the system. On the contrary, Eq. (2.2) is quite compatible with strong damping [Cal83b]. For further reference, we shall review in the sequel some results concerning the damped linear oscillator and the dissipative two–level system, obtained within the oscillator bath model. We will follow the treatment of Agarwal [Aga73, Aga74].

2.2.1 Fully coupled model

In this section, we shall describe the quantum relaxation of a harmonic oscillator and a two–level system interacting with an oscillator bath . To this end, we will utilize statistical methods which were originally developed in the context of stochastic processes [Kam81]. More precisely, we shall write down a master equation for the reduced density operator of the system and a Fokker–Planck equation for the corresponding phase–space distribution. Both equations give the time evolution of the system S.

Damped harmonic oscillator

We consider a linear harmonic oscillator with mass M and frequency ω. The corresponding Hamiltonian H_S reads

$$H_S = \frac{p^2}{2M} + \frac{1}{2}M\omega^2 x^2 . \tag{2.3}$$

Expressing the position and momentum operators in terms of the annihilation and creation operators a_i and $a_i^\dagger$ (we put $\hbar = 1$ throughout) [Mes62],

$$x_i = \sqrt{\frac{1}{2M\omega}}\,(a_i^\dagger + a_i) , \qquad p_i = i\sqrt{\frac{M\omega}{2}}\,({a_i}^\dagger - a_i) , \tag{2.4}$$

we may write the Hamiltonian (2.2) as

$$H_{\text{FC}} = \omega a^\dagger a + \sum_i \omega_i a_i^\dagger a_i + \sum_i \kappa_i \left(a + a^\dagger\right)\left(a_i + {a_i}^\dagger\right) , \tag{2.5}$$

with

$$\kappa_i = \frac{1}{2}\sqrt{\frac{1}{M\omega m_i \omega_i}}\, g_i \ . \tag{2.6}$$

The damped oscillator whose evolution is determined by the Hamiltonian (2.5) is called "fully coupled oscillator" (FC oscillator) [Lin91] or "non–rotating wave oscillator" (NRW oscillator) [Mun96]. The latter designation refers to an approximation of (2.5), the so called rotating wave approximation, which will be presented in the next section.

Let $\hat{\rho}(t)$ be the density operator for system plus bath. The von Neumann equation for $\hat{\rho}(t)$ is given by

$$\hat{\rho}(t) = U(t)\hat{\rho}(0)U^\dagger(t) \ , \tag{2.7}$$

where the time–evolution operator $U(t) = e^{-iHt}$ obeys Dyson's equation

$$U(t) = U_0(t) - i\int_0^t dt_1 U_0(t-t_1) W U(t_1) \ , \tag{2.8}$$

and where $U_0(t) = e^{-iH_0 t}$ denotes the free time–evolution operator. We define the reduced operator for the system S by

$$\hat{\rho}_S(t) = \mathrm{tr}_B\left[\hat{\rho}(t)\right] \ , \tag{2.9}$$

where the trace is taken over the bath states.

Before we proceed to discuss the derivation of the equations of motion of the system S, let us specify the three assumptions involved:

(i) The system S and the bath B are initially uncorrelated. Then

$$\hat{\rho}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_B(0), \tag{2.10}$$

is the product of the initial density operators $\hat{\rho}_S(0)$ and $\hat{\rho}_B(0)$ for system and bath, respectively.

(ii) The bath is at $t = 0$ in thermal equilibrium at temperature T,

$$\hat{\rho}_B(0) = \exp(-\beta H_B)/\mathrm{tr}_B\left[\exp(-\beta H_B)\right] \ , \tag{2.11}$$

with $\beta = (kT)^{-1}$.

(iii) The bath oscillators are closely spaced in frequency so that summation can be replaced by an integration by introducing the density of states $\rho(\omega_i)$, such that $\rho(\omega_i)d\omega_i$ gives the number of oscillators with frequencies lying between ω_i and $\omega_i + d\omega_i$,

$$\sum_i \ldots = \int d\omega_i\, \rho(\omega_i) \ldots \ . \tag{2.12}$$

Starting from the von Neumann equation (2.7), a master equation for the reduced density operator $\hat{\rho}_S(t)$ may be obtained by suitably eliminating the bath coordinates. This can be done by employing a method developed by Zwanzig [Zwa61] that makes use of a projection operator chosen to project $\hat{\rho}_S(t)$ out of the total density operator $\hat{\rho}(t)$. We introduce the time–independent projection operator $\mathcal{P}$ given by

$$\mathcal{P} \ldots = \hat{\rho}_B(0)\mathrm{tr}_B[\ldots] \ . \tag{2.13}$$

Upon multiplying both sides of Eq. (2.9) by $\mathcal{P}$, one readily finds

$$\mathcal{P}\hat{\rho}(t) = \hat{\rho}_B(0)\hat{\rho}_S(t) \ . \tag{2.14}$$

Combining Eqs. (2.7) and (2.14) and using standard projection techniques, together with a weak–coupling, short–memory approximation (Born–Markov approximation), the master equation for $\hat{\rho}_S(t)$ is found to be (ignoring the level shift) [Aga71a],

$$\begin{aligned} \frac{d\hat{\rho}_S}{dt} &= -i\omega[a^\dagger a, \hat{\rho}_S] \\ &- \gamma\Big(a^\dagger a\hat{\rho}_S - 2a\hat{\rho}_S a^\dagger + \hat{\rho}_S a^\dagger a + a^2\hat{\rho}_S - a\hat{\rho}_S a - a^\dagger\hat{\rho}_S a^\dagger + \hat{\rho}_S {a^\dagger}^2\Big) \\ &- \gamma n_{th}\Big(2[a^\dagger,[a,\hat{\rho}_S]] + [a^\dagger,[a^\dagger,\hat{\rho}_S]] + [a,[a,\hat{\rho}_S]]\Big) \ , \end{aligned} \tag{2.15}$$

where γ is the damping coefficient,

$$\gamma = \pi\rho(\omega)|\kappa(\omega)|^2 \ , \tag{2.16}$$

and n_{th} is the average occupation number at frequency ω,

$$n_{th} = \frac{1}{e^{\beta\omega} - 1} \ . \tag{2.17}$$

The master equation (2.15) determines the time development of the damped harmonic oscillator and describes its approach to equilibrium as a result of the coupling to the heat bath. Alternatively, one may describe the relaxation of the damped oscillator from the point of view of the corresponding Fokker-Planck equation, which gives the time evolution of the probability density function for the system. This can be accomplished by using a quasi–probability representation for the density operator, for example the Wigner representation. With the help of the Weyl transform [Hil84],

$$O(x,p) = \int dy\, \langle x - \frac{y}{2}|O|x + \frac{y}{2}\rangle\, e^{ipy}, \tag{2.18}$$

which associates to any system operator O a phase space function $O(x,p)$, one can easily convert the operator master equation (2.15) into a Fokker–Planck equation. This yields [Aga71a, Cal83a]

$$\frac{\partial\rho_{\text{FC}}(t)}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{p}{M}\rho_{\text{FC}}\right] + \frac{\partial}{\partial p}\Big[(M\omega^2 x + 2\gamma p)\,\rho_{\text{FC}}\Big] + 2M\omega\gamma\,(n_{th} + \frac{1}{2})\frac{\partial^2\rho_{\text{FC}}}{\partial p^2} \ , \tag{2.19}$$

where the Wigner function $\rho_{\text{FC}}(x,p,t)$ is the Weyl transform of $\hat{\rho}_S(t)/2\pi$. The Fokker–Planck equation (2.19) is stochastically equivalent to the Langevin equations [Aga71a]

$$\dot{x} = \frac{p}{M} \ , \qquad \dot{p} = -2\gamma p - M\omega^2 x + F_{\text{FC}}(t) \ , \tag{2.20}$$

where $F_{\text{FC}}(t)$ is a real Gaussian random process with the properties

$$\langle F_{\text{FC}}(t)\rangle = 0 \ , \qquad \langle F_{\text{FC}}(t)F_{\text{FC}}(t')\rangle = 2D_{\text{FC}}\,\delta(t - t') \ . \tag{2.21}$$

Here the diffusion coefficient D_{FC} is given by

$$D_{\text{FC}} = 2M\omega\gamma(n_{th} + \frac{1}{2}) \ . \tag{2.22}$$

It is worthwhile to note that the Eqs. (2.20) are of the same form as the equations describing the Brownian motion of a classical oscillator and that in the classical limit Eq. (2.22) reduces to the well–known Einstein relation [Ris89]

$$D_{\mathrm{FC}} = 2M\gamma kT \ . \tag{2.23}$$

This shows that the Hamiltonian (2.5) reproduces the results of classical Brownian motion in the high–temperature limit and that it can therefore be considered as describing a quantum generalization of Brownian motion.

Dissipative two–level system

Let us now consider a two–level system with upper (lower) level $|+\rangle$ ($|-\rangle$, respectively). We introduce the Pauli spin matrices σ_x, σ_y and σ_z [Mes62]. The Hamiltonian of the system takes then the form

$$H_S = \frac{1}{2}\omega_0\,\sigma_z \ , \tag{2.24}$$

where ω_0 is the energy separation between the two levels. We introduce the raising and lowering operators $\sigma_\pm = \frac{1}{2}(\sigma_x \pm \sigma_y)$ which satisfy

$$\begin{aligned} \sigma_+ &= |+\rangle\langle-| \ , \qquad & \sigma_+\sigma_- &= |+\rangle\langle+| = \frac{1}{2}(1+\sigma_z) \ , \\ \sigma_- &= |-\rangle\langle+| \ , \qquad & \sigma_-\sigma_+ &= |-\rangle\langle-| = \frac{1}{2}(1-\sigma_z) \ . \end{aligned} \tag{2.25}$$

The total Hamiltonian for the two–level system coupled to the bath of harmonic oscillators takes the form

$$H_{\mathrm{FC}} = \frac{1}{2}\omega_0\sigma_z + \sum_i \omega_i a_i^\dagger a_i + \sum_i \kappa_i \Big(\sigma_+ + \sigma_-\Big)\Big(a_i + {a_i}^\dagger\Big) \ , \tag{2.26}$$

where κ_i is given by Eq. (2.6). By implementing the same procedure as in the previous section and performing similar approximations, one may derive from (2.26) a Markovian master equation for the dissipative two–level system. One arrives at

$$\begin{aligned} \frac{d}{dt}\hat\rho_S &= -i\frac{1}{2}\omega_0\,[\sigma_z, \hat\rho_S] \\ &+ \gamma\Big(\sigma_+\hat\rho_S\sigma_+ + \sigma_-\hat\rho_S\sigma_- + 2\sigma_-\hat\rho_S\sigma_+ - \sigma_+\sigma_-\hat\rho_S - \hat\rho_S\sigma_+\sigma_-\Big) \ . \end{aligned} \tag{2.27}$$

The parameter γ is defined as in Eq. (2.16). The equation (2.27) has been used by Agarwal [Aga71b, Aga74] to describe spontaneous emission (i.e. $n_{th} = 0$) of a two–level atom.

2.2.2 Rotating wave approximation model

We now turn to the discussion of a common approximation of the fully coupled Hamiltonians (2.5) and (2.26), known as rotating wave approximation (RWA). This approximation has originally been applied to magnetic spin resonance [Abr62] and consists in omitting rapidly varying off–resonant terms. The rotating wave approximation is best understood by the following considerations [Lou73]. We will restrict ourselves to the two–level system Hamiltonian (2.26), but it should be obvious that our arguments equally apply, with trivial modifications of notation, to the oscillator Hamiltonian (2.5). When there is no system–bath coupling, $\kappa_i = 0$, the Heisenberg operators have a time dependence given by

$$\begin{aligned} \sigma_+(t) &= \sigma_+(0)e^{i\,\omega_0 t}\ , \qquad & a_i(t) = a_i(0)e^{-i\,\omega_i t}\ , \\ \sigma_-(t) &= \sigma_-(0)e^{-i\,\omega_0 t}\ , \qquad & {a_i}^\dagger(t) = {a_i}^\dagger(0)e^{i\,\omega_i t}\ , \end{aligned} \tag{2.28}$$

such that, near resonance $\omega_0 \simeq \omega_i$, the (off–resonant) interaction terms $\sigma_- a_i$ and $\sigma_+ {a_i}^\dagger$ are rapidly oscillating in time at frequencies $\pm(\omega_0 + \omega_i)$, whereas the (resonant) terms $\sigma_+ a_i$ and $\sigma_- {a_i}^\dagger$ are practically constant. For $\kappa_i \ll \omega_i \simeq \omega$, it is expected that this is still the case when the coupling is turned on. Then, since the high–frequency terms will almost average to zero, one may neglect the latter with respect to the near resonant terms. This approximation is equivalent to decomposing the linearly polarized field into two opposite circularly polarized waves and keeping only the one rotating in the same sense as the spin precession. This explains the name rotating wave approximation. For the case of a spin $\frac{1}{2}$ system in a magnetic field, it was shown by Bloch and Siegert [Blo40] that the effect of the counter–rotating terms is indeed negligible provided the field is not too strong. Important early studies of the oscillator bath model in the rotating wave approximations include the works of Senitzky [Sen60, Sen61], Lax [Lax66a, Lax66b], Louisell and Marburger [Lou67], and Agarwal [Aga69].

Damped harmonic oscillator

The RWA Hamiltonian for the damped harmonic oscillator may be obtained from the FC Hamiltonian (2.2) by suppressing the non–resonant terms aa_i and $a^\dagger {a_i}^\dagger$ in the interaction. This gives

$$H_{\text{RWA}} = \omega a^\dagger a + \sum_i \omega_i {a_i}^\dagger a_i + \sum_i \kappa_i \left(a{a_i}^\dagger + a^\dagger a_i\right)\ . \tag{2.29}$$

By using the same techniques and approximations as those presented in Section 2.2.1, one may derive from (2.2) a master and a Fokker–Planck equation for the RWA oscillator [Aga73]. This leads to the following master equation for the reduced density operator

$$\begin{aligned} \frac{d\hat\rho_S(t)}{dt} = &- i\omega\,[a^\dagger a, \hat\rho_S(t)] - \gamma\Big(a^\dagger a\hat\rho_S(t) - 2a\hat\rho_S a^\dagger + \hat\rho_S(t)a^\dagger a\Big) \\ &- 2\gamma n_{th}\Big(a^\dagger a\hat\rho_S(t) - a\hat\rho_S(t)a^\dagger - a^\dagger\hat\rho_S(t)a + \hat\rho_S(t)aa^\dagger\Big)\ , \end{aligned} \tag{2.30}$$

and the corresponding Fokker–Planck equation can be written in the form

$$\begin{aligned} \frac{\partial\rho_{\text{RWA}}(t)}{\partial t} = &+ \frac{\partial}{\partial x}\left[\left(-\frac{p}{M} + \gamma x\right)\rho_{\text{RWA}}\right] + \frac{\partial}{\partial p}\left[\left(M\omega^2 x + \gamma p\right)\rho_{\text{RWA}}\right] \\ &+ \gamma\left(n_{th} + \frac{1}{2}\right)\left[\frac{1}{M\omega}\frac{\partial^2\rho_{\text{RWA}}}{\partial x^2} + M\omega\frac{\partial^2\rho_{\text{RWA}}}{\partial p^2}\right]\ . \end{aligned} \tag{2.31}$$

In the above equations γ and n_{th} are defined by Eqs. (2.16) and (2.17), respectively, and $\rho_{\text{RWA}}(x,p,t)$ is the Weyl transform of $\hat\rho_S(t)/2\pi$. As in the preceding section, the Fokker–Planck equation (2.31) is equivalent to a Langevin equation which can be written as

$$\dot x = \frac{p}{M} - \gamma x + F^x_{\text{RWA}}(t) \qquad \text{and} \qquad \dot p = -M\omega^2 x - \gamma p + F^p_{\text{RWA}}(t)\ , \tag{2.32}$$

where $F^x_{\mathrm{RWA}}(t)$ and $F^p_{\mathrm{RWA}}(t)$ are two independent real Gaussian random functions with zero mean and

$$\langle F^x_{\mathrm{RWA}}(t)F^x_{\mathrm{RWA}}(t')\rangle = \frac{2\gamma(n_{th}+\frac{1}{2})}{M\omega}\,\delta(t-t'),$$
$$\langle F^p_{\mathrm{RWA}}(t)F^p_{\mathrm{RWA}}(t')\rangle = 2\gamma(n_{th}+\frac{1}{2})M\omega\,\delta(t-t')\ . \tag{2.33}$$

Eqs. (2.32) clearly show that the RWA oscillator is not a mechanical oscillator because $\dot{x} \neq p/M$. We will discuss this point in detail in Chapter 6.

Dissipative two–level system

We conclude this section by considering the RWA form of the dissipative two–level system (2.26). In analogy to the Hamiltonian (2.29) for the damped oscillator, the RWA Hamiltonian for the two–level system is given by

$$H_{\mathrm{FC}} = \frac{1}{2}\omega_0\sigma_z + \sum_i \omega_i a_i^\dagger a_i + \sum_i \kappa_i\Big(\sigma_+ a_i + \sigma_- a_i{}^\dagger\Big)\ , \tag{2.34}$$

and the master equation satisfied by $\hat{\rho}_S(t)$ can be found to be of the form [Aga74]

$$\begin{aligned}\frac{d\hat{\rho}_S(t)}{dt} = &-i\frac{1}{2}\omega_0\,[\sigma_z,\hat{\rho}_S] - \gamma\Big(\sigma_+\sigma_-\hat{\rho}_S(t) - 2\sigma_-\hat{\rho}_S\sigma_+ + \hat{\rho}_S(t)\sigma_+\sigma_-\Big)\\ &-2\gamma n_{th}\Big(\sigma_+\sigma_-\hat{\rho}_S(t) - \sigma_-\hat{\rho}_S(t)\sigma_+ - \sigma_+\hat{\rho}_S(t)\sigma_- + \hat{\rho}_S(t)\sigma_-\sigma_+\Big)\ .\end{aligned} \tag{2.35}$$

The damping constant γ is again defined by Eq. (2.16) and n_{th} is given by Eq. (2.17) with ω replaced by ω_0.

2.3 Random band-matrix model

The present section is concerned with the presentation and discussion of the random–matrix model for quantum Brownian motion. More specifically, we shall study the properties of a small quantum system S coupled to a large heat bath via a random band–matrix interaction.

Random–matrix theory (RMT) was originally introduced by Wigner to describe spectral fluctuations of quantum many–body systems such as nuclei and has since been applied successfully in a wide range of other fields such as quantum chaos and disordered mesoscopic systems [Guh98]. To the best of our knowledge, a random–matrix approach to relaxation has been first developed in nuclear physics in the context of deeply inelastic heavy–ion collisions [Ko76, Aga77, Aga79]. Since then, related models have been used to describe relaxation of a non–degenerate two–level system [Mel88, Pey91], dissipation in complex quantum systems [Wil90, Bul96] and, more recently, the dynamics of a simple quantum system in a complex environment [Bul98].

2.3.1 Non–symmetrized coupling

We write the Hamiltonian of the composite system in the form

$$H = H_S\otimes \mathbb{1}_B + \mathbb{1}_S\otimes H_B + Q\otimes V = H_0 + W\ , \tag{2.36}$$

where, as in Section 2.2, H_S and H_B describe the system S and the bath B, respectively, and $W = Q \otimes V$ is the system–bath interaction. In the present approach, in contrast to the Caldeira–Leggett model (2.2), the actual form of H_B is not specified. In particular, there are no bath oscillators. We denote by $|m\rangle$, $m = 0, \ldots, N_S$ ($|a\rangle$, $a = 0, \ldots, N_B$) the eigenstates of the system (bath) Hamiltonian with eigenvalues E_m (ε_a, respectively),

$$H_S|m\rangle = E_m|m\rangle \ , \qquad\qquad H_B|a\rangle = \varepsilon_a|a\rangle \ . \tag{2.37}$$

Here N_S and N_B are the dimensions of the Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_B$ of the system and the bath, respectively. The product states $|ma\rangle$ form a complete set for the composite system. We make assumptions which are similar to those used in the oscillator bath model. Namely,

(i) we assume that the interaction is turned on at $t = 0$ and that at that initial time S and B are not correlated, $\hat{\rho}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_B(0)$.

(ii) we assume further that at all times $t \geq 0$ the bath is in thermal equilibrium at temperature T,

$$\hat{\rho}_B = \frac{1}{Z}\sum_a e^{-\beta\varepsilon_a}|a\rangle\langle a| \ , \tag{2.38}$$

where Z is the canonical partition function.

(iii) finally, we take the heat bath sufficiently large so that the energy eigenstates ε_a are closely spaced. The energy spectrum may then be characterized by the density of states $\rho(\varepsilon)$.

We show below that expression (2.38) is equivalent to

$$\hat{\rho}_B = |a^*\rangle\langle a^*| \qquad \underline{\text{and}} \qquad \rho(\varepsilon^*) = \rho_0 e^{\beta\varepsilon^*} \ , \tag{2.39}$$

where the state $|a^*\rangle$ is defined by the temperature T.

In the Hamiltonian (2.36), Q is an operator which acts on the system S, and V is a Gaussian random band–matrix acting on the bath. The first two moments of V are given by

$$\overline{V_{ab}} = 0 \qquad \overline{V_{ab}V_{cd}} = (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\overline{V_{ab}}^2 \ , \tag{2.40}$$

where the matrix V_{ab} respects time–reversal symmetry and has non–zero elements only in a band of width Δ along the diagonal. More specifically, we adopt a form first given in Ref. [Ko76]. This paper also contains a detailed justification of the form of Eq. (2.40). This form has been widely used later, cf. Refs. [Aga77, Aga79, Bul96, Bul98].

$$\overline{V_{ab}}^2 = A_0 \left[\rho(\varepsilon_a)\rho(\varepsilon_b)\right]^{-\frac{1}{2}} e^{-\frac{(\varepsilon_a - \varepsilon_b)^2}{2\Delta^2}} \ . \tag{2.41}$$

Here A_0 is the strength of the coupling, $\rho(\varepsilon)$ the density of states of the bath, and Δ the bandwidth. For W, this implies

$$\text{(I)} \quad \overline{W_{ab}^{mn}} = 0 \quad \overline{W_{ab}^{mn}W_{cd}^{pq}} = (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})Q_{mn}Q_{pq}\overline{V_{ab}}^2 \ . \tag{2.42}$$

The form of the Hamiltonian (2.36) is a generalization of the Hamiltonian considered in Refs. [Mel88, Pey91]. There it was motivated by the observation that in relaxation problems, the process is frequently found to be insensitive to the details of the interaction. One may therefore construct an ensemble of interactions and calculate the average of the observable over this ensemble. We show in the next chapters that the relaxation of a Brownian system is indeed independent of the specific form of the coupling to the bath.

2.3.2 Symmetrized coupling

In Eq. (2.42), only the part of the interaction acting on the bath is modeled as a random matrix. This is physically sensible since only the bath is supposed to be a complex system. We observe that, as a consequence, the variance (I) of W has the inconvenient feature of being not symmetric in the variables of both the system and the bath. Therefore, we also consider a symmetrized form (II) of W where the entire interaction behaves as a random matrix. This form may be thought of as an approximation to the full form (I). We show later that form (II) leads to the rotating wave approximation. It is given by

$$\text{(II)} \quad \overline{W_{ab}^{mn}} = 0 \quad \overline{W_{ab}^{mn} W_{cd}^{pq}} = (\delta_{ac}\delta_{bd}\delta_{mp}\delta_{nq} + \delta_{ad}\delta_{bc}\delta_{mq}\delta_{np})|Q_{mn}|^2 \overline{V_{ab}}^2 \ . \tag{2.43}$$

In the sequel, we shall refer to form (II) as the symmetrized random–matrix coupling (SRM).

2.3.3 Parametrization of the bath

The equivalence of Eqs. (2.38) and (2.39) is closely related to the equivalence of the microcanonical and the canonical ensemble in the thermodynamic limit $N_{th} \to \infty$ [Hua80]. We thus consider a thermodynamical system in contact with a heat bath. The (canonical) partition function is given by

$$Z(\beta) = \int_0^\infty d\varepsilon\, \rho(\varepsilon) e^{-\beta\varepsilon} = \int_0^\infty d\varepsilon\, e^{-\beta\varepsilon + \ln\rho(\varepsilon)} \ . \tag{2.44}$$

Since ε and $\ln\rho(\varepsilon)$ grow with N_{th}, the integral can be evaluated by a saddle–point approximation. Expanding the integrand up to second order around its maximum ε^*, we obtain

$$Z(\beta) = \rho(\varepsilon^*) e^{-\beta\varepsilon^*} \int_0^\infty d\varepsilon\, \exp\left(\frac{1}{2}(\varepsilon - \varepsilon^*)^2 \left(\frac{\partial^2 \ln\rho(\varepsilon)}{\partial\varepsilon^2}\right)_{\varepsilon=\varepsilon^*}\right) \ . \tag{2.45}$$

The distribution in energy is a Gaussian centered at ε^* with a width

$$\Delta\varepsilon = \left(-\frac{\partial^2 \ln\rho(\varepsilon)}{\partial\varepsilon^2}\right)_{\varepsilon=\varepsilon^*}^{-\frac{1}{2}} = \sqrt{kT^2 C_V} \ . \tag{2.46}$$

We have used the fact that $k\, \ln\rho(\varepsilon)$ is the microcanonical entropy. For $N_{th} \to \infty$, $\Delta\varepsilon/\varepsilon^* \sim 1/\sqrt{N_{th}}$ becomes negligibly small and the Gaussian approaches a δ-function. Hence,

$$\hat{\rho}_B = \frac{1}{Z} \sum_a e^{-\beta\varepsilon_a} |a\rangle\langle a| \simeq |a^*\rangle\langle a^*| \tag{2.47}$$

where $|a^*\rangle$ is the eigenvector corresponding to the eigenvalue ε^*. Moreover, according to Eq. (2.45) the density of states can be approximated locally by

$$\rho(\varepsilon^*) = \rho_0\, e^{\beta\varepsilon^*} \ , \tag{2.48}$$

where ρ_0 is given by

$$\rho_0 = Z(\beta)/\sqrt{\pi k T^2 C_V} \ . \tag{2.49}$$

Chapter 3

Averaging: Diagrammatic method

3.1 Introduction

There are two basic equations which govern the time development of a quantum system coupled to a (large) quantum environment [Mes62]: Dyson's equation for the evolution operator $U(t)$ and von Neumann's equation for the density operator $\hat{\rho}(t)$. They both depend on the total Hamiltonian H. In the random–matrix approach to quantum Brownian motion presented in Chapter 2, a part of the interaction between the system and the heat bath was modeled by a Gaussian random matrix. Because of the presence of a stochastic quantity in the Hamiltonian, the time–evolution operator and, consequently, the density operator are themselves random variables. We therefore have to determine their average over the random–matrix ensemble. Three different tools are available to calculate the mean values of these two operators [Guh98]. The first method consists in a diagrammatic perturbation expansion. The two others methods, the replica trick and the supersymmetric approach, are based on a non–perturbative field-theoretical formulation. Supersymmetry is the most powerful of the three averaging techniques and is thus most appropriate to treat further generalizations of the random–matrix model. It will be introduced in Chapter 7.

The aim of the present chapter is to use the diagrammatic method to calculate the ensemble average of the operators $U(t)$ and $\hat{\rho}(t)$, in the limit of weak coupling. This averaging procedure consists in [Aga75a, Aga75b] (i) expanding the operators in powers of the random interaction W (Born series) then (ii) averaging each term using the Wick theorem for Gaussian random variables and finally (iii) summing up the whole series. This method is most conveniently implemented in Fourier space. We shall always work in the limit in which the dimension N_B of the bath matrices tends to infinity and consistently omit terms of order N_B^{-1} and smaller.

The results obtained in this chapter will be used in Chapter 4 as starting point for the derivation of the master equation for the system S. In Section 3.2 we shall calculate the ensemble average $\overline{U}(t)$ of the time–evolution operator. We will begin with the symmetrized form of the variance defined in Section 2.3. In this case, a solution of the averaged Dyson equation can be found in closed form. For the non-symmetrized coupling, it will turn out that the diagrammatic method can not easily be implemented as such. Nevertheless, a variant of the method, valid only for weak–coupling, will be developed. This will permit us to determine the time derivative of the matrix elements of $\overline{U}(t)$. In Section 3.3, we shall then evaluate the averaged density operator $\overline{\rho}(t)$.

3.2 The average time–evolution operator

To begin with, we introduce the propagator $K(t)$ which is defined by

$$K(t) = U(t)\theta(t) = e^{-iHt}\theta(t) \tag{3.1}$$

and which obeys the Dyson equation

$$K(t) = K_0(t) - i\int_{-\infty}^{\infty} dt_1 K_0(t-t_1) W K(t_1) \ , \tag{3.2}$$

where $K_0(t) = e^{-iH_0t}\theta(t)$ is the free propagator and $\theta(t)$ the unit step function. Unlike the Dyson equation (2.8) satisfied by the time–evolution operator $U(t)$, the above equation is given in terms of a convolution integral and its Fourier transform is thus easily calculated. This is our motivation for considering the propagator instead of the evolution operator.

To determine the average $\overline{K}(t) = \overline{U}(t)\theta(t)$, we use the energy representation and introduce the following pair of Fourier transforms

$$G(E) = \frac{1}{i}\int_{-\infty}^{\infty} dt\, e^{iEt} K(t) \ , \qquad K(t) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, e^{-iEt} G(E) \ . \tag{3.3}$$

The transforms of (3.1) and (3.2) are then given by

$$G(E) = \frac{1}{E + i\eta - H} \ , \qquad \eta \to 0^+ \ , \tag{3.4}$$

and

$$G(E) = G_0(E) + G_0(E) W G(E) \ , \tag{3.5}$$

respectively. In Eq. (3.4) an infinitesimal positive increment η has been added to ensure the convergence of the Fourier integral.

Born series

The first step in evaluating the ensemble average $\overline{G}$ of the propagator is to expand G in a power series with respect to W. By successive iteration of Eq. (3.5), we obtain

$$G = G_0 + G_0WG_0 + G_0WG_0WG_0 + \ldots = \sum_{s=0}^{\infty} G_0(WG_0)^s \ . \tag{3.6}$$

Wick theorem

To perform the average of a typical term in this series, $\overline{G_0(WG_0)^s}$, we use Eqs. (2.42), (2.43) and Wick's theorem. According to this theorem (see e.g. [Kam81, Ris89]), *a product of s Gaussian variables with zero mean values has zero average if s is odd, and for even s the average is given by summing over all possible ways of taking the average of pairs of W's.* The W's belonging to such an averaged pair are then said to be "contracted". We exemplify this rule by the first non-trivial term appearing in the expansion (3.6): the term of fourth–order. In this case we have three "Wick contractions", i.e., three different ways of forming pairs of W's,

$$\overline{(WG_0)^4} = WG_0\,WG_0\,WG_0\,WG_0 + WG_0\,WG_0\,WG_0\,WG_0 + WG_0\,WG_0\,WG_0\,WG_0 \ . \tag{3.7}$$

In the foregoing expression, the average of a pair is indicated by a "contraction line" connecting the W's. The Wick theorem is exact and entails no approximation. An approximation is introduced, and the number of terms contributing to fixed s is considerably reduced, by omitting terms which are relatively small of order N_B^{-1}. We recall that we work in the limit in which N_B is infinitely large. Due to the fact that W's with different indices are uncorrelated, see Eqs. (2.42) and (2.43), the last term on the r.h.s. of (3.7) is of order N_B^{-1} compared with the other two and can thus be neglected. This can easily be seen by direct evaluation of the matrix elements of the right side of Eq. (3.7): We find that the first two terms are written as a sum over $N_B \times N_B$ elements, whereas the last one contains only N_B matrix elements. As a general rule, the terms of leading order in N_B^{-1} are those in which contraction lines do not intersect [Ver84]. Application of this rule to the expansion (3.6) leads to

$$\begin{aligned}\overline{G} &= G_0 + G_0 W G_0 W G_0 + G_0 W G_0 W G_0 W G_0 W G_0 \\ &\quad + G_0 W G_0 W G_0 W G_0 W G_0 + \ldots \cdot \end{aligned} \tag{3.8}$$

Eq. (3.8) can also be written as an integral equation for $\overline{G}$,

$$\overline{G} = G_0 + G_0 W \overline{G} W \overline{G} \, . \tag{3.9}$$

This equation is valid in the general case, including strong coupling.

Weak coupling

In the limit of weak coupling, a further reduction of the number of terms in the series (3.8) occurs. We show below that in this limit the second term on the r.h.s. of Eq. (3.7) also gives a negligible contribution. More generally, when the coupling between the system and the bath is weak, only contractions between adjacent pairs of W's have to be taken into account. The average of Eq. (3.6) then takes the following form

$$\begin{aligned}\overline{G} &= G_0 + G_0 W G_0 W G_0 + G_0 W G_0 W G_0 W G_0 W G_0 + \ldots \\ &= G_0 + G_0 W G_0 W \overline{G} \cdot \end{aligned} \tag{3.10}$$

By comparing the integral equations (3.9) and (3.10), we observe that the weak–coupling limit amounts to formally replacing $\overline{G}$ by G_0 between the contracted pair of W's.

Resummation

After having performed the average of each term in the series expansion of G, we now come to the remaining step: the resummation of the series. The weak–coupling expansion (3.10) is geometric and can readily be summed up. We obtain

$$\begin{aligned}\overline{G} &= G_0 \left(1 + W G_0 W G_0 + (W G_0 W G_0)^2 + (W G_0 W G_0)^3 + \ldots\right) \\ &= G_0 \frac{1}{1 - \overline{W G_0 W G_0}} \, . \end{aligned} \tag{3.11}$$

We note that this is a closed expression for the averaged propagator $\overline{G}$ [1]. It depends solely on the free propagator $G_0(E) = (E + i\eta - H_0)^{-1}$ and on the second moment of the random interaction.

[1]For strong coupling, this is replaced by the Pastur equation [Pas72], $\overline{G} = G_0 \frac{1}{1 - \overline{W G_0 W \overline{G}}}$, instead.

3.2.1 Case (II): Symmetrized coupling

Let us now move to the explicit evaluation of the ensemble–averaged propagator $\overline{G}$. We start with the approximated form (II) of the coupling. A direct consequence of the symmetrized variance of W is that the SRM propagator is diagonal in the energy basis. To demonstrate this, we calculate the matrix elements of $\overline{G}$. From Eq. (2.43), we easily find that

$$\overline{W_{bb_1}^{nn_1} W_{b_1 b'}^{n_1 n'}} = \delta_{nn'} \delta_{bb'} \overline{W_{bb_1}^{nn_1 2}} \ . \tag{3.12}$$

Moreover, since the free propagator G_0 is itself diagonal,

$$\langle nb|G_0(E)|n'b'\rangle = \delta_{nn'}\delta_{bb'} \frac{1}{E + i\eta - (E_n + \varepsilon_b)} = \delta_{nn'}\delta_{bb'}(G_0)_{nb} \ , \tag{3.13}$$

we get

$$\begin{aligned} \langle nb|\overline{WG_0W}|n'b'\rangle &= \sum_{n_1 b_1} \overline{W_{bb_1}^{nn_1}(G_0)_{n_1 b_1} W_{b_1 b'}^{n_1 n'}} (G_0)_{n'b'} \\ &= \delta_{nn'}\delta_{bb'}(G_0)_{nb} \sum_{n_1 b_1} \overline{W_{bb_1}^{nn_1 2}} (G_0)_{n_1 b_1} \ . \end{aligned} \tag{3.14}$$

The matrix elements of $\overline{G}$ in Eq. (3.11) may hence be written in the form

$$\begin{aligned} \langle nb|\overline{G}(E)|n'b'\rangle &= \frac{\delta_{nn'}\delta_{bb'}}{E + i\eta - (E_n + \varepsilon_b) - \sum_{n_1 b_1} \overline{W_{bb_1}^{nn_1 2}}(G_0)_{n_1 b_1}} \\ &= \frac{\delta_{nn'}\delta_{bb'}}{E + i\eta - (E_n + \varepsilon_b) - R_{nb}(E)} \ , \end{aligned} \tag{3.15}$$

which shows that $\overline{G}$ is diagonal in the unperturbed energy basis $|nb\rangle$. In the last line of Eq. (3.15), we have introduced the self–energy $R(E)$ which is defined by

$$R_{nb}(E) = \sum_{n_1 b_1} \overline{W_{bb_1}^{nn_1 2}} \frac{1}{E + i\eta - (E_{n_1} + \varepsilon_{b_1})} \ . \tag{3.16}$$

We see that the whole dependence on the interaction W of the averaged propagator is contained in $R(E)$. The self–energy therefore represents the effects of the coupling between the system and its environment [Gol64]. On using the Plemelj identity,

$$\lim_{\eta \to 0} \frac{1}{E \pm i\eta - E_0} = P\frac{1}{E - E_0} \mp i\pi\delta(E - E_0) \ , \tag{3.17}$$

where P denotes the Cauchy principal value, we can rewrite Eq. (3.16) as

$$R_{nb}(E) = \Delta_{nb}(E) - i\frac{\Gamma_{nb}(E)}{2} \ , \tag{3.18}$$

where

$$\begin{aligned} \Delta_{nb}(E) &= P\sum_{n_1 b_1} \frac{\overline{W_{bb_1}^{nn_1 2}}}{E - (E_{n_1} + \varepsilon_{b_1})} \ , \\ \Gamma_{nb}(E) &= 2\pi \sum_{n_1 b_1} \overline{W_{bb_1}^{nn_1 2}} \delta(E - (E_{n_1} + \varepsilon_{b_1})) \ . \end{aligned} \tag{3.19}$$

For sufficiently high temperature, $\Gamma_{nb}(E)$ and $\Delta_{nb}(E)$ depend weakly on energy (see Chapter 5) and may be approximated by their values at $E = E_n + \varepsilon_b$. We then have

$$\begin{aligned} \Delta_{nb} &= \Delta_{nb}(E_n+\varepsilon_b) = P\sum_{n_1 b_1} \frac{\overline{W^{nn_1\,2}_{bb_1}}}{E_n+\varepsilon_b-(E_{n_1}+\varepsilon_{b_1})} \; , \\ \Gamma_{nb} &= \Gamma_{nb}(E_n+\varepsilon_b) = 2\pi\sum_{n_1 b_1} \overline{W^{nn_1\,2}_{bb_1}}\,\delta(E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1}) \; . \end{aligned} \tag{3.20}$$

For the averaged propagator (3.15), this yields

$$\overline{G}_{nb}(E) = \frac{1}{E+i\eta-(E_n+\varepsilon_b+\Delta_{nb})+i\frac{\Gamma_{nb}}{2}} \; . \tag{3.21}$$

Transforming back to time representation with the help of (3.3), we obtain

$$\overline{K}_{nb}(t) = e^{-i(E_n+\varepsilon_b+\Delta_{nb})t-\frac{\Gamma_{nb}}{2}t}\theta(t) \; , \tag{3.22}$$

from which we finally derive

$$\overline{U}_{nb}(t) = e^{-i(E_n+\varepsilon_b+\Delta_{nb})t-\frac{\Gamma_{nb}}{2}t} \; . \tag{3.23}$$

This is the averaged SRM evolution operator evaluated in the limit of weak coupling between the system and the bath. The physical interpretation of the real and imaginary parts of the matrix elements of the shift operator R, Δ_{nb} and Γ_{nb}, follows immediately from Eq. (3.23): Δ_{nb} is the level shift of the state $|nb\rangle$ induced by the coupling between the system and the bath and Γ_{nb} is the corresponding decay width. Both Δ_{nb} and Γ_{nb} give a measure of the intensity of the coupling. Accordingly, these quantities are expected to be "small" in the weak–coupling limit. We shall make this statement more quantitative in the next section.

Weak–coupling condition

In the limit of weak coupling, contractions between non–adjacent pairs of W's are negligible. This can be shown by inspecting the contributions from the various products of Wick contractions. We consider here the simplest case and compare [Aga75a, Aga75b]

$$\langle nb|\overbrace{W G_0 W}|nb\rangle \qquad \text{and} \qquad \langle nb|\overbrace{W G_0 \overbrace{W G_0 W} G_0 W}|nb\rangle \; . \tag{3.24}$$

For the symmetrized random–matrix interaction, we are thus led to compare

$$\sum_{n_1 b_1} \overline{W^{nn_1\,2}_{bb_1}}(G_0)_{n_1 b_1} \qquad \text{with} \qquad \sum_{n_1 b_1} \overline{W^{nn_1\,2}_{bb_1}}(G_0)^2_{n_1 b_1} \sum_{n_2 b_2} \overline{W^{n_1 n_2\,2}_{b_1 b_2}}(G_0)_{n_2 b_2} \; , \tag{3.25}$$

where Eq. (3.12) has been used. This gives the following weak-coupling condition

$$\sum_{n_1 b_1} \overline{W^{nn_1\,2}_{bb_1}}(G_0)_{n_1 b_1} \ll E - H_{0nb} \; . \tag{3.26}$$

Inequality (3.26) has a simple interpretation. It means that the self–energy $R_{nb}(E)$ has to be small in comparison with $E - H_{0nb}$ (cf. Eqs. (3.13) and (3.15)).

Actually, it is enough to compare the imaginary parts of the two terms appearing in (3.24), because the real parts represent level shifts and hence do not contribute to the decay width. We may then, equivalently, compare

$$\Gamma_{nb}(E) \qquad \text{with} \qquad \sum_{n_1 b_1} \overline{W^{nn_1\,2}_{bb_1}} (G_0)^2_{n_1 b_1} \Gamma_{n_1 b_1}(E) \ . \tag{3.27}$$

This will allow us to express the weak–coupling condition in terms of the parameters of the system S (the mean level spacing ω and the relaxation coefficient γ) and the parameters entering the random–matrix model (the bandwidth Δ and the temperature T of the heat bath). This will be carried out explicitly in Section 5.2 for the case that the system S is a harmonic oscillator. Here, we shall state the general rule without proof. Assuming that $|Q_{nn_1}|^2$ vanishes unless the states n_1 and n are close in energy, then, in the high–temperature limit $\omega \ll \Delta \ll kT$, $\Gamma_{nb}(E)$ may be replaced by $\Gamma_{nb}(E_n + \varepsilon_b)$, and in the weak–coupling limit $\gamma \ll \Delta$, the second term in (3.24) may be omitted in comparison with the first. For a detailed discussion of these parameters and the interpretation of the above inequalities, we refer to Section 4.4.

Although we have restricted the discussion of the weak–coupling condition to the SRM case, it should be clear that the conclusions we arrived at in the present section are valid in general, since averaging depends only on the (bath) random operator V but not on the properties of the deterministic (system) operator Q.

3.2.2 Case (I): Non-symmetrized coupling.

In this section, we consider the full, non–symmetrized, form (I) of the random–matrix coupling. In contrast to the SRM case, the averaged propagator is not diagonal in the energy representation and we have instead

$$\langle nb|\overline{G}(E)|n'b'\rangle = \delta_{bb'} \langle nb|\overline{G}(E)|n'b\rangle \ . \tag{3.28}$$

As a consequence, the series (3.10) is not geometric anymore and cannot be summed easily. We thus have to find another method to evaluate the average time–evolution operator $\overline{U}(t)$. We shall actually calculate only the time derivative of the matrix elements of $\overline{U}(t)$. This will suffice for the derivation of the master equation for the system S, which will be carried through in the next chapter. We start with an observation concerning the averaging procedure. Iterating the Dyson equation (3.5) once,

$$G = G_0 + G_0 W + G_0 W G_0 W G \ , \tag{3.29}$$

and taking the ensemble average,

$$\overline{G} = G_0 + G_0 \overline{W} + G_0 \overline{W G_0 W G} = G_0 + G_0 \overline{W G_0 W}\ \overline{G} \ , \tag{3.30}$$

yields a result which is identical to Eq. (3.10). We recall that Eq. (3.10) has been obtained after resummation of an infinite diagrammatic expansion. Here we simply iterated Dyson's equation once. That both expressions coincide is of course only true for weak coupling, when only contractions between adjacent pairs of W's have to be taken into account and results from Wick's theorem (see Section 3.2). However, since we are interested in the weak–coupling limit, the latter method provides an easy means to perform the ensemble average, avoiding the resummation of the series (3.10). In the following, we shall make use of this simple method to compute the average of the evolution operator. Contrary to what we did in the preceding section, we shall work in the time domain. More precisely, we will consider the Dyson equation (2.8), written in the interaction picture. We have

$$i\frac{d\tilde{U}(t)}{dt} = \tilde{W}(t)\tilde{U}(t) \tag{3.31}$$

where

$$\tilde{U}(t) = e^{iH_0 t}U(t) \qquad \text{and} \qquad \tilde{W}(t) = e^{iH_0 t}We^{-iH_0 t} \ . \tag{3.32}$$

Using the initial condition, $\tilde{U}(0) = U(0) = 1$, Eq. (3.31) can also be written in integral form,

$$\tilde{U}(t) = 1 - i\int_0^t dt_1 \tilde{W}(t_1)\tilde{U}(t_1) \ . \tag{3.33}$$

Inserting this equation into the r.h.s. of (3.31) and calculating the average of the matrix elements, we obtain

$$\frac{d\overline{\langle nb|\tilde{U}(t)|n'b\rangle}}{dt} = -\int_0^t dt_1 \sum_{\substack{n_1 b_1 \\ n_2 b_2}} \overline{\langle nb|\tilde{W}(t)|n_1b_1\rangle\langle n_1b_1|\tilde{W}(t_1)|n_2b_2\rangle}\ \overline{\langle n_2b_2|\tilde{U}(t_1)|n'b\rangle} \ . \tag{3.34}$$

From Eq. (2.42), we find that $\overline{W_{bb_1}^{nn_1}W_{b_1b_2}^{n_1n_2}} = \delta_{bb_2}\, Q_{nn_1}Q_{n_1n_2}\overline{V_{bb_1}}^2$ and hence

$$\begin{aligned}\frac{d\overline{\langle nb|\tilde{U}(t)|n'b\rangle}}{dt} &= -\int_0^t d\tau \sum_{n_1b_1n_2} e^{i(E_n-E_{n_2})t}e^{i(E_{n_2}+\varepsilon_b-E_{n_1}-\varepsilon_{b_1})\tau} \\ &\quad \times Q_{nn_1}Q_{n_1n_2}\overline{V_{bb_1}}^2\ \overline{\langle n_2b|\tilde{U}(t-\tau)|n'b\rangle} \ ,\end{aligned} \tag{3.35}$$

where we have set $\tau = t - t_1$. In the weak–coupling limit, $d\overline{\tilde{U}}/dt$ is small and $\overline{\tilde{U}(t-\tau)}$ may be replaced by $\overline{\tilde{U}(t)}$ [2]. This approximation amounts to neglecting memory effects and is thus a Markov approximation. For large t, we may further use the identity

$$\int_0^\infty d\tau e^{i(E-E_0)\tau} = iP\frac{1}{E-E_0} + \pi\delta(E-E_0) \ . \tag{3.36}$$

Neglecting the level shift given by the principal value, this results in

$$\frac{d\overline{\langle nb|\tilde{U}(t)|n'b\rangle}}{dt} = -\frac{1}{2}\sum_{n_1n_2} W^{(1)}_{nn_1n_1n_2}e^{i(E_n-E_{n_2})t}\ \overline{\langle n_2b|\tilde{U}(t)|n'b\rangle} \ , \tag{3.37}$$

where we have defined the generalized transition probability per unit time (to be compared with the Golden Rule result Eq. (4.5))

$$W^{(1)}_{nn_1n_1n_2} = 2\pi\sum_{b_1} Q_{nn_1}Q_{n_1n_2}\overline{V_{bb_1}}^2\,\delta(E_{n_2}+\varepsilon_b-E_{n_1}-\varepsilon_{b_1}) \ . \tag{3.38}$$

[2] This intuitive argument can be made more rigorous by substituting $y = A_0 t$, $\tilde{u}(y) = \tilde{U}(t)$, where A_0 is the strentgh of the coupling, and taking the van Hove limit $A_0 \to 0$, $t \to \infty$, $A_0\,t$ fixed (see Section 5 of Ref. [Zwa61]).

Returning to the Schrödinger picture, Eq. (3.37) takes the form

$$\begin{aligned}\frac{d\overline{\langle nb|U(t)|n'b\rangle}}{dt} &= - \quad i(E_n+\varepsilon_b)\,\overline{\langle nb|U(t)|n'b\rangle}\\ &- \quad \frac{1}{2}\sum_{n_1n_2} W^{(1)}_{nn_1n_1n_2}\,\overline{\langle n_2b|U(t)|n'b\rangle}\ . \end{aligned}\tag{3.39}$$

This is the expected expression for the time derivative of the matrix elements of $\overline{U}(t)$ for the unsymmetrized random–matrix coupling (I). We observe that Eq. (3.39) cannot be integrated in closed form, since the right–hand side does not only depend on $\overline{\langle nb|U(t)|n'b\rangle}$ but also on all the other matrix elements $\overline{\langle n_2b|U(t)|n'b\rangle}$, $n_2\neq n$. In a similar fashion, one can calculate the time derivative of $\overline{U}^\dagger(t)$. We find

$$\begin{aligned}\frac{d\overline{\langle nb|U^\dagger(t)|n'b\rangle}}{dt} &= \quad i(E_{n'}+\varepsilon_b)\,\overline{\langle nb|U^\dagger(t)|n'b\rangle}\\ &- \quad \frac{1}{2}\sum_{n_1n_2} W^{(2)}_{n_2n_1n_1n'}\,\overline{\langle nb|U^\dagger(t)|n_2b\rangle}\ , \end{aligned}\tag{3.40}$$

with

$$W^{(2)}_{n_1n'n_2n_1} = 2\pi\sum_{b_1} Q_{n_1n'}Q_{n_2n_1}\overline{V_{bb_1}}^2\,\delta(E_{n_2}+\varepsilon_b-E_{n_1}-\varepsilon_{b_1})\ . \tag{3.41}$$

We note that $W^{(2)}_{n_1n'n_2n_1}$ is not simply the complex conjugate of $W^{(1)}_{nn_1n_1n_2}$.

In concluding this section, it is instructive to rederive the average SRM evolution operator $\overline{U}(t)$ by implementing the simple method developed here. This will shed some light on the approximations involved. We start with the derivative of the time–evolution operator as given by Eq. (3.34). We use the symmetrized variance (2.43) and get $\overline{W^{nn_1}_{bb_1}W^{n_1n_2}_{b_1b_2}} = \delta_{nn_2}\delta_{bb_2}\overline{W^{nn_1\,2}_{bb_1}}$. This yields

$$\frac{d\overline{\langle nb|\tilde{U}(t)|nb\rangle}}{dt} = -\sum_{n_1b_1}\overline{W^{nn_1\,2}_{bb_1}}\int_0^t d\tau e^{i(E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1})\tau}\,\overline{\langle nb|\tilde{U}(t-\tau)|nb\rangle}\ . \tag{3.42}$$

We then make the Markov approximation, employ identity (3.36) and arrive at

$$\begin{aligned}\frac{d\overline{\tilde{U}_{nb}}(t)}{dt} &= -iP\sum_{n_1b_1}\frac{\overline{W^{nn_1\,2}_{bb_1}}}{E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1}}\,\overline{\tilde{U}_{nb}}(t)\\ &\quad -\pi\sum_{n_1b_1}\overline{W^{nn_1\,2}_{bb_1}}\delta(E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1})\,\overline{\tilde{U}_{nb}}(t)\\ &= (-i\Delta_{nb}-\frac{\Gamma_{nb}}{2})\,\overline{\tilde{U}_{nb}}(t)\ ,\end{aligned}\tag{3.43}$$

where we have used the definitions (3.20) of the level shift Δ_{nb} and the decay width Γ_{nb}. Finally, we integrate Eq. (3.43) taking into account the initial condition $\overline{\tilde{U}}(0)=1$. This leads to

$$\overline{\tilde{U}_{nb}}(t) = e^{(-i\Delta_{nb}-\frac{\Gamma_{nb}}{2})t}\ , \tag{3.44}$$

or, written in the Schrödinger picture,

$$\overline{U_{nb}}(t) = e^{-i(E_n+\varepsilon_b+\Delta_{nb})t-\frac{\Gamma_{nb}}{2}t}\ . \tag{3.45}$$

This expression for the ensemble–averaged evolution operator is identical to the result (3.23) obtained using the diagrammatic method. However, we emphasize that the approach used in the present section is restricted to the weak–coupling case, whereas the diagrammatic method has a much wider range of applicability (including the strong–coupling limit).

We now investigate the validity of the Markov approximation. We do this by substituting Eq. (3.44) into the r.h.s. of Eq. (3.42). We obtain

$$\frac{d\overline{\langle nb|\tilde{U}(t)|nb\rangle}}{dt} = -e^{-(i\Delta_{nb}+\frac{\Gamma_{nb}}{2})t}\sum_{n_1 b_1}\overline{W_{bb_1}^{nn_1 2}}\int_0^t d\tau e^{(i(E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1}+\Delta_{nb}))+\frac{\Gamma_{nb}}{2})\tau} \tag{3.46}$$

Under the conditions

$$\begin{aligned}\Gamma_{nb} &\ll E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1}\ ,\\ \Delta_{nb} &\ll E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1}\ ,\end{aligned} \tag{3.47}$$

we can ignore Γ_{nb} and Δ_{nb} in the integrand, and with the help of identity (3.36), we find that Eq. (3.46) reduces to Eq. (3.43). The weak–coupling conditions (3.47) are very similar to those used by Wigner and Weisskopf in their early work on spontaneous emission [Wei31]. We also see from these considerations that the weak–coupling condition (3.26) is actually equivalent to a Markov approximation.

3.3 The average density operator

In this section, we shall apply the diagrammatic technique to calculate the mean value of the total density operator $\hat{\rho}(t)$ for system plus bath. As in Section 3.2, we will introduce the propagator $K(t)$ and work in Fourier space. The density operator at times t, t' obeys the following von Neumann equation

$$\hat{\rho}(t,t')\theta(t)\theta(t') = K(t)\hat{\rho}(0)K^\dagger(t')\ , \tag{3.48}$$

where we have used the definition (3.1) for the propagator $K(t)$. We take the double Fourier transform

$$F(E,E') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt dt' e^{i(Et-E't')} f(t,t') \tag{3.49}$$

and obtain

$$\hat{\rho}(E,E') = G(E)\hat{\rho}(0)G^\dagger(E')\ . \tag{3.50}$$

Here, the Fourier transform $G(E)$ of the propagator is given by Eq. (3.5). We next expand $\hat{\rho}(E,E')$ in a Born series and take the ensemble average. We use the expansion (3.6) for $G(E)$ and its Hermitian conjugate and get

$$\overline{\rho}(E,E') = \sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\overline{G_0(WG_0)^s\hat{\rho}(0)(G_0^\dagger W)^r G_0^\dagger}\ . \tag{3.51}$$

In performing the average of this quantity, we use again Wick's theorem and retain only products of pairs of W's. Two W's are said to be "cross-contracted" if one of them occurs in G, the other in $G^\dagger$. We use the general rule that terms containing contraction lines which intersect or cross-contraction lines which cross any other contraction lines give negligible contributions and can thus be omitted. This means that contributions like $WG_0WG_0WG_0WG_0\cdots WG_0^\dagger WG_0^\dagger$ and $WG_0W\cdots WG_0^\dagger WG_0^\dagger$ can be neglected. As a result, we find that $\overline{G}$ appears sandwiched between two W's which are contracted across $\hat{\rho}(0)$, and the general term in the expansion (3.48) attains the form

$$\overline{G}W\cdots W\overline{G}\hat{\rho}(0)\overline{G}^\dagger W\cdots W\overline{G}^\dagger \ . \tag{3.52}$$

The average density operator $\overline{\rho}$ may thus be written as

$$\begin{aligned}\overline{\rho} = \overline{G}\hat{\rho}(0)\overline{G}^\dagger \quad &+ \quad \overline{G}W\overline{G}\hat{\rho}(0)\overline{G}^\dagger W\overline{G}^\dagger \\ &+ \quad \overline{G}W\overline{G}\,W\overline{G}\hat{\rho}(0)\,\overline{G}^\dagger W\overline{G}^\dagger W\overline{G}^\dagger + \cdots \end{aligned} \tag{3.53}$$

or

$$\overline{\rho} = \overline{G}\hat{\rho}(0)\overline{G}^\dagger + \overline{G}W\overline{G}\overline{\rho}\,\overline{G}^\dagger W\overline{G}^\dagger \ . \tag{3.54}$$

3.4 Summary

In this chapter, we have calculated the ensemble average of $U(t)$ and $\hat{\rho}(t)$, the evolution and density operators for system–plus–bath, using a diagrammatic perturbation expansion. The obtained averaged equations (3.10) and (3.54) completely determine the dynamics of the composite system in the weak–coupling limit. Transforming back to the time representation, we may rewrite these equations as

$$\overline{U}(t) = U_0(t) - \int_0^t dt_1 \int_0^{t_1} dt_2\ U_0(t-t_1)W\,U_0(t_1-t_2)\ W\,\overline{U}(t_2)\ , \tag{3.55}$$

and

$$\begin{aligned}\overline{\rho}(t,t') \quad &= \quad \overline{U}(t)\hat{\rho}(0)\overline{U}^\dagger(t') \\ &+ \quad \int_0^t d\tau \int_0^{t'} d\tau'\ \overline{U}(t-\tau)W\,\overline{\rho}(\tau,\tau')\ W\,\overline{U}^\dagger(t'-\tau')\ . \end{aligned} \tag{3.56}$$

Eqs. (3.55) and (3.56) are the averaged form of the Dyson and von Neumann equation, respectively. These two equations will be used in the next chapter to derive the averaged master equation describing the system S.

In the case of the symmetrized random–matrix coupling, Eq. (3.55) can be solved in closed form: $\overline{U}(t)$ is diagonal in the energy eigenbasis and its diagonal matrix elements are given by Eq. (3.23). In contradistinction, for the non–symmetrized coupling, a closed solution of the averaged Dyson equation is not available. Nevertheless, an expression for the time derivative of $\overline{U}(t)$, Eq. (3.39), may be obtained by using a variant of the diagrammatic method, valid solely in the limit of weak–coupling.

Chapter 4

Derivation of the master equation

4.1 Introduction

This chapter deals with the description of the relaxation of the quantum system S resulting from its coupling to the heat bath. The difficulty one is confronted with when dealing with complex systems, is how to achieve such a description in a manageable way. Even though the microscopic equations (3.55) and (3.56), which were derived in Chapter 3, specify the averaged time development of the composite system in full detail, they are not very useful as they stand. They are too complex and contain more information than is actually needed. Two approximate and more tractable evolution equations are currently used in the literature [Kam97] [1]: The master equation and the quantum Langevin equation. In the present chapter, we shall focus on the master equation and we will defer the discussion of the Langevin approach to Chapter 6.

To obtain a Markovian master equation for the reduced density operator $\hat{\rho}_S(t)$ from the averaged equations (3.55) and (3.56), we shall proceed as follows: (i) solve the Dyson equation (3.55), (ii) substitute the resulting $\overline{U}(t)$ into the von Neumann equation (3.56), (iii) take the trace over the bath degrees of freedom and (iv) differentiate with respect to time. As we already know from the discussion in Section 3.2.2, it is not always possible to perform step (i). This is the case, in particular, for the unsymmetrized variance of Eq. (2.42). Nevertheless, a master equation can still be derived in the weak–coupling limit.

As a preliminary step to the derivation of the master equation, we shall first evaluate, in Section 4.2, the ensemble average of the transition probability per unit time. This calculation up to second order in the random interaction W yields the averaged form of Fermi's Golden Rule. In Section 4.3, we shall then carry out the derivation of the Markovian master equation. We shall again consider the two forms of the random–matrix coupling presented in Section 2.3. We will conclude this chapter by a discussion of the different times scales appearing in the random–matrix model, in Section 4.4.

[1] A new approach, based on a stochastic Schrödinger equation, has been developed recently [Str99a, Str99b].

4.2 Fermi's Golden Rule

We suppose in this section that S is initially in some eigenstate $|m\rangle$ with $\hat{\rho}_S(0) = |m\rangle\langle m|$, and we ask for the ensemble–averaged probability to find the quantum system in another state $|n\rangle$ at a later time t. We have

$$\overline{P_n}(t) = \langle n|\overline{\rho}_S(t)|n\rangle = \sum_b \langle nb|\overline{\rho}(t)|nb\rangle \ , \tag{4.1}$$

where the total density operator $\overline{\rho}(t)$ is given by Eq. (3.56) with $t = t'$. In order to obtain an expression for $\overline{\rho}(t)$ up to second order in W, we replace $\overline{U}(t)$ by $U_0(t)$ and $\overline{\rho}(\tau, \tau')$ by $U_0(\tau)\hat{\rho}(0)U_0^\dagger(\tau')$ in the r.h.s. of the averaged von Neumann equation (3.56). This results in

$$\overline{\rho}(t) = \int_0^t d\tau \int_0^t d\tau' \, U_0(t-\tau) \overline{W \, U_0(\tau)\hat{\rho}(0)U_0^\dagger(\tau') \, W} \, U_0^\dagger(t-\tau') \ , \tag{4.2}$$

where we have omitted the first term which does not contribute to $\overline{P_n}(t)$ unless $n = m$. Inserting Eq. (4.2) into Eq. (4.1) and evaluating the integrals, we get

$$\overline{P_n}(t) = \sum_b \overline{|\langle nb|W|ma\rangle|^2} \; 4 \left(\frac{\sin(E_n + \varepsilon_b - E_m - \varepsilon_a)\frac{t}{2}}{E_n + \varepsilon_b - E_m - \varepsilon_a} \right)^2 . \tag{4.3}$$

For times much larger than the collision time, $t \gg t_\Delta$, the factor $4(\sin^2 \frac{1}{2}xt)/x^2$ is sharply peaked at $x = 0$ and may be approximated by $2\pi t \, \delta(x)$. This yields

$$\overline{P_n}(t) = 2\pi t \sum_b \overline{|\langle nb|W|ma\rangle|^2} \; \delta(E_n + \varepsilon_b - E_m - \varepsilon_a) \ . \tag{4.4}$$

The transition probability per unit time is then defined as

$$W_{nm} = \frac{\overline{P_n}(t)}{t} = 2\pi \sum_b \overline{|\langle nb|W|ma\rangle|^2} \; \delta(E_n + \varepsilon_b - E_m - \varepsilon_a) \ . \tag{4.5}$$

This is the ensemble–averaged version of Fermi's Golden Rule. We see that Eq. (4.5) agrees with the result obtained from ordinary second–order perturbation theory [Mes62].

4.3 Derivation of the master equation

We now turn to the derivation of the Markovian master equation satisfied by $\overline{\rho}_S(t)$. The equation applies provided the coupling between system and bath is weak. More precisely, we use the following assumptions.

i) The time t obeys the inequalities $t_\Delta \ll t \ll t_P$. Here $t_\Delta = 1/\Delta$ is the duration time of a single action of the interaction and t_P is the Poincaré recurrence time of the system. This condition is always needed to describe a relaxation process in terms of a transport equation.

ii) For all states $|n\rangle$ and $|b\rangle$, the bandwidth Δ has to satisfy the inequalities $\omega, \gamma \ll \Delta \ll kT$. Here γ is the relaxation constant and ω denotes the mean level spacing of the system S. For the harmonic oscillator, γ is defined in Eq. (5.10). For other systems, an analogous definition applies. Condition ii) requires weak coupling between bath and system and ensures the validity of the Markov approximation. It also requires the temperature T to be larger than a minimum temperature $kT_m = \Delta$ and may, therefore, also be seen as defining a semiclassical approximation.

These assumptions are discussed in more detail in Section 4.4. In the next two subsections, we treat the two forms of the random–matrix interaction separately.

4.3.1 Case (II): Symmetrized coupling

We begin with the simpler case, that is, with the approximated form (II). Because the variance of the random–matrix coupling is symmetric in the system–bath variables, the averaged Dyson equation (3.55) for the evolution operator can be solved in closed form (see Section 3.2.1, Eq. (3.23)): $\overline{U}(t)$ is diagonal in the energy basis and given by

$$\overline{U}_{nb}(t) = e^{-i(E_n+\varepsilon_b)t-\frac{\Gamma_{nb}}{2}t} , \tag{4.6}$$

where we have neglected the level shift Δ_{nb}. The decay width Γ_{nb} is given by Eq. (3.20). To obtain the averaged master equation, we proceed as indicated in the introduction. We first substitute $\overline{U}(t)$ from Eq. (4.6) into the averaged von Neumann equation (3.56). We then take the trace over the bath and arrive at

$$\begin{aligned}\sum_b \langle nb|\overline{\rho}(t,t')|n'b\rangle &= \sum_b \overline{U}_{nb}(t)\,\langle nb|\hat{\rho}(0)|n'b\rangle\,\overline{U}^{\dagger}_{n'b}(t') \\ &\quad + \delta_{nn'}\int_0^t d\tau \int_0^{t'} d\tau' \sum_{bn_1b_1} \overline{U}_{nb}(t-\tau) \\ &\quad\quad \times \overline{|\langle nb|W|n_1b_1\rangle|^2}\langle n_1b_1|\overline{\rho}(\tau,\tau')|n_1b_1\rangle \overline{U}^{\dagger}_{n'b}(t'-\tau') \;. \end{aligned} \tag{4.7}$$

Here we have used (2.43) to write

$$\overline{\langle nb|W|n_1b_1\rangle\langle n_2b_2|W|n'b\rangle} \simeq \delta_{nn'}\delta_{n_1n_2}\delta_{b_1b_2}\,\overline{|\langle nb|W|n_1b_1\rangle|^2} \;. \tag{4.8}$$

By arguments similar to those presented in Section 3.2, one can show that the terms we have omitted in Eq. (4.8) give a negligible contribution in the limit N_B to infinity. We now take the time derivatives with respect to t and t', use

$$\frac{d}{dt}\overline{U}_{nb}(t) = -\left(\frac{\Gamma_{nb}}{2} + i(E_n+\varepsilon_b)\right)\overline{U}_{nb}(t), \tag{4.9}$$

and its Hermitian conjugate, and find

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}\right)\sum_b\langle nb|\overline{\rho}(t,t')|n'b\rangle &= -i(E_n-E_{n'})\sum_b\langle nb|\overline{\rho}(t,t')|n'b\rangle \\ &-\frac{1}{2}\sum_b\left(\Gamma_{nb}+\Gamma_{n'b}\right)\langle nb|\overline{\rho}(t,t')|n'b\rangle \\ &+\delta_{nn'}\sum_{bn_1b_1}\overline{|\langle nb|W|n_1b_1\rangle|^2}\int_{-t}^{0} dt_1 e^{(i(E_n+\varepsilon_b)+\frac{\Gamma_{nb}}{2})t_1}\langle n_1b_1|\overline{\rho}(t+t_1,t')|n_1b_1\rangle \\ &+\delta_{nn'}\sum_{bn_1b_1}\overline{|\langle nb|W|n_1b_1\rangle|^2}\int_{-t'}^{0} dt_1' e^{(-i(E_n+\varepsilon_b)+\frac{\Gamma_{nb}}{2})t_1'}\langle n_1b_1|\overline{\rho}(t,t'+t_1')|n_1b_1\rangle \;, \end{aligned} \tag{4.10}$$

where we have put $t_1 = \tau - t$ and $t_1' = \tau' - t'$.

The right–hand side of Eq. (4.10) is easily interpreted. The first term describes the free motion. The second term (the "loss term") corresponds to transitions which *deplete* the states n and n' whereas the two last terms ("gain terms") correspond to transitions which *feed* the state n (we shall discuss the absence of off–diagonal gain terms in more detail in Chapter 6). The gain terms are non–local in time and, therefore, involve memory effects. In the limit of weak coupling, however, the process becomes Markovian and the memory effects play no role [Wei80]. To see this, we note that for short times (to zeroth order in W), Eq. (3.56) reduces to

$$\overline{\rho}(t,t') = U_0(t)\hat{\rho}(0){U_0}^{\dagger}(t') \ . \tag{4.11}$$

We use this form in the gain terms and accordingly approximate

$$\langle n_1 b_1|\overline{\rho}(t+t_1,t')|n_1 b_1\rangle \qquad \text{by} \qquad e^{-i(E_{n_1}+\varepsilon_{b_1})t_1}\langle n_1 b_1|\overline{\rho}(t,t')|n_1 b_1\rangle \tag{4.12}$$

and

$$\langle n_1 b_1|\overline{\rho}(t,t'+t_1')|n_1 b_1\rangle \qquad \text{by} \qquad \langle n_1 b_1|\overline{\rho}(t,t')|n_1 b_1\rangle e^{i(E_{n_1}+\varepsilon_{b_1})t_1'} \ . \tag{4.13}$$

Setting $t = t'$, we obtain

$$\begin{aligned}
\frac{d}{dt}\langle n|\overline{\rho}_S(t)|n'\rangle &= -i(E_n - E_{n'})\langle n|\overline{\rho}_S(t)|n'\rangle - \frac{1}{2}\sum_b \left(\Gamma_{nb} + \Gamma_{n'b}\right)\langle nb|\overline{\rho}(t,t')|n'b\rangle \\
&+ \delta_{nn'} \sum_{bn_1b_1} \overline{|\langle nb|W|n_1b_1\rangle|^2}\langle n_1b_1|\overline{\rho}(t)|n_1b_1\rangle \\
&\times \left[\int_{-t}^{0} dt_1 e^{(i(E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1})+\frac{\Gamma_{nb}}{2})t_1} + \int_{-t}^{0} dt_1 e^{(-i(E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1})+\frac{\Gamma_{nb}}{2})t_1}\right] \ .
\end{aligned} \tag{4.14}$$

Condition ii) implies that $\Gamma_{nb} \ll E_n + \varepsilon_b - E_{n_1} - \varepsilon_{b_1}$; see also Eq. (3.47). We hence neglect Γ_{nb} in the integrands of Eq. (4.14). This amounts to replacing $\overline{U}(t)$ by $U_0(t)$ and is similar to what we did in Section 4.2 when we calculated the average of the survival probability $\overline{P_n}(t)$. Moreover, for sufficiently large times, we have

$$\int_{-t}^{t} dt_1 e^{i(E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1})t_1} \simeq 2\pi\delta(E_n + \varepsilon_b - E_{n_1} - \varepsilon_{b_1}) \ . \tag{4.15}$$

For times much larger than the collision time, $t \gg t_\Delta$, Eq. (4.14) thus reduces to

$$\begin{aligned}
\frac{d}{dt}\langle n|\overline{\rho}_S(t)|n'\rangle &= -i\langle n|\left[H_S,\overline{\rho}_S(t)\right]|n'\rangle \\
&-\pi\sum_{bn_1b_1}\overline{|\langle nb|W|n_1b_1\rangle|^2}\,\delta(E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1})\,\langle nb|\overline{\rho}(t)|n'b\rangle \\
&-\pi\sum_{bn_1b_1}\overline{|\langle n'b|W|n_1b_1\rangle|^2}\,\delta(E_{n'}+\varepsilon_b-E_{n_1}-\varepsilon_{b_1})\,\langle nb|\overline{\rho}(t)|n'b\rangle \\
&+2\pi\,\delta_{nn'}\sum_{bn_1b_1}\overline{|\langle nb|W|n_1b_1\rangle|^2}\delta(E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1})\,\langle n_1b_1|\overline{\rho}(t)|n_1b_1\rangle \ ,
\end{aligned} \tag{4.16}$$

where we have used the definition (3.20) of the decay width. It is clear that the transition probabilities per unit time, W_{nn_1} and W_{n_1n}, as given by their Golden Rule expressions

(4.5),

$$
\begin{aligned}
W_{nn_1} &= 2\pi \sum_b \overline{|\langle nb|W|n_1b_1\rangle|^2}\,\delta(E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1})\ ,\\
W_{n_1n} &= 2\pi \sum_{b_1} \overline{|\langle nb|W|n_1b_1\rangle|^2}\,\delta(E_n+\varepsilon_b-E_{n_1}-\varepsilon_{b_1})\ ,
\end{aligned}
\tag{4.17}
$$

do not depend explicitly on the bath variables b and b_1. Consequently, the traces over the bath can be carried out and we finally obtain the SRM Markovian master equation

$$
\begin{aligned}
\frac{d}{dt}\langle n|\overline{\rho}_S(t)|n'\rangle &= -i\,\langle n|\left[H_S,\overline{\rho}_S(t)\right]|n'\rangle \\
&\quad -\sum_{n_1}\left(\frac{W_{n_1n}+W_{n_1n'}}{2}\right)\langle n|\overline{\rho}_S(t)|n'\rangle + \delta_{nn'}\sum_{n_1}W_{nn_1}\langle n_1|\overline{\rho}_S(t)|n_1\rangle\ .
\end{aligned}
\tag{4.18}
$$

It may seem curious that, in the evaluation of the gain term, it is necessary to invoke the Markov approximation while in the loss term, the limit of weak coupling apparently suffices. To explain this fact, we recall that we always work in the limit of infinite matrix dimension N_B. The loss term in the master equation (4.18) is obtained from single–side Wick contractions, symbolically written as $\overline{VV}(\ ):(\ )$ or $(\ ):(\ )\overline{VV}$, whereas the gain term is generated by Wick contractions $\overline{V(\ ):(\ )V}$ which connect matrix elements in both amplitudes. Selection among the first type of Wick contractions is affected by both, the limit $N_B\to\infty$ and the weak–coupling limit. From our discussion in Section 3.2, we know that of the three Wick contractions that correspond to the terms of fourth order, VVVV, VVVV, and VVVV, the last is neglected because $N_B\to\infty$ and the second, because of weak coupling. In contradistinction, the form of the gain terms in Eq. (4.10) is determined entirely by the limit $N_B\to\infty$. Hence, an additional step is needed to implement the weak–coupling limit.

4.3.2 Case (I): Non-symmetrized coupling

For the non–symmetric form (2.42) of the random–matrix coupling, the averaged time–evolution operator is not diagonal in energy representation. We recall that we have instead

$$
\langle nb|\overline{U}(t)|n'b'\rangle = \delta_{bb'}\langle nb|\overline{U}(t)|n'b\rangle\ .
\tag{4.19}
$$

Therefore, the matrix elements $\langle nb|\overline{U}(t)|n'b'\rangle$ cannot be given in closed form (see Section 3.2.2). However, the time derivative of these quantities can be obtained explicitly in the limit of weak coupling. This suffices to obtain the master equation. Aside from this difference, the derivation proceeds in complete analogy to that given in the previous subsection. In analogy to Eq. (4.7), we obtain

$$
\begin{aligned}
\sum_b\langle nb|\overline{\rho}(t,t')|n'b\rangle &= \sum_{bn_1n_2}\langle nb|\overline{U}(t)|n_1b\rangle\langle n_1b|\hat{\rho}(0)|n_2b\rangle\langle n_2b|\overline{U}^\dagger(t')|n'b\rangle \\
&\quad\times\int_0^t d\tau\int_0^{t'}d\tau'\sum_{bn_1}\sum_{\substack{n_2b_2\\ n_3n_4}}\langle nb|\overline{U}(t-\tau)|n_1b\rangle Q_{n_1n_2}Q_{n_3n_4}\overline{V_{bb_2}}^2 \\
&\quad\times\langle n_2b_2|\overline{\rho}(\tau,\tau')|n_3b_2\rangle\langle n_4b|\overline{U}^\dagger(t'-\tau')|n'b\rangle\ ,
\end{aligned}
\tag{4.20}
$$

where we have used that, according to Eq. (2.42),

$$\overline{\langle n_1 b|W|n_2 b_2\rangle\langle n_3 b_3|W|n_4 b\rangle} \simeq \delta_{b_2 b_3} Q_{n_1 n_2} Q_{n_3 n_4} \overline{V_{bb_2}}^2 \ . \tag{4.21}$$

We take the double time derivative, use Eqs. (3.39) and (3.40) for the time derivative of the averaged evolution operator, the Markov approximation in the gain terms, and the identity

$$\int_0^\infty d\tau e^{ix\tau} = iP\frac{1}{x} + \pi\delta(x) \ . \tag{4.22}$$

We neglect the level shift due to the principal–value integral. All these steps yield the Markovian master equation

$$\begin{aligned}
\frac{d}{dt}\langle n|\overline{\rho}_S(t)|n'\rangle &= -i\langle n|\left[H_S, \overline{\rho}_S(t)\right]|n'\rangle \\
&-\frac{1}{2}\sum_{n_1 n_2} W^{(1)}_{nn_1n_1n_2}\langle n_2|\overline{\rho}_S(t)|n'\rangle - \frac{1}{2}\sum_{n_1n_2} W^{(2)}_{n_1n'n_2n_1}\langle n|\overline{\rho}_S(t)|n_2\rangle \\
&+\frac{1}{2}\sum_{n_1n_2} W^{(3)}_{nn_1n_2n'}\langle n_1|\overline{\rho}_S(t)|n_2\rangle + \frac{1}{2}\sum_{n_1n_2} W^{(4)}_{nn_1n_2n'}\langle n_1|\overline{\rho}_S(t)|n_2\rangle \ ,
\end{aligned} \tag{4.23}$$

where we have defined the generalized transition probabilities

$$\begin{aligned}
W^{(1)}_{nn_1n_1n_2} &= 2\pi\sum_{b_1} Q_{nn_1}Q_{n_1n_2}\overline{V_{bb_1}}^2\ \delta(E_{n_2}+\varepsilon_b - E_{n_1} - \varepsilon_{b_1}) \ , \\
W^{(2)}_{n_1n'n_2n_1} &= 2\pi\sum_{b_1} Q_{n_1n'}Q_{n_2n_1}\overline{V_{bb_1}}^2\ \delta(E_{n_2}+\varepsilon_b - E_{n_1} - \varepsilon_{b_1}) \ , \\
W^{(3)}_{nn_1n_2n'} &= 2\pi\sum_{b} Q_{nn_1}Q_{n_2n'}\overline{V_{bb_1}}^2\ \delta(E_{n_1}+\varepsilon_{b_1} - E_{n} - \varepsilon_{b}) \ , \\
W^{(4)}_{nn_1n_2n'} &= 2\pi\sum_{b} Q_{nn_1}Q_{n_2n'}\overline{V_{bb_1}}^2\ \delta(E_{n_2}+\varepsilon_{b_1} - E_{n'} - \varepsilon_{b}) \ .
\end{aligned} \tag{4.24}$$

The master equation (4.23) shows a structure which is quite similar to the SRM master equation (4.18): the first term on the right side describes the free motion without coupling, the next two terms are the loss terms which deplete the states n and n' and the last terms correspond to the gain terms which feed the states n and n'. It is interesting to note, that the unsymmetrized random–matrix coupling (2.42) leads to an averaged master equation, Eq. (4.23), in which gain and loss terms are symmetric, whereas the symmetrized coupling (2.43) leads to a master equation, Eq. (4.18), in which this symmetry is lost.

4.4 Discussion: Time Scales

In view of the derivation given in the previous section, we discuss the various time scales appearing in our random–matrix model. These time scales play an essential role in defining the range of validity of the master equation [Wei80, Rau96].

According to the statistical *ansatz* in Eq. (2.41), the interaction V connects eigenstates of H_B within an energy interval $\sim \Delta$. Thus, the bandwidth Δ can be visualized as the average amount of energy exchanged during a single action of V, and $t_\Delta = 1/\Delta$ can be interpreted as the duration time of a single action of V (i.e., the time needed to transfer the energy Δ). A statistical description in terms of a master equation can be valid only for times $t \gg t_\Delta$.

Any dynamical process in a finite–sized system will return (close) to its initial state after a characteristic time, the Poincaré recurrence time t_P (in the example of a two–level system, the Poincaré time corresponds to the Rabi period $t_P = 2\pi/(E_+ - E_-)$, where $(E_+ - E_-)$ is the energy difference between the two (perturbed) levels [Mes62]). When the bath is much larger than the system, the recurrence time is essentially determined by the mean level spacing D of the bath, $t_P \sim 1/D$. Obviously, t_P tends to infinity with the size of the bath. This is the condition of irreversibility. The inequality $t \ll t_P$ must be fulfilled in order to have relaxation.

The weak–coupling condition ii) requires that the relaxation constant γ be much smaller than the amount Δ of energy transferred during a single action of the interaction, $\gamma \ll \Delta$ (see also Section 3.2.1). This condition has a simple interpretation in terms of the times that correspond to these energies. The relaxation time $t_R = 1/\gamma$ must be much larger than the time $t_\Delta = 1/\Delta$ needed for a single action of V, $t_\Delta \ll t_R$.

There are two time scales which determine the memory time of the heat bath [Asl85, Lin84]: The time t_Δ (the inverse of the frequency cutoff of the bath in the Caldeira-Leggett model) and the time $t_B = 1/kT$. The latter is purely quantum in origin. For high (low) temperature, thermal (quantum) fluctuations dominate and the memory time of the bath is given by t_Δ (by t_B, respectively). A crossover between thermal and quantum fluctuations occurs at the crossover temperature $kT_m = \Delta$. To guarantee the validity of the Markov approximation, the temperature must be much larger than the crossover temperature, $\Delta \ll kT$. This condition can be rephrased in terms of length scales: The range of the interaction must be much larger than the thermal de Broglie wavelength of the Brownian particle $\lambda_{dB} = 1/\sqrt{4MkT}$.

4.5 Summary

The present chapter has been concerned with the derivation of a Markovian master equation for the reduced density operator $\overline{\rho}_S(t)$, starting from the averaged Dyson and von Neumann equations (3.55) and (3.56). Our central result is embodied in the master equations (4.18) and (4.23), obtained for the symmetrized and non-symmetrized form of the random–matrix coupling, respectively. The equations are valid in a domain of parameter values specified by the inequalities (i) and (ii) of Section 4.3. Eqs. (4.18) and (4.23) are completely determined once the transition probabilities (4.17) and (4.24) are known. Both equations show the typical gain–loss structure. However, the SRM master equation has the peculiar feature of having no off-diagonal gain terms. This is due to the form (2.43) of the second moment which supposes that the entire interaction acts as a random matrix: In our approach, the gain term for these non–diagonal elements vanishes because of the Kronecker delta for the states of the system S appearing in condition (II).

Before we end this chapter, let us make a final comment on the fact that the averaged master equations (4.18) and (4.23) are naturally given in the energy basis $|n\rangle$ of the system S. It is well known that full random matrices which belong to GOE are, by construction, basis independent, since they are invariant under any orthogonal transformation [Meh91]. On the other hand, random band–matrices single out a specific basis in Hilbert space, namely the basis in which they have a band structure. In the case of our model, this preferred basis corresponds to the energy eigenbasis $|b\rangle$ of the bath Hamiltonian, as shown by Eq. (2.41). This, together with the conservation of the total energy, explains why Eqs. (3.55) and (3.56) are expressed, after the bath has been traced out, in the energy eigenbasis of S.

Chapter 5

Applications

5.1 Introduction

In the previous chapters, we have developed a formalism which allows us, in principle, to calculate the ensemble–averaged density operator for a system interacting with a heat bath. The present chapter is devoted to the illustration of the results obtained so far. More specifically, we shall apply the averaged Markovian master equations (4.18) and (4.23) to two simple quantum systems: The damped harmonic oscillator [Dek81] and the dissipative two–level system [Leg87]. These two systems are the paradigm of a quantum dissipative system and have been the object of an intensive study in past years (see [Wei93] and references therein). We shall compare our master equations with those derived by Agarwal [Aga73, Aga74] within the oscillator bath model.

We shall begin with the study of the damped harmonic oscillator in Section 5.2. In Section 5.2.1, we will consider the SRM master equation and compare it with the corresponding RWA equation. The range of validity of the SRM equation will then be investigated and we will derive an expression for the weak–coupling condition in terms of the parameters of our random–matrix model. Positivity of the density matrix will also be briefly discussed. In the following section, we shall deal with the unsymmetrized random coupling. In particular, we shall compare the resulting master equation with the Caldeira–Leggett equation and discuss the fluctuation–dissipation theorem. Finally, in Section 5.3, we will examine the case of the dissipative two–level system, again for the two forms of the random–matrix interaction.

5.2 First Application: Harmonic oscillator

Let us consider a one–dimensional harmonic oscillator with mass M and frequency ω. The corresponding Hamiltonian H_S is given by Eq. (2.3) and the energy spectrum reads [Mes62]

$$E_n = (n + 1/2)\ \omega, \qquad n = 1, 2, \dots\ . \tag{5.1}$$

We assume a coupling linear in the position of the system S and, accordingly, set $Q = x$. We introduce the usual creation and annihilation operators $a^\dagger$ and a. We recall that in terms of these ladder operators, the position operator x may be expressed as

$$x = \sqrt{\frac{1}{2M\omega}}\,(a + a^\dagger)\ . \tag{5.2}$$

Due to the tridiagonal structure of this operator in energy representation, we will find that the elements of the matrix $W_{ab}^{mn} = Q_{mn}V_{ab}$ vanish unless $|m-n| = 1$, defining thus a one–step process.

5.2.1 Case (II): Symmetrized coupling

Derivation of the SRM master equation

The transition probabilities in Eq. (4.17) are easily evaluated with the aid of Eqs. (2.41) and (2.43). For times $t \ll t_P$, we can replace the sum over b by an integral and obtain

$$\begin{aligned} W_{nn_1} &= 2\pi \sum_b |Q_{nn_1}|^2 \overline{V_{bb_1}}^2 \, \delta(E_n + \varepsilon_b - E_{n_1} - \varepsilon_{b_1}) \\ &= 2\pi \, |Q_{nn_1}|^2 A_0 \int d\varepsilon_b \left[\frac{\rho(\varepsilon_b)}{\rho(\varepsilon_{b_1})}\right]^{\frac{1}{2}} e^{-\frac{(\varepsilon_b - \varepsilon_{b_1})^2}{2\Delta^2}} \delta(E_n + \varepsilon_b - E_{n_1} - \varepsilon_{b_1}) \\ &= 2\pi \, |Q_{nn_1}|^2 A_0 \, e^{\frac{\beta}{2}(E_{n_1} - E_n)} e^{-\frac{(E_{n_1} - E_n)^2}{2\Delta^2}} \quad . \end{aligned} \tag{5.3}$$

In the last line, we have replaced the density of states $\rho(\varepsilon)$ by the expression (2.39). We make use of the identities

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \text{and} \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \ , \tag{5.4}$$

together with Eq. (5.2) for $Q = x$. This gives

$$Q_{nn_1} = \frac{1}{\sqrt{2M\omega}}\Big(\sqrt{n}\,\delta_{n_1,n-1} + \sqrt{n+1}\,\delta_{n_1,n+1}\Big) \ . \tag{5.5}$$

Substituting (5.5) in Eq. (5.3), we obtain for the only non–vanishing terms

$$\begin{aligned} W_{nn-1} &= W_0 n e^{-\frac{\beta}{2}\omega} \ , & W_{nn+1} &= W_0(n+1)e^{\frac{\beta}{2}\omega} \ , \\ W_{n-1n} &= W_0 n e^{\frac{\beta}{2}\omega} \ , & W_{n+1n} &= W_0(n+1)e^{-\frac{\beta}{2}\omega} \ , \end{aligned} \tag{5.6}$$

where we have set

$$W_0 = \frac{A_0\pi}{M\omega} e^{-\frac{\omega^2}{2\Delta^2}} \ , \tag{5.7}$$

and used the formula of the energy spectrum. The averaged master equation for the damped harmonic oscillator takes then the form

$$\begin{aligned} &\frac{d}{dt}\langle n|\overline{\rho}_S(t)|n'\rangle = -i\langle n|[H_S, \overline{\rho}_S(t)]|n'\rangle \\ &\quad -\frac{1}{2}\Big(W_{n-1n} + W_{n+1n} + W_{n'-1n'} + W_{n'+1n'}\Big)\langle n|\overline{\rho}_S(t)|n'\rangle \\ &\quad +\Big(W_{nn+1}\,\langle n+1|\overline{\rho}_S(t)|n+1\rangle + W_{nn-1}\,\langle n-1|\overline{\rho}_S(t)|n-1\rangle\Big)\,\delta_{nn'} \ , \end{aligned} \tag{5.8}$$

or, equivalently,

$$\begin{aligned} \frac{d}{dt}\langle n|\overline{\rho}_S(t)|n'\rangle &= -i\omega(n-n')\langle n|\overline{\rho}_S(t)|n'\rangle - \gamma\Big[(n+n')(2n_{th}+1) + 2n_{th}\Big]\langle n|\overline{\rho}_S(t)|n'\rangle \\ &+\Big(2\gamma(n_{th}+1)(n+1)\,\langle n+1|\overline{\rho}_S(t)|n+1\rangle + 2\gamma n_{th} n\,\langle n-1|\overline{\rho}_S(t)|n-1\rangle\Big)\,\delta_{nn'} \ . \end{aligned} \tag{5.9}$$

Here, we have defined the relaxation constant,

$$\gamma = W_0 \sinh \frac{\beta}{2}\omega \ , \tag{5.10}$$

and n_{th} denotes the average number of quanta at temperature T,

$$n_{th} = \frac{1}{e^{\beta\omega} - 1} \ . \tag{5.11}$$

The SRM master equation (5.9) is given in energy representation. This form is not very suitable for a comparison with the RWA master equation. An operator equation can be obtained with of the help of the identities (5.4). Employing them it is straightforward to check that (5.9) is the energy representation of the following equation

$$\begin{aligned}\frac{d\overline{\rho}_S(t)}{dt} &= - \ i\omega\,[a^\dagger a, \overline{\rho}_S(t)] - \gamma\Big(a^\dagger a\overline{\rho}_S(t) - 2\mathcal{D}a\overline{\rho}_S a^\dagger + \overline{\rho}_S(t)a^\dagger a\Big) \\ &- \ 2\gamma n_{th}\Big(a^\dagger a\overline{\rho}_S(t) - \mathcal{D}a\overline{\rho}_S(t)a^\dagger - \mathcal{D}a^\dagger\overline{\rho}_S(t)a + \overline{\rho}_S(t)aa^\dagger\Big) \ . \end{aligned} \tag{5.12}$$

In Eq. (5.12) we have introduced the projector $\mathcal{D}$ which selects the diagonal part of the operator on which it operates,

$$(\mathcal{D}O)_{nn'} = O_{nn}\,\delta_{nn'} \ . \tag{5.13}$$

An important property of the SRM master equation (5.12) is that the dynamics of the off–diagonal terms decouples from that of the occupation probabilities $\overline{P}_n(t) = \langle n|\overline{\rho}_S(t)|n\rangle$. This is easily demonstrated by setting $n = n'$ in (5.9). We find that the diagonal elements obey the equation

$$\begin{aligned}\frac{d}{dt}\overline{P}_n(t) &= 2\gamma n_{th} n\overline{P}_{n-1}(t) + 2\gamma(n_{th}+1)(n+1)\overline{P}_{n+1}(t) \\ &- \ 2\gamma\Big[n(2n_{th}+1) + n_{th}\Big]\overline{P}_n(t) \ . \end{aligned} \tag{5.14}$$

Eq. (5.14) does not depend on non–diagonal matrix elements and is of the form of a Pauli master equation [Pau28]. In this form, the gain–loss structure for the probability of each state n becomes particularly clear (see Fig. 5.1): The first line is the gain due to transitions from the states $n-1$ and $n+1$, and the second term is the loss due to transitions in these two states. The steady state solution of Eq. (5.14) reads

$$2\gamma(n_{th}+1)n\overline{P}_n = 2\gamma n_{th} n\overline{P}_{n-1} \ . \tag{5.15}$$

This leads to the thermal distribution for $\overline{P}_n$,

$$\overline{P}_n = \frac{n_{th}}{n_{th}+1}\,\overline{P}_0 = \frac{e^{-n\beta\omega}}{1 - e^{-\beta\omega}} \ . \tag{5.16}$$

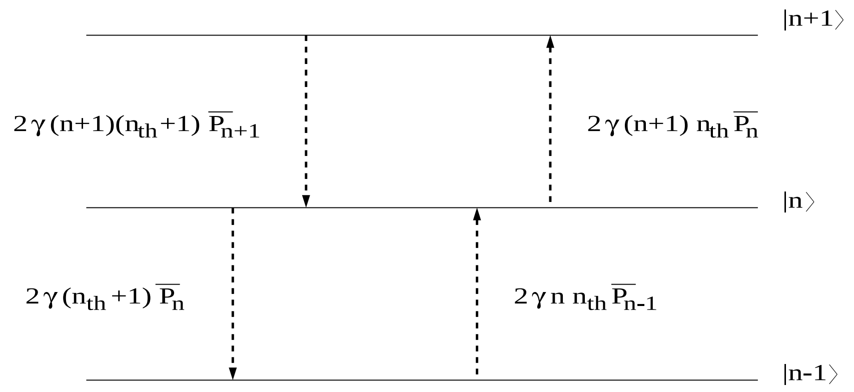

Figure 5.1: Probability flows in the Pauli equation (5.14) (taken from Ref. [Aga73]).

Comparison with the RWA master equation

Let us now compare the RWA and SRM master equations (2.30) and (5.12). First, we note that an essential feature common to these approximate equations is that the diagonal and off–diagonal elements do not mix. The resulting simplification which occurs in the description of the open system is one of the reasons why the RWA Hamiltonian is so widely used in quantum optics [1] (see, e.g. [Mey91]). The rationales for this decoupling are different, however. In the RWA case, it is due to the omission of the rapidly varying terms in the interaction Hamiltonian, whereas in the SRM case, it stems from the fact that the entire interaction is taken as a random–matrix. Second, we observe that the evolution equations for the diagonal terms are identical: In both cases the occupation probabilities obey Eq. (5.14). This can easily be understood by direct comparison of Eqs. (2.30) and (5.12). We see that the two master equations coincide term by term except for the gain terms $a\overline{\rho}_S(t)a^\dagger$ and $a^\dagger\overline{\rho}_S(t)a$. In the SRM equation, they are replaced by their projection onto the diagonal, $\mathcal{D}a\overline{\rho}_S(t)a^\dagger$ and $\mathcal{D}a^\dagger\overline{\rho}_S(t)a$. Hence, only the diagonal parts of these equations are the same. That the coincidence between the SRM and RWA equations does not extent to the time dependence of the non–diagonal elements $\langle n|\overline{\rho}_S(t)|n'\rangle$, $n \neq n'$, is due to the form (2.43) of the approximated variance of the random interaction and the resulting absence of gain terms in the coherence.

Weak–coupling condition

As already indicated in Chapter 4, the averaged Markovian master equation (5.12) is only valid when the coupling between system and bath is "weak". Here we wish to investigate what "weak" really means. More specifically, we want to express a weak–coupling condition in terms of the parameters of the random–matrix interaction, Eq. (2.41). Following the discussion at the end of Section 3.2.1, we compare

$$\Gamma_{nb}(E) \qquad \text{and} \qquad \sum_{n_1 b_1} \overline{W^{nn_1\,2}_{bb_1}}(G_0)^2_{n_1 b_1}\Gamma_{n_1 b_1}(E)\ . \tag{5.17}$$

[1] One may mention, for example, the Jaynes–Cummings model for a dissipative two–level system [Jay63], which is only exactly solvable under the rotating wave approximation.

These quantities are calculated as in the preceding section. For the decay width $\Gamma_{nb}(E)$ this gives

$$\begin{aligned}
\Gamma_{nb}(E) &= 2\pi \sum_{n_1} |Q_{nn_1}|^2 \sum_{b_1} \overline{V_{bb_1}}^2 \, \delta(E - E_{n_1} - \varepsilon_{b_1}) \\
&= \frac{\pi A_0}{M\omega} \left\{ n \exp\left(-\frac{1}{2\Delta^2} \left(E - E_n - \varepsilon_b + \omega - \frac{\beta\Delta^2}{2} \right)^2 + \frac{\beta^2\Delta^2}{8} \right) \right. \\
&+ \left. (n+1) \exp\left(-\frac{1}{2\Delta^2} \left(E - E_n - \varepsilon_b - \omega - \frac{\beta\Delta^2}{2} \right)^2 + \frac{\beta^2\Delta^2}{8} \right) \right\} . \qquad (5.18)
\end{aligned}$$

We shall assume in the following that $n \gg 1$, so that $n+1$ may be replaced by n in the second line of the above expression. We first show that $\Gamma_{nb}(E)$ depends weakly on energy. From Eq. (5.18) we see that $\Gamma_{nb}(E)$ is written as a sum of two Gaussians, each with width Δ: The first Gaussian is centered at $E_1 = E_n + \varepsilon_b - \omega + \frac{\beta\Delta^2}{2}$ and the second one at $E_2 = E_n + \varepsilon_b + \omega + \frac{\beta\Delta^2}{2}$. The two maxima of the Gauss curves are thus separated by a distance 2ω.

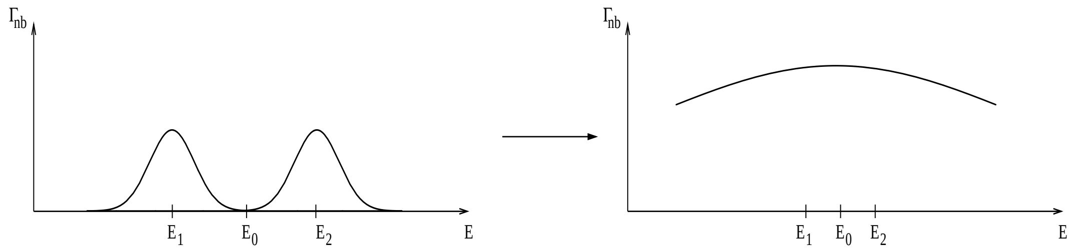

Figure 5.2: Two Gaussians centered at E_1 and E_2 coalesce in a single Gaussian centered at E_0 in the limit $\omega \ll \Delta \ll kT$ (see text for details).

Under the conditions in which

(a) the bandwidth Δ is much larger than the level spacing ω of the system,

$$\omega \ll \Delta \; , \qquad (5.19)$$

(b) and the temperature T of the heat bath is much larger than a minimum temperature $kT_m = \Delta$,

$$\Delta \ll kT \qquad \text{or, alternatively,} \qquad \beta\Delta^2 \ll \Delta \; , \qquad (5.20)$$

one may omit $\pm\,\omega - \beta/2\Delta^2$ in the argument of the exponentials in Eq. (5.18). It thus follows that the two curves coalesce in one single Gaussian of width Δ and centered at $E_0 = E_n + \varepsilon_b$ (see Fig. 5.2). Moreover, it is possible to neglect the variation of $\Gamma_{nb}(E)$ with energy if the height of this Gaussian, $\Gamma_{nb} = \Gamma_{nb}(E_n + \varepsilon_b) \simeq 2\gamma n$, is small compared to its width Δ. Consequently, if $\gamma \ll \Delta$, one can approximate the decay width by its maximum value, $\Gamma_{nb}(E) \simeq \Gamma_{nb}$. Next, we compute $\sum_{n_1 b_1} \overline{W_{bb_1}^{nn_1\,2}} (G_0)^2_{n_1 b_1} \Gamma_{n_1 b_1}(E)$. The evaluation of the latter is very similar to that of the decay width. We obtain

$$\begin{aligned} \sum_{n_1 b_1} \overline{W_{bb_1}^{nn_1\,2}} (G_0)^2_{n_1 b_1} \Gamma_{n_1 b_1}(E) &= \pi \sum_{n_1} |Q_{nn_1}|^2 \, \Gamma_{n_1 b_1} \sum_{b_1} \overline{V_{bb_1}^{\;2}} \, \delta'(E - E_{n_1} - \varepsilon_{b_1}) \\ &\simeq \Gamma_{n_1 b_1} \left[\frac{\pi A_0}{M\omega\Delta} n \right] . \end{aligned} \tag{5.21}$$

From Eq. (5.21), we deduce that the second term in Eq. (5.17) can be neglected with respect to the decay width, if

$$\frac{\pi A_0}{M\omega} \ll \Delta \; , \tag{5.22}$$

or, equivalently,

$$\gamma \ll \Delta \; , \tag{5.23}$$

where we have used Eq. (5.10) and condition (a). We have thus shown that the SRM master equation for the damped harmonic oscillator is valid provided the following weak–coupling–high–temperature condition is fulfilled

$$\omega, \gamma \ll \Delta \ll kT \; . \tag{5.24}$$

We emphasize again that condition (5.24) is neither restricted to the approximated form of the random–matrix interaction nor to the specific case of the harmonic oscillator. Indeed, since the symmetrization of the variance does not affect the random bath operator V, it is clear that (5.24) also applies for the unsymmetrized form of the the random–matrix coupling. Moreover, the preceding arguments may be easily generalized to systems other than the harmonic oscillator, by requiring that $|Q_{nn_1}|^2$ vanishes unless the states n_1 and n are close in energy and by identifying ω with the mean level spacing.

Positivity

By employing an approach to quantum dissipation based on quantum dynamical semigroups, Lindblad has shown that the general form of a Markovian master equation, which satisfies complete positivity, is given by [Lin76a]

$$\frac{d\rho_S(t)}{dt} = -i[H, \rho_S(t)] - \frac{1}{2} \sum_k \left(L_k^\dagger L_k \rho_S(t) + \rho_S(t) L_k^\dagger L_k - 2 L_k \rho_S(t) L_k^\dagger \right) \; , \tag{5.25}$$

where L_k are operators acting in the Hilbert space of the system S. For the harmonic oscillator, Lindblad also proved that Markovian master equations cannot simultaneously fulfill the following three conditions [Lin76b]: (i) The reduced density operator is positive definite for all times $t > 0$; (ii) for $t \to \infty$, the reduced density operator attains thermodynamic equilibrium; (iii) in the classical limit, the equation is equivalent to the Langevin equation. Upon rewriting Eq. (2.30) in the following form

$$\begin{aligned} \frac{d\hat\rho_S(t)}{dt} = &- i\omega \, [a^\dagger a, \hat\rho_S(t)] - \gamma n_{th} \left(a a^\dagger \hat\rho_S(t) + \hat\rho_S(t) a a^\dagger - 2 a^\dagger \hat\rho_S(t) a \right) \\ &- \gamma (n_{th} + 1) \left(a^\dagger a \hat\rho_S(t) + \hat\rho_S(t) a a^\dagger - 2 a \hat\rho_S(t) a^\dagger \right) \; , \end{aligned} \tag{5.26}$$

we recognize immediately that the RWA master equation is of the Lindblad type with $L_1 \sim a$ and $L_2 \sim a^\dagger$. This guarantees the positivity of the reduced density operator. As a consequence, the quantum–classical correspondence with the Langevin equation is lost, however, since as shown by Eq. (5.16), the RWA equation leads to thermal equilibrium (cf. also the discussion in Section 6.3.1).

We now examine if the SRM master equation (5.12) is of the Lindblad form. By a transformation similar to that used in Eq. (5.25), Eq. (5.12) can be restated as

$$\begin{aligned}\frac{d\overline{\rho}_S(t)}{dt} &= - i\omega\,[a^\dagger a, \overline{\rho}_S(t)] - \gamma n_{th}\Big(aa^\dagger\overline{\rho}_S(t) + \overline{\rho}_S(t)aa^\dagger - 2\mathcal{P}a^\dagger\overline{\rho}_S(t)a\Big) \\ &- \gamma(n_{th}+1)\Big(a^\dagger a\overline{\rho}_S(t) + \overline{\rho}_S(t)aa^\dagger - 2\mathcal{P}a\overline{\rho}_S(t)a^\dagger\Big)\ . \end{aligned} \tag{5.27}$$

Eq. (5.27) clearly shows that the SRM equation is not of the form (5.25). This implies that the reduced density operator may become negative for some particular initial conditions. As we will see in the next section, this is also the case for the Caldeira–Leggett master equation.

5.2.2 Case (I): Non-symmetrized coupling

Let us consider the averaged master equation (4.23) that corresponds to the unsymmetrized random–matrix coupling. We first evaluate the generalized transition probabilities of Eq. (4.24). We proceed in complete analogy to case (II). We find, for instance,

$$W^{(1)}_{nn_1n_1n_2} = 2\pi A_0\, Q_{nn_1}Q_{n_1n_2}e^{\frac{\beta}{2}(E_{n_2}-E_{n_1})}e^{-\frac{(E_{n_2}-E_{n_1})^2}{2\Delta^2}}\ . \tag{5.28}$$

Using the explicit form (5.5) of the matrix elements of Q, we get

$$\begin{aligned} W^{(1)}_{nn-1n-1n} &= W_0 n e^{\frac{\beta\omega}{2}}\ , & W^{(1)}_{nn+1n+1n+2} &= W_0\sqrt{(n+1)(n+2)}e^{\frac{\beta\omega}{2}}\ , \\ W^{(1)}_{nn+1n+1n} &= W_0(n+1)e^{-\frac{\beta\omega}{2}}\ , & W^{(1)}_{nn-1n-1n-2} &= W_0\sqrt{n(n-1)}e^{-\frac{\beta\omega}{2}}\ , \end{aligned} \tag{5.29}$$

where W_0 is given by Eq. (5.7). Proceeding analogously for the other transition probabilities and inserting the result into the master equation, we obtain

$$\begin{aligned} &\frac{d}{dt}\langle n|\overline{\rho}_S(t)|n'\rangle = -i\omega(n-n')\langle n|\overline{\rho}_S(t)|n'\rangle) \\ &-\gamma n_{th}\left(\sqrt{n(n-1)}\,\langle n-2|\overline{\rho}_S|n'\rangle + (n+1)\langle n|\overline{\rho}_S|n'\rangle\right) \\ &-\gamma(n_{th}+1)\left(n\langle n|\overline{\rho}_S|n'\rangle + \sqrt{(n+1)(n+2)}\,\langle n+2|\overline{\rho}_S|n'\rangle\right) \\ &-\gamma n_{th}\left(\sqrt{n'(n'+1)}\,\langle n|\overline{\rho}_S|n'-2\rangle + (n'+1)\langle n|\overline{\rho}_S|n'\rangle\right) \\ &-\gamma(n_{th}+1)\left(n'\langle n|\overline{\rho}_S|n'\rangle + \sqrt{(n'+1)(n'+2)}\,\langle n|\overline{\rho}_S|n'+2\rangle\right) \\ &+\gamma n_{th}\left(\sqrt{nn'}\langle n-1|\overline{\rho}_S|n'-1\rangle + \sqrt{n(n'+1)}\langle n-1|\overline{\rho}_S|n'+1\rangle\right) \\ &+\gamma(n_{th}+1)\left(\sqrt{(n+1)n'}\langle n+1|\overline{\rho}_S|n'-1\rangle\right) \end{aligned}$$

$$
\begin{aligned}
&+\gamma n_{th}\left(\sqrt{nn'}\langle n-1|\overline{\rho}_S|n'-1\rangle+\sqrt{(n+1)n'}\langle n+1|\overline{\rho}_S|n'-1\rangle\right)\\
&+\gamma(n_{th}+1)\left(\sqrt{n(n'+1)}\langle n-1|\overline{\rho}_S|n'\rangle\right)\\
&+2\gamma(n_{th}+1)\left(\sqrt{(n+1)(n'+1)}\langle n+1|\overline{\rho}_S|n'+1\rangle\right)\ .
\end{aligned}
\tag{5.30}
$$

It is easy to check that this result coincides with the master equation (2.15) in energy representation derived (without RWA) by Agarwal for a harmonic oscillator linearly coupled to a bath of harmonic oscillators,

$$
\begin{aligned}
\frac{d\overline{\rho}_S}{dt} &= -i\omega[a^\dagger a,\overline{\rho}_S]\\
&- \gamma\left(a^\dagger a\overline{\rho}_S-2a\overline{\rho}_S a^\dagger+\overline{\rho}_S a^\dagger a+a^2\overline{\rho}_S-a\overline{\rho}_S a-a^\dagger\overline{\rho}_S a^\dagger+\overline{\rho}_S {a^\dagger}^2\right)\\
&- \gamma n_{th}\left(2[a^\dagger,[a,\overline{\rho}_S]]+[a^\dagger,[a^\dagger,\overline{\rho}_S]]+[a,[a,\overline{\rho}_S]]\right)\ .
\end{aligned}
\tag{5.31}
$$

Besides the master equation, there is an important relation which expresses the close relationship existing between relaxation and fluctuations at equilibrium. This fluctuation–dissipation relation will be the subject of the following section.

5.2.3 Fluctuation–Dissipation Theorem and High–Temperature Limit

Upon substituting $a=\left(\sqrt{M\omega}\,x+i/\sqrt{M\omega}\,p\right)/\sqrt{2}$ and its Hermitian conjugate in (5.31), we find the Caldeira–Leggett master equation [Cal83a, Cal89] (see also [Koh97])

$$
\frac{d\overline{\rho}_S}{dt}=-i\omega\left[H_S,\overline{\rho}_S\right]-i\gamma\left[x,\{p,\overline{\rho}_S\}\right]-D\left[x,[x,\overline{\rho}_S]\right]\ ,
\tag{5.32}
$$

where the diffusion constant D is given by

$$
D=\gamma M\omega\coth\left(\frac{\beta\omega}{2}\right)\ .
\tag{5.33}
$$

On the other hand, using Eq. (5.7), the damping coefficient γ can be written as

$$
\gamma=\frac{A_0\pi}{M\omega}e^{-\frac{\omega^2}{2\Delta^2}}\sinh\frac{\beta}{2}\omega\ .
\tag{5.34}
$$

We see that a small bandwidth Δ tends to exponentially decrease the damping coefficient. In view of our *ansatz* for the interaction between system and bath, this is not surprising. Eqs. (5.33) and (5.34) are both a statement of the fluctuation–dissipation theorem between the friction coefficient γ and the diffusion coefficient D. To be consistent, the two expressions should coincide. We find that this is the case only in the limit of large bandwidth, $\omega\ll\Delta$ and in the limit of high temperatures, $\beta\omega\ll 1$. In these limits, Eqs. (5.33) and (5.34) become identical and they reduce to the Einstein relation

$$
\gamma=\frac{D}{2MkT}\ .
\tag{5.35}
$$

Here the diffusion constant D is determined by the strength of the coupling,

$$
D=A_0\pi\ .
\tag{5.36}
$$

We have thus demonstrated, by a self–consistency argument, that the averaged master equation (5.31) is valid in the large–bandwidth, high–temperature limit, $\omega \ll \Delta$ and $\omega \ll kT$, in agreement with Eq. (5.24) [2].

We finally mention that the Caldeira–Leggett equation (5.32) does not have the Lindblad form (5.25) [Dio93]. This implies that the positivity of the reduced density operator can be violated for certain initial states. In Ref. [Pei98], it was shown that positivity is guaranteed provided the dispersion $\sigma_{xx} = \langle x^2\rangle - \langle x\rangle^2$ of the initial wave packet obeys the condition

$$\sigma_{xx} \geq {\lambda_{dB}}^2 \ . \tag{5.37}$$

5.3 Second Application: Two–level system

As a second illustration, we consider a two–level system with upper (lower) level $|+\rangle$ ($|-\rangle$, respectively) and level separation ω_0. The corresponding Hamiltonian of the system is given by Eq. (2.24). We write

$$E_n = \frac{1}{2}\omega_0 n, \qquad n = \pm 1 \ , \tag{5.38}$$

and take $Q = \sigma_x$. We first derive the averaged master equation for the non–symmetrized coupling.

5.3.1 Case (I): Non-symmetrized coupling

We begin with Eq. (4.23) and proceed as in Section 5.2 for the damped harmonic oscillator. For the generalized transition probabilities we find

$$\begin{aligned} W^{(1)}_{+--+} &= W_0\, e^{\frac{\beta}{2}\omega_0} \ , \\ W^{(1)}_{-++-} &= W_0\, e^{-\frac{\beta}{2}\omega_0} \ , \end{aligned} \tag{5.39}$$

and similar expressions for $W^{(2)}$, $W^{(3)}$ and $W^{(4)}$. Here W_0 is given by Eq. (5.7) with ω replaced by ω_0 and M set to unity. Thus,

$$\begin{aligned} \frac{d}{dt}\langle +|\overline{\rho}_S|+\rangle &= -\frac{1}{2}W_0\, e^{\frac{\beta}{2}\omega_0}\langle +|\overline{\rho}_S|+\rangle - \frac{1}{2}W_0\, e^{\frac{\beta}{2}\omega_0}\langle +|\overline{\rho}_S|+\rangle \\ &\quad + \frac{1}{2}e^{-\frac{\beta}{2}\omega_0}\langle -|\overline{\rho}_S|-\rangle + \frac{1}{2}e^{-\frac{\beta}{2}\omega_0}\langle -|\overline{\rho}_S|-\rangle \\ &= -2\gamma(n_{th}+1)\langle +|\overline{\rho}_S|+\rangle + 2\gamma n_{th}\langle -|\overline{\rho}_S|-\rangle \ , \end{aligned} \tag{5.40}$$

and, in a similar way,

$$\begin{aligned} \frac{d}{dt}\langle -|\overline{\rho}_S|-\rangle &= -2\gamma n_{th}\langle -|\overline{\rho}_S|-\rangle + 2\gamma(n_{th}+1)\langle +|\overline{\rho}_S|+\rangle \ , \\ \frac{d}{dt}\langle -|\overline{\rho}_S|+\rangle &= i\omega_0\langle -|\overline{\rho}_S|+\rangle - \gamma(2n_{th}+1)\Big[\langle -|\overline{\rho}_S|+\rangle - \langle +|\overline{\rho}_S|-\rangle\Big] \ , \\ \frac{d}{dt}\langle +|\overline{\rho}_S|-\rangle &= -i\omega_0\langle +|\overline{\rho}_S|-\rangle - \gamma(2n_{th}+1)\Big[\langle +|\overline{\rho}_S|-\rangle - \langle -|\overline{\rho}_S|+\rangle\Big] \ . \end{aligned} \tag{5.41}$$

[2] One should mention that the Caldeira–Leggett master equation has been shown to be valid even at low temperatures for a weakly damped oscillator, $\gamma \ll \omega$ [Cal89].

In the previous equations, γ and n_{th} are given by Eq. (5.10) and (5.11), respectively, with ω replaced by ω_0 and M set to unity. We introduce the raising and lowering operators σ_+ and σ_- which obey the relations (2.25). Simple manipulations with these spin matrices, together with the closure relation

$$\overline{\rho}_S = \sum_{\{m,n\}=\{+,-\}} |m\rangle\langle m|\overline{\rho}_S|n\rangle\langle n| \ , \tag{5.42}$$

show that the master equation for the dissipative two–level system may be written as

$$\begin{aligned}
\frac{d}{dt}\overline{\rho}_S &= -2\gamma(n_{th}+1)\,\sigma_+\sigma_-\overline{\rho}_S\sigma_+\sigma_- + 2\gamma n_{th}\,\sigma_+\overline{\rho}_S\sigma_- \\
&\quad -i\omega_0\,\sigma_+\sigma_-\overline{\rho}_S\sigma_-\sigma_+ - \gamma(2n_{th}+1)\Big[\sigma_+\sigma_-\overline{\rho}_S\sigma_-\sigma_+ - \sigma_+\overline{\rho}_S\sigma_+\Big] \\
&\quad +i\omega_0\,\sigma_-\sigma_+\overline{\rho}_S\sigma_+\sigma_- - \gamma(2n_{th}+1)\Big[\sigma_-\sigma_+\overline{\rho}_S\sigma_+\sigma_- - \sigma_-\overline{\rho}_S\sigma_-\Big] \\
&\quad -2\gamma n_{th}\,\sigma_-\sigma_+\overline{\rho}_S\sigma_-\sigma_+ + 2\gamma(n_{th}+1)\,\sigma_-\overline{\rho}_S\sigma_+ \ ,
\end{aligned} \tag{5.43}$$

or, in a simpler fashion,

$$\begin{aligned}
\frac{d}{dt}\overline{\rho}_S &= -i\frac{1}{2}\omega_0\,[\sigma_z,\overline{\rho}_S] \\
&+ \gamma n_{th}\Big(2\sigma_+\overline{\rho}_S\sigma_- + \sigma_+\overline{\rho}_S\sigma_+ + \sigma_-\overline{\rho}_S\sigma_- - \overline{\rho}_S\sigma_-\sigma_+ - \sigma_-\sigma_+\overline{\rho}_S\Big) \\
&+ \gamma(n_{th}+1)\Big(\sigma_+\overline{\rho}_S\sigma_+ + \sigma_-\overline{\rho}_S\sigma_- + 2\sigma_-\overline{\rho}_S\sigma_+ - \sigma_+\sigma_-\overline{\rho}_S - \overline{\rho}_S\sigma_+\sigma_-\Big) \ .
\end{aligned} \tag{5.44}$$

For $n_{th} = 0$, Eq. (5.44) reduces to the Agarwal equation for spontaneous emission of a two–level atom [Aga71b, Aga74]. It may seem surprising that Eq. (5.44) applies for $T = 0$ where the inequality $\Delta \ll T$ is clearly violated and non-Markovian effects are present. However, it was shown in Ref. [Unr89], using a coupling to a scalar field, that the high–temperature master equation is obeyed even at low temperatures provided the time t is larger than the memory time of the bath, $t \gg t_B$. This implies that condition (ii) of Section 4.1 may be replaced by the less restrictive condition $t_B \ll t \ll t_P$ (see also Footnote 2).

5.3.2 Case (II): Symmetrized coupling

It is possible to quickly write down the SRM master equation for the dissipative two–level without any calculation. We note that Eq. (5.44) is the same as Eq. (5.30) for the damped harmonic oscillator with the substitution $a \to \sigma_-$, $a^\dagger \to \sigma_+$. A similar correspondence exists between the RWA equations for the harmonic oscillator and the two–level system [Car93]. We can therefore simply make the substitution $a \to \sigma_-$, $a^\dagger \to \sigma_+$ in Eq. (5.12) to write

$$\begin{aligned}
\frac{d\overline{\rho}_S(t)}{dt} = &-i\frac{1}{2}\omega_0\,[\sigma_z,\overline{\rho}_S] - \gamma\Big(\sigma_+\sigma_-\overline{\rho}_S(t) - 2\mathcal{D}\sigma_-\overline{\rho}_S\sigma_+ + \overline{\rho}_S(t)\sigma_+\sigma_-\Big) \\
&- 2\gamma n_{th}\Big(\sigma_+\sigma_-\overline{\rho}_S(t) - \mathcal{D}\sigma_-\overline{\rho}_S(t)\sigma_+ - \mathcal{D}\sigma_+\overline{\rho}_S(t)\sigma_- + \overline{\rho}_S(t)\sigma_-\sigma_+\Big) \ .
\end{aligned} \tag{5.45}$$

This is the SRM master equation (5.44) applied to the two–level system. As for the damped harmonic oscillator, this equation coincides with the corresponding RWA master equation except for the gain terms which are equal to zero for the off–diagonal elements.

5.4 Summary

We have applied the averaged Markovian master equations (4.18) and (4.23) to two cases, the damped harmonic oscillator and the two–level system. We have assumed that the system–bath interaction is linear in the position coordinate of the quantum system. For the damped oscillator and case (I), we have obtained the same equation (5.31) as Agarwal who considered a harmonic oscillator coupled to a heat bath which was modeled as an infinite set of harmonic oscillators. In the limit of high temperature, this equation coincides with the master equation (5.32) derived by Caldeira and Leggett. For the SRM case (II), our master equation (5.14) for the diagonal elements of the reduced density operator is identical to the Lax–Louisell master equation evaluated in the rotating wave approximation. This is because in case (II) we impose conditions upon the interaction matrix elements of the position coordinate of the quantum system. These conditions are tantamount to the RWA. For the non–diagonal elements of the reduced density operator, these same conditions imply the vanishing of the gain terms. In this point our result differs from the one obtained by Lax and Louisell. In contrast to the RWA equation, Eqs. (5.31) and (5.32) are not of the Lindblad form. This means that for initial conditions which do not satisfy condition (5.37), the reduced density operator is not positive definite for all times. The range of validity of the averaged master equations (4.18) and (4.23) is expressed by the weak–coupling condition (5.24). Although derived for the specific case of the SRM damped harmonic oscillator, this condition is completely general.

For the two–level system and case (I), our master equation (5.44) reduces, at $T = 0$, to the Agarwal equation for spontaneous emission of a two–level atom. In case (II), as for the damped oscillator, this coincidence is restricted to the diagonal matrix elements.

We conclude that Markovian master equations for quantum Brownian motion derived in the weak coupling limit possess universal validity [Lut99]: These equations are independent of the specific microscopic model used for their derivation. This is not true of approximations such as the rotating wave approximation. Typically, such approximations violate certain invariance requirements (translational invariance in the case of the RWA) and thereby lose universal validity.

Chapter 6

Symmetrized random–matrix coupling

6.1 Introduction

This chapter discusses the symmetrized form of the random–matrix interaction. As we saw in the preceding chapter when we studied the damped harmonic oscillator and the dissipative two–level system, the averaged master equation obtained using this approximated random–matrix coupling bears a strong resemblance with the master equation derived from the RWA Hamiltonian, but they are not identical. In both cases we have a decoupling of the time evolution of the diagonal and the non–diagonal matrix elements, but only the diagonal parts coincide. The non-diagonal parts differ in that there is no gain term in the random–matrix model. This is due to the form of the symmetrized variance (2.43). It is our goal here to investigate the physical consequences of such a difference. To this end, we find it convenient to use an alternative description of quantum relaxation based on the quantum Langevin equation. For simplicity we shall restrict ourselves to the damped harmonic oscillator.

Section 6.2 begins with a general presentation of the quantum Langevin equation and its relation to the master equation. In Section 6.2.1, we briefly summarize various results concerning the RWA quantum Langevin equation for the annihilation operator. This will make it possible to compare with the SRM quantum Langevin equation to be derived in Section 6.2.2. In the following section, we shall express both Langevin equations in terms of position and momentum of the system S. We will then consider their classical limit by using the Weyl transform. This will permit us, in particular, to get some new insight into the loss of correspondence of the RWA equation with the classical Langevin equation for Brownian motion, mentioned at the end of Section 5.2.1.

6.2 Quantum Langevin equation

Just as there are two fundamental ways of describing the evolution of an isolated quantum system, the Schödinger picture and the Heisenberg picture, there are two basic ways to deal with a quantum system in contact with a heat bath [Mey91, Coh92]. The first one is based on the Schödinger picture and leads to an equation for the density operator, the *master equation.* This is the approach we have used so far. The second one is based on the

Heisenberg picture and leads to an equation for a given quantum operator which is similar to the classical Langevin equation. In particular, it contains a "quantum noise operator" akin to the classical fluctuating force. For this reason, this equation is called *quantum Langevin equation.* It has been shown recently [Kam97] that both descriptions coincide for the oscillator bath model. This means that, although each approach individually involves approximations, the resulting equations — the master equation and the Langevin equation — contain the same physical information about the behavior of the quantum system. This will be illustrated and emphasized in the coming section.

6.2.1 RWA coupling

We consider a quantum harmonic oscillator of mass M and frequency ω coupled to a bath of harmonic oscillators according to the rotating wave Hamiltonian (2.31). The Heisenberg equation of motion for the annihilation operator $a(t)$ is written as

$$\dot{a}(t) = i[H, a(t)] = -i\omega a(t) - i\sum_i \kappa_i a_i(t) \ . \tag{6.1}$$

After eliminating the bath operators $a_i(t)$ and performing the Markov approximation, Eq. (6.1) becomes [Mey91, Coh92] (see also [Aga73])

$$\dot{a}(t) = -i\omega a(t) - \gamma a(t) + \hat{F}_{\mathrm{RWA}}(t) \ . \tag{6.2}$$

This is the quantum Langevin equation obeyed by the operator $a(t)$. The relaxation constant γ is defined in Eq. (5.10) and the noise operator $\hat{F}_{\mathrm{RWA}}(t)$ is a delta correlated complex Gaussian random operator with the properties

$$\begin{aligned} &\langle \hat{F}_{\mathrm{RWA}}(t)\rangle = \langle \hat{F}^\dagger_{\mathrm{RWA}}(t)\rangle = \langle \hat{F}_{\mathrm{RWA}}(t)\hat{F}_{\mathrm{RWA}}(t')\rangle = \langle \hat{F}^\dagger_{\mathrm{RWA}}(t)\hat{F}^\dagger_{\mathrm{RWA}}(t')\rangle = 0 \ , \\ &\langle \hat{F}(t)^\dagger_{\mathrm{RWA}}\hat{F}_{\mathrm{RWA}}(t')\rangle = 2D_{\mathrm{RWA}}\,\delta(t-t') \ , \end{aligned} \tag{6.3}$$

where $D_{\mathrm{RWA}} = \gamma\, n_{th}$ is the diffusion constant. In Eq. (6.3) the symbol $\langle\ \rangle$ denotes the thermal average of a given system operator in the Heisenberg picture,

$$\langle O(t)\rangle = \frac{1}{Z}\,\mathrm{tr}_B\Big(O(t)\exp(-H_B/kT)\Big) \ . \tag{6.4}$$

We can now use the quantum Langevin equation (6.2) to compute the evolution of the expectation value of the operator $a(t)$. Since the mean value of $\hat{F}_{\mathrm{RWA}}(t)$ is equal to zero, we readily find

$$\frac{d}{dt}\langle a(t)\rangle = -i\omega\langle a(t)\rangle - \gamma\langle a(t)\rangle \ . \tag{6.5}$$

Here we see that the mean value of the annihilation operator tends exponentially towards zero with a relaxation time $t_R = 1/\gamma$

$$\langle a(t)\rangle = \langle a(0)\rangle e^{-(i\omega+\gamma)t} \ . \tag{6.6}$$

One can also calculate the expectation of the number operator $a^\dagger(t)a(t)$ to obtain

$$\frac{d}{dt}\langle a^\dagger(t)a(t)\rangle = -2\gamma\,\langle a^\dagger(t)a(t)\rangle + 2\gamma\, n_{th} \ . \tag{6.7}$$

A useful way to interpret this result is to reexpress it as

$$\frac{d}{dt}\langle a^\dagger(t)a(t)\rangle = -2\gamma\,(n_{th}+1)\ \langle a^\dagger(t)a(t)\rangle + 2\gamma\, n_{th}\left(\langle a^\dagger(t)a(t)\rangle + 1\right)\ . \tag{6.8}$$

The rate of change of the mean number $\langle a^\dagger(t)a(t)\rangle$ is seen to result from the balance between emission from the system into the bath and from the bath into the system (cf. the discussion of the diagonal part of the RWA master equation in Section 5.2.1). Equation (6.7) can be solved easily to give

$$\langle a^\dagger(t)a(t)\rangle = \langle a^\dagger(0)a(0)\rangle\, e^{-2\gamma t} + n_{th}(1-e^{-2\gamma t})\ . \tag{6.9}$$

Thus, for large times, the average number of quanta of the harmonic oscillator reaches the equilibrium value $n_{th} = (\exp(\omega/kT)-1)^{-1}$.

One may also use the master equation satisfied by the reduced density operator $\hat\rho_S(t)$ to compute these expectation values. Indeed, let O be any system operator in the Schrödinger picture and let $\langle O(t)\rangle$ be its average value at time t. Then,

$$\langle O(t)\rangle = \mathrm{tr}_S\left(O\hat\rho_S(t)\right)\ . \tag{6.10}$$

We note that in the Heisenberg picture Eq. (6.4), the operator O is time dependent, while in the Schrödinger picture Eq. (6.10), the time dependence is carried by the density operator. One can easily verify, using the RWA master equation (2.31) for $\hat\rho_S(t)$, that the resulting expressions for the mean values of the operators a and $a^\dagger a$ are identical to Eqs. (6.5) and (6.6). The object of the next section is to make use of this important remark to obtain the SRM Langevin equation from the averaged master equation (5.8).

6.2.2 SRM coupling

Our purpose is to derive the averaged Markovian quantum Langevin equation for the annihilation operator a, corresponding to the symmetrized form of the random–matrix coupling (2.43). To this end, we shall not start from the random–matrix Hamiltonian (2.36), but instead we will use the averaged Markovian master equation (5.8) obtained in Section 5.2. As we learned in the preceding section, the two approaches are completely equivalent. However, the advantage of the latter method clearly lies in the fact that the average over the random–matrix ensemble has already been performed. In analogy to the RWA Eqs. (6.2) and (6.3), we shall assume that the SRM Markovian quantum Langevin equation obeyed by a is of the general form

$$\dot a(t) = -i\omega a(t) - \gamma a(t) + \hat F_{\mathrm{SRM}}(t)\ , \tag{6.11}$$

where $\hat F_{\mathrm{SRM}}(t)$ is a delta correlated noise operator satisfying

$$\begin{aligned} \langle \hat F_{\mathrm{SRM}}(t)\hat F_{\mathrm{SRM}}(t')\rangle &= \langle \hat F^\dagger_{\mathrm{SRM}}(t)\hat F^\dagger_{\mathrm{SRM}}(t')\rangle = 0\ ,\\ \langle \hat F(t)^\dagger_{\mathrm{SRM}}\hat F_{\mathrm{SRM}}(t')\rangle &= 2D_{\mathrm{SRM}}\,\delta(t-t')\ . \end{aligned} \tag{6.12}$$

The mean value $\langle \hat F_{\mathrm{SRM}}(t)\rangle$ of the noise operator and the diffusion constant D_{SRM} are the two quantities we wish to determine. We use Eq. (5.8) to calculate the time derivative of the thermal average of the annihilation and number operators

$$\begin{aligned}
\langle a(t)\rangle &= \mathrm{tr}_S\left(a\,\overline{\rho}_S(t)\right) = \sum_n \sqrt{n+1}\,\langle n+1|\overline{\rho}_S(t)|n\rangle\ , \\
\langle a^\dagger a(t)\rangle &= \mathrm{tr}_S\left(a^\dagger a\,\overline{\rho}_S(t)\right) = \sum_n n\,\langle n|\overline{\rho}_S(t)|n\rangle\ .
\end{aligned} \tag{6.13}$$

This leads to

$$\begin{aligned}
\frac{d}{dt}\langle a(t)\rangle = &-(i\omega+\gamma)\langle a(t)\rangle + 4\gamma\, n_{th}\langle a(t)\rangle \\
&- 2\gamma\,(2n_{th}+1)\sum_n n\,\sqrt{n+1}\,\langle n+1|\overline{\rho}_S(t)|n\rangle\ ,
\end{aligned} \tag{6.14}$$

and

$$\frac{d}{dt}\langle a^\dagger a(t)\rangle = -2\gamma\,\langle a^\dagger a(t)\rangle + 2\gamma\, n_{th}\ . \tag{6.15}$$

At this stage it is instructive to compare these two equations with the corresponding RWA equations (6.5) and (6.7). We first see that the SRM equation (6.15) for the number operator is identical with the RWA equation (6.7). This is not surprising since this quantity depends only on the diagonal matrix elements of $\overline{\rho}_S$ (see Eq. (6.13)) and we already know that these matrix elements coincide in both cases. It also shows that the SRM harmonic oscillator reaches thermal equilibrium at the same rate as the RWA oscillator. On the other hand, Eq. (6.14), which depends on the non–diagonal matrix elements of $\overline{\rho}_S$, is different from its RWA counterpart. In addition to the first term, which corresponds to the drift operator $\hat{D}_{\mathrm{RWA}}(t) = -(i\omega+\gamma)a(t)$, there is a term which does not appear in Eq. (6.5). This term is equal to the mean value of the noise operator $\hat{F}_{\mathrm{SRM}}(t)$ (we recall that $\langle\hat{F}_{\mathrm{RWA}}(t)\rangle$ vanishes). We therefore conclude that the absence of gain terms in the coherence of the reduced density operator leads to a non–zero centered noise operator. This has the consequence that the random process described by the SRM quantum Langevin equation is non–stationary, since the mean value is explicitly time dependent.

We now proceed to evaluate the mean of the operator $\hat{F}_{\mathrm{SRM}}(t)$. We begin by calculating the time dependence of the last term on the r.h.s. of (6.14),

$$h(t) = 2\gamma\,(2n_{th}+1)\sum_n n\,\sqrt{n+1}\,\langle n+1|\overline{\rho}_S(t)|n\rangle\ . \tag{6.16}$$

Since the diagonal and off–diagonal matrix elements of the master equation do not mix, this can be done exactly. The matrix elements $\langle n+1|\overline{\rho}_S(t)|n\rangle$ satisfy the simple differential equation

$$\frac{d}{dt}\langle n+1|\overline{\rho}_S(t)|n\rangle = -\Big(i\omega+\gamma(2n+1)+4\gamma n_{th}(n+1)\Big)\langle n+1|\overline{\rho}_S(t)|n\rangle\ . \tag{6.17}$$

which has the solution

$$\langle n+1|\overline{\rho}_S(t)|n\rangle = \langle n+1|\overline{\rho}_S(0)|n\rangle\ \exp\Big[-\Big(i\omega+\gamma(4n_{th}+1)+2\gamma n(2n_{th}+1)\Big)t\Big]\ . \tag{6.18}$$

By substituting this result into Eq. (6.16), we find that

$$h(t) = 2\gamma\,(2n_{th}+1)\langle n+1|\overline{\rho}_S(0)|n\rangle\ \exp\Big[-\Big(i\omega+\gamma(4n_{th}+1)\Big)t\Big]\ u\Big(2\gamma(2n_{th}+1)t\Big)\ , \tag{6.19}$$

where the function $u(y)$ is given by

$$u(y) = \int_0^\infty dn\, \langle n+1|\overline{\rho}_S(0)|n\rangle\, n\, \sqrt{n+1}\, e^{-yn}\ . \tag{6.20}$$

Combining Eqs. (6.19) and (6.20), we can write the solution of Eq. (6.14) in the form

$$\begin{aligned} \langle a(t)\rangle &= \exp\Big[-\Big(i\omega + \gamma(4n_{th}+1)\Big)t\Big] \\ &\times \Big(\langle a(0)\rangle - 2\gamma(2n_{th}+1)\int_0^t dt' u\Big(2\gamma(2n_{th}+1)t'\Big)\Big)\ , \end{aligned} \tag{6.21}$$

and we get the following expression for the mean value of $\hat{F}_{\text{SRM}}(t)$

$$\langle \hat{F}_{\text{SRM}}(t)\rangle = -\gamma 4 n_{th} \langle a(t)\rangle - h(t)\ . \tag{6.22}$$

Let us now give an estimate of the function $u(y)$ in Eq. (6.20). We assume that, initially, we have a wave packet localized at some point $n = n_0$ in the spectrum. We write

$$\langle n+1|\overline{\rho}_S(0)|n\rangle \sim \exp\left(-\frac{(n-n_0)^2}{r^2}\right)\ . \tag{6.23}$$

We further assume that the overlap of this Gaussian with the exponential $\exp(-yn)$ in (6.20) is negligible. This is for instance the case if $n_0 = 100$, $r = 4$. Then, $u(y)$ is very small and can be set to zero for all practical purposes. Eqs. (6.21) and (6.22) may thus be rewritten as

$$\langle a(t)\rangle = \langle a(0)\rangle\ \exp\Big[-\Big(i\omega + \gamma(4n_{th}+1)\Big)t\Big]\ , \tag{6.24}$$

and

$$\langle \hat{F}_{\text{SRM}}(t)\rangle = -\gamma 4 n_{th} \langle a(t)\rangle\ . \tag{6.25}$$

Discussion

The effect of the lack of off–diagonal gain terms in the master equation (5.8) on the dynamics of the SRM oscillator is twofold. First, we note from Eq. (6.18) that the non–diagonal matrix elements of $\overline{\rho}_S$ decay exponentially on a time scale given by $t_D = (\gamma(4n_{th}+1) + 2\gamma n(2n_{th}+1))^{-1}$. This shows that the reduced density operator becomes diagonal in the energy basis as a consequence of the coupling to the environment. This phenomenon is known as decoherence and t_D is called the decoherence time [Zur81, Zur91, Giu96]. We see that t_D is inversely proportional to the state n and to the temperature T of the heat bath. This implies that t_D is in general much smaller than the relaxation time $t_R = \gamma^{-1}$. In contradistinction, the RWA oscillator becomes diagonal in the coherent–state basis [Wal85]. The difference is of course due to the form (2.43) of the approximated variance which singles out the energy eigenbasis of the SRM oscillator. The second effect, as already indicated above, is that the mean value of the noise operator $\hat{F}_{\text{SRM}}(t)$ is non–zero. However, in the limit of low temperatures, n_{th} and, consequently, $\langle \hat{F}_{\text{SRM}}(t)\rangle$ become vanishingly small. This shows that in this limit the RWA and SRM oscillators are identical (cf. Eqs. (6.6) and (6.24)). On the contrary, for high temperatures, (6.18) is a large quantity and cannot be omitted. As a result, the decay of the mean value of the annihilation operator (6.25) is much faster in the SRM case, with a decay rate rate equal to $\gamma(4n_{th}+1)$. This is approximately $4n_{th}$ times larger than for the RWA oscillator. It is interesting to compare this with the number operator. As we have already seen, the decay rate of $\langle a^\dagger(t)a(t)\rangle$ is the same in both cases. This means that for the SRM oscillator the annihilation operator relaxes much faster than the number operator (see Figs. 6.1 and 6.2). This is a consequence of the fact that $\langle a(t)\rangle$ is determined by the mean value of the noise operator, whereas $\langle a^\dagger(t)a(t)\rangle$ depends on its variance, and that these two quantities can be specified independently.

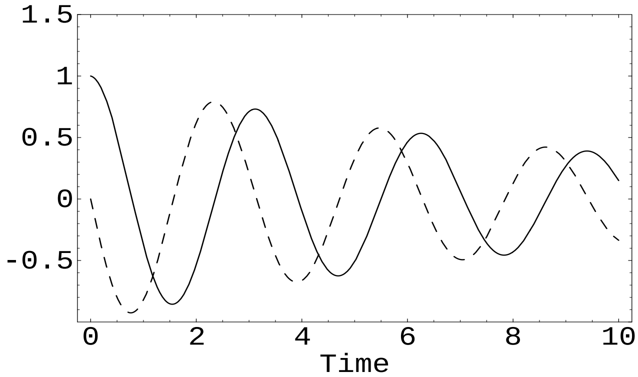

Figure 6.1: RWA oscillator: $\langle x(t)\rangle/\langle x(0)\rangle$ (continuous line) and $\langle p(t)\rangle/\langle x(0)\rangle$ (dashed line) for $\langle p(0)\rangle = 0$, $M = 1$, $\omega = 2$, $\gamma = 0,1$.

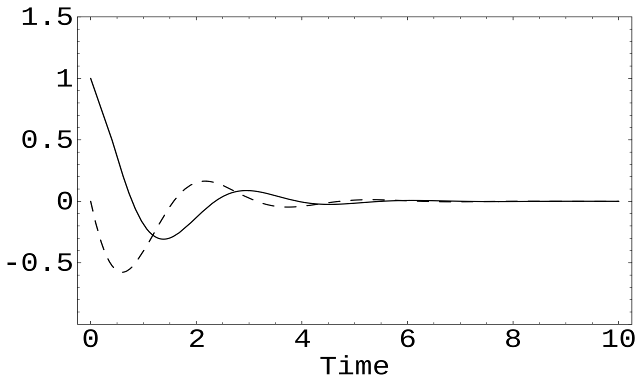

Figure 6.2: SRM oscillator: $\langle x(t)\rangle/\langle x(0)\rangle$ (continuous line) and $\langle p(t)\rangle/\langle x(0)\rangle$ (dashed line) for $\langle p(0)\rangle = 0$, $M = 1$, $\omega = 2$, $\gamma = 0,1$, $n_{th} = 8$.

Finally, we calculate the SRM diffusion constant D_{SRM}. We do this by identifying the time derivative of $\langle a^\dagger a\rangle$ obtained from the Langevin equation (6.15), with the known result Eq. (6.13). We use Eq. (6.15) and the identity

$$a(t) = a(t-\Delta t) + \int_{t-\Delta t}^{t} dt'\, a(t') \tag{6.26}$$

to obtain the correlation function

$$\begin{aligned}\langle \hat{F}^\dagger_{\text{SRM}}(t)a(t)\rangle &= \langle \hat{F}^\dagger_{\text{SRM}}(t)a(t-\Delta t)\rangle \\ &+ \int_{t-\Delta t}^{t} dt' \left(\langle \hat{F}^\dagger_{\text{SRM}}(t)\hat{D}_{\text{RWA}}(t')\rangle + \langle \hat{F}^\dagger_{\text{SRM}}(t)\hat{F}_{\text{SRM}}(t')\rangle\right) \; .\end{aligned} \tag{6.27}$$

Since the operator $a(t')$ at time t' cannot be affected by a fluctuation at a later time t, the first term on the r.h.s. of (6.27) is zero. Similarly, the correlation $\langle \hat{F}^{\dagger}_{\mathrm{SRM}}(t)\hat{D}_{\mathrm{RWA}}(t')\rangle$ is zero except at the point $t = t'$ which is a set of measure zero. We are hence left with [Mey91]

$$\langle \hat{F}^{\dagger}_{\mathrm{SRM}}(t)a(t)\rangle = \int_{t-\Delta t}^{t} dt' \, \langle \hat{F}^{\dagger}_{\mathrm{SRM}}(t)\hat{F}_{\mathrm{SRM}}(t')\rangle = D_{\mathrm{SRM}} \ , \tag{6.28}$$

where Eq. (6.12) have been used. In an similar fashion, one finds

$$\langle a^{\dagger}(t)\hat{F}_{\mathrm{SRM}}(t)\rangle = D_{\mathrm{SRM}} \ . \tag{6.29}$$

We further have

$$\frac{d}{dt}a^{\dagger}(t)a(t) = \frac{da^{\dagger}(t)}{dt}a(t) + a^{\dagger}(t)\frac{da(t)}{dt} \ . \tag{6.30}$$

Taking the average on both sides of (6.30) and using (6.15), we find

$$\begin{aligned} \frac{d}{dt}\langle a^{\dagger}(t)a(t)\rangle &= -2\gamma\langle a^{\dagger}(t)a(t)\rangle + \langle \hat{F}^{\dagger}_{\mathrm{SRM}}(t)a(t)\rangle + \langle a^{\dagger}(t)\hat{F}_{\mathrm{SRM}}(t)\rangle \\ &= -2\gamma\langle a^{\dagger}(t)a(t)\rangle + 2D_{\mathrm{SRM}} \ . \end{aligned} \tag{6.31}$$

By direct comparison with Eq. (6.13), we infer that the SRM and RWA diffusion constants are equal

$$D_{\mathrm{SRM}} = D_{\mathrm{RWA}} = \gamma n_{th} \ . \tag{6.32}$$

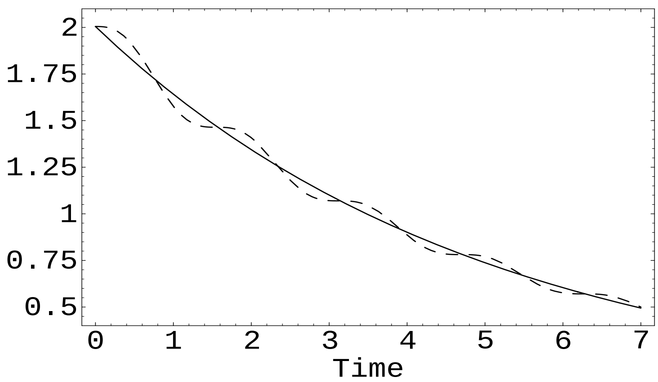

Figure 6.3: Mean energy $\langle E(t)\rangle$. RWA/SRM oscillator (continuous line), FC oscillator (dashed line) for $\langle p(0)\rangle = 0$, $M = 1$, $\omega = 2$, $\gamma = 0,1$.

6.3 Classical Langevin equation

In this section, we focus on the classical aspect of the quantum Langevin equation. The Weyl transform (2.18) will serve as our main tool to obtain the classical analogue of a quantum equation. Again, we shall begin our discussion with the RWA oscillator.

6.3.1 RWA coupling

One of the interesting features of the Heisenberg picture is that the equations of motion look like the corresponding classical equations of motion. This can be easily demonstrated by expressing these equations in terms of the position and momentum operators x and p

$$x = \sqrt{\frac{1}{2M\omega}}\left(a^\dagger + a\right) \qquad\qquad p = i\sqrt{\frac{M\omega}{2}}\left(a^\dagger - a\right) \ . \tag{6.33}$$

From Eq. (6.2) together with its Hermitian conjugate, one obtains

$$\dot{x} = \frac{p}{M} - \gamma x + \hat{F}^x_{\mathrm{RWA}}(t) \qquad \text{and} \qquad \dot{p} = -M\omega^2 x - \gamma p + \hat{F}^p_{\mathrm{RWA}}(t) \ . \tag{6.34}$$

In Eq. (6.34), we have introduced $\hat{F}^x_{\mathrm{RWA}}(t) = (2/M\omega)^{\frac{1}{2}}$ Re $\hat{F}_{\mathrm{RWA}}(t)$ and $\hat{F}^p_{\mathrm{RWA}}(t) = (2M\omega)^{\frac{1}{2}}$ Im $\hat{F}_{\mathrm{RWA}}(t)$, two independent real Gaussian random operators with zero means and variance

$$\begin{aligned}
\langle \hat{F}^x_{\mathrm{RWA}}(t)\hat{F}^x_{\mathrm{RWA}}(t')\rangle &= \frac{2\gamma n_{th}}{M\omega}\,\delta(t-t'),\\
\langle \hat{F}^p_{\mathrm{RWA}}(t)\hat{F}^p_{\mathrm{RWA}}(t')\rangle &= 2\gamma n_{th} M\omega\,\delta(t-t') \ .
\end{aligned} \tag{6.35}$$

As announced, the quantum Langevin equations (6.34) for x and p have the same form as their classical counterparts, but they are still operator equations. They may be converted into equations for ordinary variables by an appropriate phase–space transformation $\{(x)_{op} \to x \ , \ (p)_{op} \to p\}$, for instance the Weyl transformation. This yields

$$\dot{x} = \frac{p}{M} - \gamma x + F^x_{\mathrm{RWA}}(t) \qquad \text{and} \qquad \dot{p} = -M\omega^2 x - \gamma p + F^p_{\mathrm{RWA}}(t) \ , \tag{6.36}$$

where $F^x_{\mathrm{RWA}}(t)$ and $F^p_{\mathrm{RWA}}(t)$ are now two independent real Gaussian random functions with zero mean and

$$\begin{aligned}
\langle F^x_{\mathrm{RWA}}(t)F^x_{\mathrm{RWA}}(t')\rangle &= \frac{2\gamma(n_{th}+\frac{1}{2})}{M\omega}\,\delta(t-t'),\\
\langle F^p_{\mathrm{RWA}}(t)F^p_{\mathrm{RWA}}(t')\rangle &= 2\gamma(n_{th}+\frac{1}{2})M\omega\,\delta(t-t') \ .
\end{aligned} \tag{6.37}$$

Note that in (6.37) n_{th} has been replaced by $n_{th}+1/2$. This is due to our choice of the Weyl ordering [1]. In the classical limit, these equations then reduce to the classical Langevin equation, where $n_{th} + 1/2$ has to be consistently replaced by its high temperature value kT/ω . The number quantum Langevin equations (6.36) were first derived by Agarwal [Aga71a, Aga73] from the corresponding RWA Fokker–Planck equation (2.31). Before we proceed to examine the SRM oscillator, we comment on the RWA Langevin equations.

We see from (6.36) that the two equations are symmetric in x and p: Friction and random forces are associated with momentum as well as with position. As a result, the time derivative of x is no longer equal to p/M, as it is the case for normal Brownian motion. This means in turn that the RWA Hamiltonian breaks translational invariance. One can show, however, [Aga71a] that in the limit of weak damping, $\gamma \ll \omega$, the solutions of the FC and the RWA Fokker-Planck equations become identical. Nevertheless, the presence

[1] We recall that in the Weyl ordering $(xy)_{op} \to (xy + yx)/2$.

of these additional terms in the x-equation is considered by some authors [For96, For97] as constituting a severe drawback of the rotating wave approximation as, for example, the Ehrenfest theorem can never be satisfied.

From a mathematical point of view, it is quite clear how this violation of translation invariance comes about. The suppression of the terms aa_i, $a^\dagger a_i^\dagger$ in the rotating wave approximation transforms a coupling, originally linear in the position x thus symmetric in a and $a^\dagger$, in a coupling symmetric in x and p. But, physically, it is less transparent how the omission of non–resonant terms which induce virtual (i.e. non–energy conserving) transitions can break translational invariance. A more appealing explanation can be found within the random–matrix model, where a similar violation occurs.

6.3.2 SRM coupling

We now turn to the discussion of the SRM case. In complete analogy to the preceding section, we first express the SRM quantum Langevin equation (6.11) in terms of the position and momentum operators and then take the Weyl transformation. We find

$$\dot{x} = \frac{p}{M} - \gamma x + F^x_{\text{SRM}}(t) \qquad \text{and} \qquad \dot{p} = -M\omega^2 x - \gamma p + F^p_{\text{SRM}}(t) \ . \tag{6.38}$$

Here $F^x_{\text{SRM}}(t)$ and $F^p_{\text{SRM}}(t)$ are two independent real Gaussian random functions with mean

$$\begin{aligned} \langle F^x_{\text{SRM}}(t)\rangle &= \sqrt{\frac{2}{M\omega}}\ \text{Re}\ \langle F_{\text{SRM}}(t)\rangle \\ \langle F^y_{\text{SRM}}(t)\rangle &= \sqrt{2M\omega}\ \text{Im}\ \langle F_{\text{SRM}}(t)\rangle \ , \end{aligned} \tag{6.39}$$

and variance

$$\begin{aligned} \langle F^x_{\text{RWA}}(t) F^x_{\text{RWA}}(t')\rangle &= \frac{2\gamma(n_{th}+\frac{1}{2})}{M\omega}\, \delta(t-t'), \\ \langle F^p_{\text{RWA}}(t) F^p_{\text{RWA}}(t')\rangle &= 2\gamma(n_{th}+\frac{1}{2}) M\omega\, \delta(t-t') \ . \end{aligned} \tag{6.40}$$

In Eq. (6.39), $\langle F_{\text{SRM}}(t)\rangle$ is the Weyl transform of $\langle \hat{F}_{\text{SRM}}(t)\rangle$. We find from Eq. (6.38) that the SRM oscillator breaks translation invariance since these equations are also symmetric in x and p. This can be explained as follows. In the symmetrized form of the random–matrix coupling the whole interaction $Q\otimes V$, including the system operator Q, behaves like a random–matrix. But Q, the position operator of the system in our case, is essentially deterministic. The reason for the violation of translation invariance is hence that the deterministic position of the system is treated as a random quantity. The fact that the latter has the properties of a random matrix, i.e. independence of the matrix elements as expressed by the Kronecker delta in (2.43), leads further to the non–vanishing of the mean of the noise operator.

6.4 Summary

Using the quantum Langevin equation approach to quantum dissipation, we have investigated the physical consequences of the lack of off–diagonal gain terms in the SRM master equation (5.8) for the damped harmonic oscillator. In particular, we have made a comparison with the related RWA oscillator. First of all, we have found that, in contrast to the RWA case, decoherence takes place in energy representation. This is due to the form (2.43) of the variance which singles out the eigenbasis of H_S. We recall that for the RWA oscillator decoherence occurs in the coherent–state basis. The second consequence is that the stochastic process described by the SRM Langevin equation (6.11) is not stationary. Indeed, the mean value of the noise operator is non–zero and explicitly time–dependent. Moreover, it depends on the initial density operator $\hat{\rho}(0)$. However, for not low–lying initial wave–packets, one may neglect this dependence on the initial conditions. The diffusion constant, on the contrary, is the same in both cases. This implies that energy transfer between system and bath is also identical for both oscillators. For very low temperatures, the mean value of $F_{\text{SRM}}(t)$ vanishes, and the dynamics of the SRM harmonic oscillator reduces to that of the RWA oscillator. On the other hand, in the high–temperature limit, this non–zero centered noise operator induces a rapid decay of the annihilation operator.

By expressing the quantum Langevin equation in terms of position and momentum of the system and employing the Weyl transform, we have obtained the classical Langevin equation. As in the RWA case, the SRM oscillator violates translational invariance. This is due to the symmetrization of the random interaction which treats the deterministic position of the system as a random quantity.

Chapter 7

Averaging: The supersymmetry method

7.1 Introduction

As already mentioned in the introduction, our second motivation for considering a random–matrix model for quantum Brownian motion is to go beyond the usual Caldeira–Leggett approach and to treat situations where the environment is strongly influenced by the system. A typical example is nuclear fission. In this case, the deformation of the nucleus strongly affects the internal (collective) degrees of freedom. As a result, the latter cannot be modeled by a set of independent harmonic oscillators [Wei89]. In the context of our random–matrix model, the influence of the system on the environment can easily be taken into account by introducing ensembles of random matrices which depend parametrically on variables of the system, for instance the position coordinate x. The second moment of the random band–matrix V in Eq. (2.40) may then be generalized to read [Ko76, Bul96]

$$\overline{V_{ab}(x)V_{cd}(x')} = (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\overline{V_{ab}}^2 \exp\left(-\frac{(x-x')^2}{2d^2}\right) , \tag{7.1}$$

where we have introduced a spatial correlation length d. For finite d, the form (7.1) implies that the bath operators V become uncorrelated as soon as the distance $|x - x'|$ is much larger than d. If, however, the correlation length becomes infinitely large, $d \to \infty$, Eq. (7.1) reduces to Eq. (2.40). We just recover the random–matrix model we have studied so far where the environment is not affected by the system. The correlation length d can thus be interpreted as a measure of the strength of the system's influence on the environment.

Before we calculate the ensemble average of the evolution equations (2.7) and (2.8) using the generalized variance (7.1), we rederive the results obtained in Chapter 3 with the diagrammatic expansion, by employing the powerful supersymmetry method [Efe97, Ver85]. The supersymmetry technique is a non–perturbative method which permits to perform the ensemble average exactly (for a pedagogical introduction to supersymmetry we refer to Ref. [Fyo95]). The first step of this averaging method consists in expressing the propagator (the one–point function) as the derivative of a generating function $Z(J)$ with respect to some source variable J [Guh98]. The generating function is written as a Gaussian functional integral over both commuting and anticommuting (Grassmann) variables. This has the important consequence that $Z(J)$ in normalized to unity, $Z(0) = 1$, and the average over the Gaussian random–matrix ensemble becomes straightforward. The second step consists in an approximate evaluation of the remaining integrals in the averaged generating function $\overline{Z(J)}$ by using a saddle–point approximation. This leads to a non–linear σ model. The average of two–point correlators can be computed by applying the same formalism by doubling the dimension of the field space.

The next section is devoted to the derivation of the non–linear σ model corresponding to the SRM coupling which has been studied in Chapter 6. We will begin in Section 7.2.1 with the introduction of the generating function and calculate its ensemble average in Section 7.2.2. In Section 7.2.3 we shall then perform the Hubbard–Stratonovitch transformation before making the saddle–point approximation in Section 7.2.4. In the following section these results will be used to evaluate the averaged one–point function and the weak–coupling limit will also be discussed. Finally, in Section 7.4, we shall be concerned with the calculation of the ensemble average of the two–point function.

7.2 Derivation of the non-linear sigma model

We recall that we aim at calculating the mean values of the propagator $G(E(1))$ and of the density operator $\hat{\rho}(E(1), E(2)) = G(E(1))\hat{\rho}(0)G^{\dagger}(E(2))$ for the symmetrized random–matrix coupling, using a field–theoretical averaging method. In order to carry out their ensemble averages, we begin by expressing these two quantities as the derivative of a generating function involving integration over both commuting and anticommuting variables. In the sequel we use the techniques described in Ref. [Ver85].

7.2.1 Generating function

In this section we introduce the basic formulas and define our notation which is close to the one used in Ref. [Ver85]. We recall that the indices m ($m = 1, \ldots, N_S$) and a ($a = 1, \ldots, N_B$) label the states in the Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_B$ of the system and the bath, respectively, and that $N = N_S.N_B$ is the dimension of the composite Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$. Let D be a $N \times N$ symmetric matrix whose matrix elements are given by

$$\begin{aligned} [D(E)]_{ma,nb} &= E\,\delta_{mn}\delta_{ab} - H_{ma,nb} \\ &= E\,\delta_{mn}\delta_{ab} - (E_m + \varepsilon_a)\,\delta_{mn}\delta_{ab} - Q_{mn}V_{ab}\ . \end{aligned} \tag{7.2}$$

The propagator and the density operator may be expressed in terms of the operator D as

$$[G(E(1))]_{ma,nb} = [D^{-1}(E(1))]_{ma,nb}\ , \tag{7.3}$$

and

$$[\hat{\rho}(E(1), E(2))]_{ma,nb} = \sum_{pcqd} [D^{-1}(E(1))]_{ma,pc} \langle pc|\hat{\rho}(0)|qd\rangle [D^{*\,-1}(E(2))]_{qd,nb}\ . \tag{7.4}$$

Both $E(1)$ and $E(2)$ are given a positive infinitesimal part η. We introduce $4N$ ordinary (commuting) real integration variables $S^1(1)_{ma}, S^2(1)_{ma}, S^1(2)_{ma}, S^2(2)_{ma}$ and $4N$ Grassmann (anticommuting) complex integration variables $\chi(1)_{ma}, \chi^*(1)_{ma}, \chi(2)_{ma}, \chi^*(2)_{ma}$, which form the elements of the $8N$ graded vector ψ,

$$\psi^T_{ma} = \left(S^1(1)_{ma}, S^2(1)_{ma}, S^1(2)_{ma}, S^2(2)_{ma}, \chi(1)_{ma}, \chi^*(1)_{ma}, \chi(2)_{ma}, \chi^*(2)_{ma}\right)\ . \tag{7.5}$$

The associated volume element is

$$[d\psi] = [dS(1)][dS(2)][d\chi(1)][d\chi(2)] \ , \tag{7.6}$$

where

$$[dS(p)] = \prod_{ma} dS^1_{ma}(p)\, dS^2_{ma}(p) \qquad \text{and} \qquad [d\chi(p)] = \prod_{ma} d\chi^*_{ma}(p)\, d\chi_{ma}(p) \ . \tag{7.7}$$

The label $p = 1, 2$ distinguishes the two (advanced–retarded) blocks which are necessary for calculating the two-point function (7.4) (for the evaluation of the one–point function the $p = 1$ block would suffice). To account for the complex conjugation sign for $p = 2$, we further introduce the 8×8 diagonal matrix $L = \mathrm{diag}(1, 1, -1, -1, 1, 1, -1, -1) = \{L_{\alpha\beta}\}$ with $1 \le \alpha, \beta \le 8$; the $8N \times 8N$ graded matrix $\mathbf{L} = L \otimes 1\!\!1_N = \mathrm{diag}(1\!\!1_N, 1\!\!1_N, -1\!\!1_N, -1\!\!1_N, 1\!\!1_N, 1\!\!1_N, -1\!\!1, -1\!\!1_N) = \{L_{\alpha\beta}\delta_{mn}\delta_{ab}\}$, where $1\!\!1_N$ is the $N \times N$ unit operator, $1\!\!1_N = 1\!\!1_S \otimes 1\!\!1_B$; and we define the mean energy $E = (E(1) + E(2))/2$ and the energy difference $\varepsilon = E(2) - E(1)$. We consider the Gaussian generating function

$$Z(E, \varepsilon, J) = \int e^{\mathcal{L}(\psi, J)} [d\psi] \ , \tag{7.8}$$

where the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2} i \psi^\dagger \mathbf{L}^{\frac{1}{2}} \mathbf{D}(J) \mathbf{L}^{\frac{1}{2}} \psi \ , \tag{7.9}$$

with

$$\mathbf{D}(J) = \mathbf{E} + i\boldsymbol{\eta} - \mathbf{H_0} - \mathbf{W} - \frac{1}{2}\boldsymbol{\varepsilon} + \mathbf{J} \ . \tag{7.10}$$

The individual terms in Eq. (7.10) are $8N \times 8N$ graded matrices defined as follows

$$\begin{aligned}
\mathbf{E} &= E\, 1\!\!1_8 \otimes 1\!\!1_N = E\, \{\delta_{\alpha\beta}\delta_{mn}\delta_{ab}\} \ , \\
\boldsymbol{\eta} &= \eta\, L \otimes 1\!\!1_N = \eta\, \{L_{\alpha\beta}\delta_{mn}\delta_{ab}\} = \eta.\mathbf{L} \ , \\
\mathbf{H_0} &= 1\!\!1_8 \otimes H_0 = \{\delta_{\alpha\beta}(E_m + \varepsilon_a)\, \delta_{mn}\delta_{ab}\} \ , \\
\mathbf{W} &= 1\!\!1_8 \otimes W = \{\delta_{\alpha\beta} W_{ma,nb}\} \ , \\
\boldsymbol{\varepsilon} &= \varepsilon\, L \otimes 1\!\!1_N = \varepsilon\, \{L_{\alpha\beta}\delta_{mn}\delta_{ab}\} = \varepsilon.\mathbf{L} \ .
\end{aligned} \tag{7.11}$$

with $1 \le \alpha, \beta \le 8$, $1 \le m, n \le N_S$, and $1 \le a, b \le N_B$. The $8N \times 8N$ graded matrix $\mathbf{J}$ (the source matrix) is obtained by block construction from the two real and symmetric matrices $J_{ma,nb}(1)$ and $J_{ma,nb}(2)$ according to

$$\begin{aligned}
\mathbf{J} = \{&-J_{ma,nb}(1), -J_{ma,nb}(1), -J_{ma,nb}(2), -J_{ma,nb}(2), \\
&+J_{ma,nb}(1), +J_{ma,nb}(1), +J_{ma,nb}(2), +J_{ma,nb}(2)\} \ .
\end{aligned} \tag{7.12}$$

This matrix is introduced in order to obtain the expression of physical interest by differentiation of $\overline{Z}$ with respect to elements of $\mathbf{J}$. Explicitly, we have [Ver85],

$$(1 - \frac{1}{2}\delta_{mn}\delta_{ab})\, \overline{[D^{-1}(E(1))]_{ma,nb}} = \frac{1}{4} \frac{\partial}{\partial J_{ma,nb}(1)} \overline{Z(E, \varepsilon, J)}\Big|_{J=0} \ , \tag{7.13}$$

and

$$(1-\frac{1}{2}\delta_{mn}\delta_{ab})(1-\frac{1}{2}\delta_{pc}\delta_{qd})\,\overline{[D^{-1}(E(1))]_{ma,nb}[D^{*-1}(E(2))]_{pc,qd}} = \frac{1}{16}\frac{\partial^2}{\partial J_{ma,nb}(1)\partial J_{pc,qd}(2)}\overline{Z(E,\varepsilon,J)}\Big|_{J=0} . \tag{7.14}$$

We note that $Z(E,\varepsilon,J)$ contains the entire physical information about the composite system and that the average of the propagator and the density operator can be calculated from $\overline{Z(E,\varepsilon,J)}$ by combining Eqs. (7.2), (7.13) and Eqs. (7.3), (7.14).

7.2.2 Ensemble average

In this section we calculate the ensemble average $\overline{Z(E,\varepsilon,J)}$ of the generating function (7.8). The stochastic quantities all reside in the graded matrix $\mathbf{W}$. We therefore need to calculate the ensemble average of $\exp(-\frac{1}{2}i\psi^\dagger\mathbf{L}^{\frac{1}{2}}\mathbf{W}\mathbf{L}^{\frac{1}{2}}\psi)$. To this end, we use the fact that for a $N\times N$ random matrix W belonging to any Gaussian distribution with zero mean, we have [Zuk94]

$$\overline{\exp\left(\pm i\,\mathrm{tr}WX\right)} = \exp\left(-\frac{1}{2}\,\overline{(\mathrm{tr}WX)^2}\right) \tag{7.15}$$

for any fixed $N\times N$ matrix X. The second moment of the trace on the r.h.s of Eq. (7.15) is easily calculated. We use the symmetrized form (2.43) of the random–matrix interaction and obtain

$$\overline{(\mathrm{tr}WX)^2} = \sum_{manb}\overline{W_{ab}^{mn\,2}}\,X_{ba}^{nm}(X_{ba}^{nm}+X_{ab}^{mn}) . \tag{7.16}$$

Explicit calculation shows that in the present case the matrix X is symmetric,

$$X_{ba}^{nm} = \sum_\alpha \psi^*_{ma,\alpha}\,L_{\alpha\alpha}\,\psi_{nb,\alpha} = \sum_\alpha \psi^*_{nb,\alpha}\,L_{\alpha\alpha}\,\psi_{ma,\alpha} = X_{ab}^{mn} , \tag{7.17}$$

and hence

$$\begin{aligned}\overline{(\mathrm{tr}WX)^2} &= 2\sum_{manb}\overline{W_{ab}^{mn\,2}}\,X_{ba}^{nm}X_{ab}^{mn} \\ &= 2\sum_{\substack{manb\\ \alpha\beta}}\overline{W_{ab}^{mn\,2}}\,(\psi^*_{ma,\alpha}\,L_{\alpha\alpha}\,\psi_{nb,\alpha})\,(\psi^*_{ma,\beta}\,L_{\beta\beta}\,\psi_{nb,\beta}) .\end{aligned} \tag{7.18}$$

This expression can be simplified by introducing a set of graded 8×8 matrices A_{ma}; $m=1,\ldots,N_S$ and $a=1,\ldots,N_B$. We define

$$A_{ma}^{\alpha\beta} = (L^{\frac{1}{2}})_{\alpha\alpha}\psi_{ma,\alpha}\psi^*_{ma,\beta}(L^{\frac{1}{2}})_{\beta\beta} , \tag{7.19}$$

which satisfies

$$\sum_{\alpha\beta}\psi^*_{ma,\alpha}\,L_{\alpha\alpha}\,\psi_{nb,\alpha}\psi^*_{ma,\beta}\,L_{\beta\beta}\,\psi_{nb,\beta} = \mathrm{trg}(A_{ma}A_{nb}) . \tag{7.20}$$

With the help of Eq. (7.20), the average of $\exp(-\frac{1}{2}i\psi^\dagger\mathbf{L}^{\frac{1}{2}}\mathbf{W}\mathbf{L}^{\frac{1}{2}}\psi)$ can be written in the form

$$\overline{\exp\left(-\frac{1}{2}i\psi^\dagger\mathbf{L}^{\frac{1}{2}}\mathbf{W}\mathbf{L}^{\frac{1}{2}}\psi\right)} = \exp\left(-\frac{1}{4}\sum_{manb}M_{ab}^{mn}\,\mathrm{trg}(A_{ma}A_{nb})\right) , \tag{7.21}$$

where we have introduced the $N \times N$ matrix M defined by $M_{ab}^{mn} = \overline{W_{ab}^{mn2}}$. Consequently, the averaged generating function can be cast into the form

$$\overline{Z(E,\varepsilon,J)} = \int \exp\left(\frac{1}{2} i\psi^\dagger \mathbf{L}^{\frac{1}{2}}(\mathbf{E} + i\boldsymbol{\eta} - \mathbf{H_0} - \frac{1}{2}\boldsymbol{\varepsilon} + \mathbf{J})\mathbf{L}^{\frac{1}{2}}\psi \right. \\ \left. -\frac{1}{4}\sum_{manb} M_{ab}^{mn}\,\mathrm{trg}(A_{ma}A_{nb})\right)\,[d\psi]\;. \tag{7.22}$$

We note that because $\overline{V_{ab}}^2$ is a banded matrix with bandwidth Δ, the matrix $M_{ab}^{mn} = \overline{V_{ab}}^2|Q_{mn}|^2$ is also a band–matrix with a bandwidth given by $\Delta.N_S$. The ratio of the bandwidth and the matrix dimension is then equal to $\Delta.N_S/N = \Delta/N_B$.

7.2.3 Hubbard–Stratonovitch transformation

In Eq. (7.22) the second term in the exponent is a quartic function of the integration variables ψ_{ma} (remember that A_{ma} depends quadratically on the components of the graded vector ψ, cf. Eq. (7.19)). As a result the integral over $[d\psi]$ is difficult to calculate as it stands. The standard way to circumvent this difficulty is the Hubbard–Stratonovitch transformation

$$\exp\left(-\frac{1}{4}\sum_{manb} M_{ab}^{mn}\,\mathrm{trg}(A_{ma}A_{nb})\right) = \\ \int \exp\left(-\frac{1}{4}\sum_{manb}(M^{-1})_{ab}^{mn}\,\mathrm{trg}(\sigma_{ma}\sigma_{nb}) - \frac{i}{2}\sum_{ma}\mathrm{trg}(\sigma_{ma}A_{ma})\right)[d\sigma]\;. \tag{7.23}$$

With the help of a Gaussian integration over an a set of auxiliary 8×8 graded matrices σ_{ma}, the transformation (7.23) reduces the quartic form in (7.22) to a quadratic one. The volume element reads $[d\sigma] = \prod_{ma}[d\sigma_{ma}]$. By direct calculation and using the definition (7.19), we find that

$$\mathrm{trg}(\sigma_{ma}A_{ma}) = \sum_{\alpha\beta}\psi_{ma,\alpha}^*(L^{\frac{1}{2}})_{\alpha\alpha}\sigma_{ma}^{\alpha\beta}(L^{\frac{1}{2}})_{\beta\beta}\psi_{ma,\beta}\;. \tag{7.24}$$

Now one can substitute Eqs. (7.23) and (7.24) into Eq. (7.22) for $\overline{Z(E,\varepsilon,J)}$ and formally change the order of integration over $[d\sigma]$ and $[d\psi]$. The expression in the exponent is again bilinear with respect to the graded vector ψ and the corresponding integration can be performed. This yields

$$\overline{Z(E,\varepsilon,J)} = \int e^{\mathcal{L}(\sigma,J)}[d\sigma]\;, \tag{7.25}$$

where the Lagrangian is given by

$$\mathcal{L}(\sigma,J) = -\frac{1}{4}\sum_{manb}(M^{-1})_{ab}^{mn}\,\underset{\alpha}{\mathrm{trg}}\,(\sigma_{ma}\sigma_{nb}) - \frac{1}{2}\underset{\alpha,ma}{\mathrm{trg}}\,\ln\mathbf{N}(J)\;, \tag{7.26}$$

with

$$\mathbf{N}(J) = \mathbf{E} + i\boldsymbol{\eta} - \mathbf{H_0} - \frac{1}{2}\boldsymbol{\varepsilon} + \mathbf{J} - \boldsymbol{\Sigma}\;, \tag{7.27}$$

and

$$\boldsymbol{\Sigma} = \{\sigma_{ma}^{\alpha\beta}\delta_{mn}\delta_{ab}\}\;. \tag{7.28}$$

The graded trace multiplying $(M^{-1})^{mn}_{ab}$ has as argument the matrix product $\sigma_{ma}\sigma_{nb}$; the symbol α denotes that the trace extends over a graded 8×8 matrix. In contradistinction, the trace over the logarithm extends in addition over the set of indices $\{m, a\}$.

7.2.4 Saddle-point approximation

We are going to evaluate the integral Eq. (7.25) with the help of a saddle-point approximation. We put $\boldsymbol{\varepsilon} = \mathbf{J} = 0$ to have the case of maximum symmetry (we keep the infinitesimal matrix $\boldsymbol{\eta}$ to ensure converge) and rewrite the Lagrangian (7.26) as

$$\mathcal{L}(\sigma) = \operatorname*{trg}_{\alpha} \left[-\frac{1}{4} \sum_{manb} (M^{-1})^{mn}_{ab}\, \sigma_{ma}\sigma_{nb} - \frac{1}{2} \sum_{ma} \ln(E^+ - H_{0ma} - \sigma_{ma}) \right] . \tag{7.29}$$

The saddle-points are determined by the condition that the Lagrangian be stationary, i.e. $\delta\mathcal{L}(\sigma) = 0$ at $\sigma = \sigma^{(0)}$. For this purpose we write $\sigma_{ma} = \sigma_{ma}{}^{(0)} + \delta\sigma_{ma}$ and get

$$\mathcal{L}(\sigma) = \mathcal{L}(\sigma^{(0)}) + \delta\mathcal{L} + \delta^2\mathcal{L} \ , \tag{7.30}$$

with the first variation of the Lagrangian given by

$$\delta\mathcal{L} = \operatorname*{trg}_{\alpha} \left[-\frac{1}{2} \sum_{manb} (M^{-1})^{mn}_{ab}\, \delta\sigma_{ma}\sigma_{nb}{}^{(0)} + \frac{1}{2} \sum_{ma} (E^+ - H_{0ma} - \sigma_{ma}{}^{(0)})^{-1} \delta\sigma_{ma} \right] , \tag{7.31}$$

and where the second variation is equal to

$$\begin{aligned} \delta^2\mathcal{L} \ &= \ \operatorname*{trg}_{\alpha} \Bigg[-\frac{1}{2} \sum_{manb} (M^{-1})^{mn}_{ab}\, \delta\sigma_{ma}\delta\sigma_{nb} \\ &+ \ \frac{1}{2} \sum_{ma} (E^+ - H_{0ma} - \sigma_{ma}{}^{(0)})^{-1} \delta\sigma_{ma} (E^+ - H_{0ma} - \sigma_{ma}{}^{(0)})^{-1} \delta\sigma_{ma} \Bigg] . \end{aligned} \tag{7.32}$$

The saddle-point equation $\delta\mathcal{L} = 0$ then reads

$$\sum_{nb} (M^{-1})^{mn}_{ab} \sigma_{nb}{}^{(0)} = (E^+ - H_{0ma} - \sigma_{ma}{}^{(0)})^{-1} \ . \tag{7.33}$$

We note that the different $\sigma_{ma}{}^{(0)}$ are coupled by the non–diagonal elements of the matrix M^{-1}. Moreover, since the saddle–point solutions satisfy the non–linear equation (7.33), the integral over $[d\sigma]$ in Eq. (7.25) will lead to a non–linear sigma model. We now consider the case of the one–point function and two–point function separately.

7.3 One-point function

In the present section, we shall apply the results obtained previously to evaluate the ensemble average of the propagator (7.3). In this case, we can restrict ourselves to introducing only a $4N$ graded vector ψ. The resulting expression for the Lagrangian $\mathcal{L}(\sigma)$ coincides with Eq. (7.26) with the only replacement of the matrix L by the identity matrix $\mathbb{1}_4 = \mathrm{diag}(1, 1, 1, 1)$. We begin by making the shift of variables $\sigma_{ma} \to \sigma_{ma} + J_{ma}$ in

Eq. (7.26). This removes the J-dependence from the logarithm. We find

$$\begin{aligned}\mathcal{L}(\sigma) &= \operatorname*{trg}_{\alpha}\left[-\frac{1}{4}\sum_{manb}(M^{-1})^{mn}_{ab}\,\sigma_{ma}\sigma_{nb}-\frac{1}{2}\sum_{ma}\ln(E^{+}-H_{0ma}-\sigma_{ma})\right]\\ &\quad+2\sum_{manb}(M^{-1})^{mn}_{ab}\,\sigma_{ma}J_{nb}\ ,\end{aligned} \tag{7.34}$$

where we have used that

$$\operatorname*{trg}_{\alpha}\left[\sum_{manb}(M^{-1})^{mn}_{ab}\,\sigma_{ma}J_{nb}\right]=\sum_{manb}(M^{-1})^{mn}_{ab}\sum_{\alpha\beta}\sigma^{\alpha\beta}_{ma}J^{\beta\alpha}_{nb}=-4\sum_{manb}(M^{-1})^{mn}_{ab}\,\sigma_{ma}J_{nb}\ . \tag{7.35}$$

Differentiating with respect to the source matrix J, we get from Eqs. (7.3) and (7.13) the following expression for the averaged propagator

$$\begin{aligned}&\overline{[G(E)]_{nb}}=\overline{[D^{-1}(E)]_{nb}}=\int\sum_{ma}(M^{-1})^{mn}_{ab}\,\sigma_{ma}\times\\ &\exp\left(\operatorname*{trg}_{\alpha}\left[-\frac{1}{4}\sum_{manb}(M^{-1})^{mn}_{ab}\,\sigma_{ma}\sigma_{nb}-\frac{1}{2}\sum_{ma}\ln(E^{+}-H_{0ma}-\sigma_{ma})\right]\right)[d\sigma]\ ,\end{aligned} \tag{7.36}$$

which has to be evaluated with the help of the saddle-point approximation. It turns out that in this case the saddle-point equation (7.33) has a single diagonal solution ${\sigma_{ma}}^{(0)}$. We further notice that $\mathcal{L}(\sigma^{(0)})=0$ and that

$$\int\exp\left(\delta^2\mathcal{L}(\sigma^{(0)})\right)[d(\delta\sigma)]=1\ . \tag{7.37}$$

The last statement can easily be understood by observing that there are no fluctuations around a non-degenerate saddle-point. Collecting everything, we arrive at

$$\overline{G}_{ma}=\sum_{nb}(M^{-1})^{mn}_{ab}\,{\sigma_{nb}}^{(0)}\ . \tag{7.38}$$

In this expression, we recognize the components of the linear equation $\overline{G}=M^{-1}\sigma^{(0)}$. This equation can readily be inverted to give $\sigma^{(0)}=M\overline{G}$ or

$${\sigma_{ma}}^{(0)}=\sum_{nb}M^{mn}_{ab}\,\overline{G_{nb}}\ . \tag{7.39}$$

Inserting the solution (7.39) into the right–hand side of the saddle point equation (7.33) and using (7.38), we finally obtain the average propagator

$$\overline{G}_{ma}=\frac{1}{E^{+}-H_{0ma}-\sum_{nb}M^{mn}_{ab}\,\overline{G}_{nb}}\ . \tag{7.40}$$

We recognize the strong coupling result (3.9).

Weak coupling

Let us now determine the solutions of the saddle-point equation (7.33) in the limit in which the system–bath coupling is weak. We begin by considering the case of vanishing coupling. In this case the matrix $M = 0$ and therefore its inverse M^{-1} diverges. The only way to obtain a finite G from Eq. (7.38) is to require that the saddle-point solution ${\sigma_{ma}}^{(0)}$ vanishes. According to Eqs. (7.33) and (7.38) the corresponding averaged propagator then reduces to the free propagator G_0,

$$\overline{G}_{ma} = \frac{1}{E^+ - H_{0ma}} = (G_0)_{ma} \ , \tag{7.41}$$

as one would expect for vanishing coupling. As we turn on the weak interaction, we assume that ${\sigma_{ma}}^{(0)}$ is still small and we therefore neglect it in the denominator of Eq. (7.33). The saddle–point equation becomes

$$\sum_{nb} (M^{-1})^{mn}_{ab} \, {\sigma_{nb}}^{(0)} \simeq (E^+ - H_{0ma}) = (G_0)_{ma} \ . \tag{7.42}$$

The solution of (7.42) is readily found to be equal to

$${\sigma_{ma}}^{(0)} = \sum_{nb} M^{mn}_{ab} \, (G_0)_{nb} \ . \tag{7.43}$$

Again, we substitute this expression for ${\sigma_{ma}}^{(0)}$ in Eq. (7.33) and make use of the relation (7.38). The corresponding average propagator can then be written in the form

$$\overline{G}_{ma} = \frac{1}{E^+ - H_{0ma} - \sum_{nb} M^{mn}_{ab} \, (G_0)_{nb}} \ . \tag{7.44}$$

This is the weak-coupling result (3.10) obtained in Section 3.2.1 using the diagrammatic method. Eq. (7.44) is valid provided ${\sigma_{ma}}^{(0)}$ is small with respect to $E^+ - H_{0ma}$ in Eq. (7.33). More specifically, the weak-coupling condition reads

$${\sigma_{ma}}^{(0)} = \sum_{nb} M^{mn}_{ab} \, (G_0)_{nb} \ll E^+ - H_{0ma} \ , \tag{7.45}$$

and is identical to (3.26). It is interesting to notice that in the limit of weak coupling the ${\sigma_{ma}}^{(0)}$'s in Eq. (7.42) decouple, in contrast to the strong coupling expression Eq. (7.33).

7.4 Two-point function

In this section, we turn to the evaluation of the averaged two–point function. Here, in contrast to the one–point function, the saddle points are not isolated, instead, there is a degenerate manifold of saddle points. This is due to the invariance of the Lagrangian (7.9) under symmetry transformations. This degeneracy of the non–linear manifold then implies the existence of massless Goldstone modes which may be obtained by expanding the Lagrangian near the extremum. Further, by expanding the resulting effective Lagrangian in an asymptotic series, the diagrammatic perturbation theory may be recovered.

Parametrization of $\boldsymbol{\Sigma}$

We begin by discussing the parametrization of the saddle–point manifold. In the absence of symmetry breaking ($\boldsymbol{\eta} = \boldsymbol{\varepsilon} = \mathbf{J} = 0$), the Lagrangian (7.9) is invariant under any linear transformation $\psi \to \mathbf{T}\psi$ which preserves the bilinear form $\psi^\dagger \mathbf{L} \psi$. The $8N \times 8N$ graded matrices $\mathbf{T} = \{T_{\alpha\beta}\delta_{mn}\delta_{ab}\}$ then satisfy

$$\mathbf{T}^{-1} = \mathbf{L}^{\frac{1}{2}} \mathbf{T} \mathbf{L}^{\frac{1}{2}} \ . \tag{7.46}$$

The corresponding transformation induced on the matrix $\boldsymbol{\Sigma}$ is given by

$$\boldsymbol{\Sigma} \to \mathbf{T}^{-1} \boldsymbol{\Sigma} \mathbf{T} \ . \tag{7.47}$$

We see from (7.47) that the group of the transformations $\mathbf{T}$ generates a full manifold of matrices $\boldsymbol{\Sigma}$. Moreover, it turns out that the integral over $d[\sigma]$ in the Hubbard–Stratonovitch transformation (7.23) does not converge if the symmetry of the Lagrangian (7.9) is not implemented correctly. In order to overcome this convergency problem one has to require that the manifold of matrices $\boldsymbol{\Sigma}$ is invariant with respect to the transformation (7.47). This leads to the following general form of the matrix $\boldsymbol{\Sigma}$ [Ver85]

$$\boldsymbol{\Sigma} = \mathbf{T}^{-1} \boldsymbol{\Sigma}_{\mathbf{D}} \mathbf{T} \ , \tag{7.48}$$

where $\boldsymbol{\Sigma}_{\mathbf{D}}$ is a diagonal matrix. The forms of $\boldsymbol{\Sigma}_{\mathbf{D}}$ and $\mathbf{T}$ are completely determined by arguments of symmetry and convergence. At this stage, it is convenient to perform a coset decomposition of the graded group of the $\mathbf{T}$ matrices. We write $\mathbf{T} = \mathbf{R}\mathbf{T}_{\mathbf{0}}$ where $\mathbf{R}$ spans the subgroup that commutes with $\mathbf{L}$. We then have

$$\boldsymbol{\Sigma} = {\mathbf{T}_{\mathbf{0}}}^{-1} \mathbf{P} \mathbf{T}_{\mathbf{0}} \ , \tag{7.49}$$

where we have defined

$$\mathbf{P} = \mathbf{R}^{-1} \boldsymbol{\Sigma}_{\mathbf{D}} \mathbf{R} \ . \tag{7.50}$$

We next consider the vicinity of the saddle point and expand $\boldsymbol{\Sigma}_{\mathbf{D}}$ around its saddle–point value $\boldsymbol{\Sigma}^{(0)}$. We write

$$\boldsymbol{\Sigma}_{\mathbf{D}} = \boldsymbol{\Sigma}^{(0)} + \delta \boldsymbol{\Sigma}_{\mathbf{D}} \ , \tag{7.51}$$

use Eqs. (7.49) and (7.50) together with the fact that $\mathbf{R}$ commutes with $\boldsymbol{\Sigma}^{(0)}$ and arrive at

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{\mathbf{G}} + \delta \boldsymbol{\Sigma} \ , \tag{7.52}$$

where

$$\boldsymbol{\Sigma}^{\mathbf{G}} = {\mathbf{T}_{\mathbf{0}}}^{-1} \boldsymbol{\Sigma}^{(0)} \mathbf{T}_{\mathbf{0}} \ , \tag{7.53}$$

and

$$\delta \boldsymbol{\Sigma} = \mathbf{T}^{-1} \delta \boldsymbol{\Sigma}_{\mathbf{D}} \mathbf{T} = {\mathbf{T}_{\mathbf{0}}}^{-1} \delta \mathbf{P} \mathbf{T}_{\mathbf{0}} \ . \tag{7.54}$$

The graded matrices $\boldsymbol{\Sigma}^{\mathbf{G}}$ represent massless (Goldstone) modes, whereas the matrices $\mathbf{P}$ represent massive modes and $\delta\mathbf{P}$ the massive fluctuations around it. We shall see below that in the limit $1 \ll \Delta \ll N_B$, the two modes decouple and the integration over $\delta\mathbf{P}$ can be done exactly with the help of the saddle–point approximation. Using (7.28), we may rewrite Eq. (7.52) as

$$\sigma_{ma} = \sigma^{G}_{ma} + \delta\sigma_{ma} \ , \tag{7.55}$$

with

$$\sigma^G_{ma} = T_0^{-1}{}_{ma} \sigma_{ma}{}^{(0)} T_{0ma} \; , \tag{7.56}$$

and

$$\delta\sigma_{ma} = T_0^{-1}{}_{ma} \delta P_{ma} T_{0ma} \; . \tag{7.57}$$

The matrix T_{0ma} can be written in the form (in the $[1,2]$ block notation)

$$T_{0ma} = \begin{pmatrix} (1 + t^{ma}_{12} t^{ma}_{21})^{\frac{1}{2}} & i\, t^{ma}_{12} \\ -i\, t^{ma}_{21} & (1 + t^{ma}_{21} t^{ma}_{12})^{\frac{1}{2}} \end{pmatrix} \; , \tag{7.58}$$

where the 4×4 graded matrices t^{ma}_{12} and t^{ma}_{21} are given in table D.3 of Ref. [Ver85]. Eq. (7.56) shows that the saddle points are degenerate and form a manifold which is parametrized in terms of the variables t^{ma}_{12}. The elements of this manifold are precisely the matrices σ^G_{ma}.
Let us consider the parametrization (7.49) rewritten as

$$\sigma_{ma} = T_0^{-1}{}_{ma} P_{ma} T_{0ma} \; . \tag{7.59}$$

If we take the independent elements of $P_{ma} = \mathrm{diag}(P_{11ma}, P_{22ma})$ and T_{0ma} as the new integration variables, the Jacobian of the transformation may be cast into the form

$$[d\sigma_{ma}] = \mathcal{F}[P_{ma}][dP_{ma}] d\mu(t) \; , \tag{7.60}$$

where $\mathcal{F}[P_{ma}]$ depends only on the eigenvalues of P_{ma}, $[dP_{ma}] = [dP_{11ma}][dP_{22ma}]$ is the usual flat measure and $d\mu(t)$ is the measure on the coset space and depends only on the variables t^{ma}_{12} [Ver85].

Decoupling of the massive modes

We substitute Eq. (7.55) into the Lagrangian (7.26) and use Eqs. (7.56) and (7.57) . In the limit $1 \ll \Delta \ll N_B$, the terms linear in $\delta\sigma_{ma}$ cancel out and we find [Fyo91, Fyo94]

$$\mathcal{L}(\sigma^G + \delta\sigma, J) = -\frac{1}{4} \underset{\alpha}{\mathrm{trg}} \sum_{manb} \left[(M^{-1})^{mn}_{ab} + 2\sigma^{(0)}_{ma} \sigma^{(0)}_{nb} \delta_{mn} \delta_{ab} \right] \delta P_{ma} \delta P_{nb} + \mathcal{L}(\sigma^G, J) \; . \tag{7.61}$$

We observe that the variables σ^G_{ma} and $\{\delta P_{ma}\}$ decouple and that the latter occur quadratically in (7.61). Furthermore, the Jacobian $\mathcal{F}[P_{ma}]$ in (7.61) can be shown to be unity [Ver85]. As a result, the Gaussian integral over $\{\delta P_{ma}\}$ can be performed exactly with the aid of the identity

$$\int \prod_{ma} [dP_{ma}] \exp\left(-\frac{1}{2} \sum_{manb} O^{mn}_{ab} \, \underset{\alpha}{\mathrm{trg}} \, \delta P_{ma} \delta P_{nb} \right) = 1 \; . \tag{7.62}$$

We are then left with an effective Lagrangian which is identical to Eq. (7.26) except for the replacement $\sigma_{ma} \to \sigma^G_{ma}$. We thus have

$$\overline{Z(E, \varepsilon, J)} = \int e^{\mathcal{L}(\sigma^G, J)} [d\mu(t)] \; , \tag{7.63}$$

where $\mathcal{L}(\sigma^G, J)$ is given by

$$\begin{aligned}\mathcal{L}(\sigma^G, J) &= -\frac{1}{4}\sum_{manb}(M^{-1})^{mn}_{ab}\left[\operatorname*{trg}_{\alpha}(\sigma^G_{ma}\sigma^G_{nb}) + \varepsilon\operatorname*{trg}_{\alpha}\sigma^G_{ma}L\right] \\ &\quad -\frac{1}{2}\operatorname*{trg}_{\alpha,ma}\ln\left(\mathbf{E}^+ - \mathbf{H_0} - \mathbf{\Sigma}^G + \mathbf{J}\right) . \end{aligned} \tag{7.64}$$

Here, the term $-\frac{1}{2}\boldsymbol{\varepsilon}$ has been removed from the logarithm with the help of the substitution $\mathbf{\Sigma} \to \mathbf{\Sigma} - \frac{1}{2}\boldsymbol{\varepsilon}$.

Asymptotic expansion

The two–point function (7.14) is obtained by double differentiation with respect to J of $\overline{Z(E,\varepsilon,J)}$ at $J = 0$. We therefore write

$$\ln\left(\mathbf{E}^+ - \mathbf{H_0} - \mathbf{\Sigma}^G + \mathbf{J}\right) = \ln\left(\mathbf{E}^+ - \mathbf{H_0} - \mathbf{\Sigma}^G\right) + \ln\left(\mathbb{1} + (\mathbf{E}^+ - \mathbf{H_0} - \mathbf{\Sigma}^G)^{-1}\mathbf{J}\right) , \tag{7.65}$$

and expand the second term in powers of J, keeping terms up to second order. We then take the trace over α, ma and expand the exponential in Eq. (7.63) in powers of J, again up to second order. All these steps yield

$$\begin{aligned}\overline{Z(E,\varepsilon,J)} = \int[d\mu(t)]\Bigg(&\frac{1}{4}\sum_{mam'a'}\operatorname*{trg}_{\alpha} B^G_{ma}J_{ma,m'a'}B^G_{m'a'}J_{m'a',ma} \\ &+\frac{1}{8}\sum_{mam'a'}\operatorname*{trg}_{\alpha} B^G_{ma}J_{ma,ma}\operatorname*{trg}_{\alpha} B^G_{m'a'}J_{m'a',m'a'}\Bigg) \\ \times\exp\Bigg(&-\frac{1}{4}\sum_{manb}(M^{-1})^{mn}_{ab}\left[\operatorname*{trg}_{\alpha}(\sigma^G_{ma}\sigma^G_{nb}) + \varepsilon\operatorname*{trg}_{\alpha}\sigma^G_{ma}L\right]\Bigg) ,\end{aligned} \tag{7.66}$$

where we have introduced the notation

$$B^G_{ma} = (E^+ - H_{0ma} - \sigma^G_{ma})^{-1} . \tag{7.67}$$

We then take the derivatives with respect to $J_{m_1a_1,m_2a_2}(1)$ and $J_{m'_1a'_1,m'_2a'_2}(2)$ and obtain

$$\begin{aligned}&\overline{[D^{-1}(E(1))]_{m_1a_1,m_2n_2}[D^{*-1}(E(2))]_{m'_1a'_1,m'_2n'_2}} = \\ &\quad\frac{1}{4}\int[d\mu(t)]\Bigg\{\frac{1}{4}\operatorname*{trg}_{\alpha} B^G_{m_1a_1}I(1)B^G_{m_2a_2}I(2)\,\delta_{m'_1m_2}\delta_{a_1a'_2}\delta_{m_1m'_2}\delta_{a_1a'_2} \\ &\qquad + \frac{1}{4}\operatorname*{trg}_{\alpha} B^G_{m'_1a'_1}I(1)B^G_{m'_2a'_2}I(2)\,\delta_{m'_1m_2}\delta_{a_1a'_2}\delta_{m_1m'_2}\delta_{a_1a'_2} \\ &\qquad + \frac{1}{4}\operatorname*{trg}_{\alpha} B^G_{m_1a_1}I(1)\,\delta_{m_1m_2}\delta_{a_1a_2}\operatorname*{trg}_{\alpha} B^G_{m'_1a'_1}I(2)\,\delta_{m'_1m'_2}\delta_{a'_1a'_2}\Bigg\} \\ &\qquad\times\ \exp\left(-\frac{1}{4}\sum_{manb}(M^{-1})^{mn}_{ab}\left[\operatorname*{trg}_{\alpha}(\sigma^G_{ma}\sigma^G_{nb}) + \varepsilon\operatorname*{trg}_{\alpha}\sigma^G_{ma}L\right]\right) .\end{aligned} \tag{7.68}$$

Here, we have defined the 8×8 graded matrices $I(1) = \mathrm{diag}(-1,-1,0,0,1,1,0,0)$ and $I(2) = \mathrm{diag}(0,0,-1,-1,0,0,1,1)$.

We now use Eq. (7.68) to generate an asymptotic expansion for the two–point function. We proceed as in Ref. [Nis86] and expand the matrix σ^G_{ma} in powers of the matrices t^{ma}_{12}.

Since we are only interested in the terms of lowest order, we only carry the expansion to second order in the t_{12}^{ma}. According to Eq. (7.58), we have

$$T_{0ma} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} t_{12}^{ma} t_{21}^{ma} & i\, t_{12}^{ma} \\ -i\, t_{21}^{ma} & \frac{1}{2}(t_{21}^{ma} t_{12}^{ma}) \end{pmatrix} . \tag{7.69}$$

We use Eqs. (7.56), (7.69) together with the identity ${T_0^{-1}}_{ma} = LT_{0ma}L$ and expand the terms multiplying the exponential in Eq. (7.68). We get

$$\begin{aligned} &\operatorname*{trg}_{\alpha} B_{m_1 a_1}^{(0)} T_{0 m_1 a_1} I(1) {T_0^{-1}}_{m_2 a_2} B_{m_2 a_2}^{(0)} T_{0 m_2 a_2} I(2) {T_0^{-1}}_{m_1 a_1} \simeq \\ &\qquad \left(\overline{G}_{m_1 a_1} \overline{G^\dagger}_{m_2 a_2} + \overline{G^\dagger}_{m_1 a_1} \overline{G}_{m_2 a_2} \right) \operatorname{trg} [t_{21}^{m_1 a_1} \Lambda t_{12}^{m_2 a_2} \Lambda] \ , \end{aligned} \tag{7.70}$$

for the first term and a similar expression for the second one. Here we have defined the 4×4 matrix $\Lambda = \operatorname{diag}(-1,-1,1,1)$ and we have made use of the relation $B_{ma}^{G} = {T_0^{-1}}_{ma} B_{ma}^{(0)} T_{0ma}$, where

$$B_{ma}^{(0)} = \begin{pmatrix} \overline{G}_{ma} & 0 \\ 0 & \overline{G^\dagger}_{ma} \end{pmatrix} . \tag{7.71}$$

The expression (7.71) for $B_{ma}^{(0)}$ is obtained by combining Eq. (7.67) with Eqs. (7.33) and (7.38) for the averaged one–point function $\overline{G}_{ma}$. For the third preexponential term (the disconnected term), we find

$$\left(\operatorname*{trg}_{\alpha} B_{m_1 a_1}^{(0)} T_{0 m_1 a_1} I(1) {T_0^{-1}}_{m_1 a_1} \right) \left(\operatorname*{trg}_{\alpha} B_{m_1' a_1'}^{(0)} T_{0 m_1' a_1'} I(2) {T_0^{-1}}_{m_1' a_1'} \right) \simeq 16\, \overline{G}_{m_1 a_1} \overline{G^\dagger}_{m_1' a_1'} . \tag{7.72}$$

We now turn to the remaining terms in the integrands of Eq. (7.68). The volume element $[d\mu(t)] = \prod_{ma} [d\mu(t_{12}^{ma})]$ is given by [Nis86]

$$\prod_{ma} [d\mu(t_{12}^{ma})] = \prod_{ma} \operatorname{detg}^{-\frac{1}{2}} (1 + t_{12}^{ma} t_{21}^{ma}) [dt_{12}^{ma}] . \tag{7.73}$$

By expanding the graded determinants in powers of the $t_{12}^{ma} t_{21}^{ma}$ and keeping only the terms of lowest order, Eq. (7.73) reduces to

$$\prod_{ma} [d\mu(t_{12}^{ma})] \simeq \prod_{ma} [dt_{12}^{ma}] . \tag{7.74}$$

By proceeding in an analogous way, one may expand the argument of the exponential in powers of the matrices t_{12}^{ma}. Up to second–order, the exponent can then be written in the form [Nis86]

$$-\frac{1}{4} \sum_{manb} (M^{-1})_{ab}^{mn} \left[\operatorname*{trg}_{\alpha} (\sigma_{ma}^{G} \sigma_{nb}^{G}) + \varepsilon \operatorname*{trg}_{\alpha} \sigma_{ma}^{G} L \right] \simeq -\frac{1}{2} \sum_{manb} \left(\Pi^{-1} \right)_{ab}^{mn} \operatorname{trg}[t_{12}^{ma} t_{21}^{nb}] \ , \tag{7.75}$$

where Π^{-1} is an ordinary symmetric $N \times N$ matrix. One may evaluate the ensemble average of the density operator (7.4) by employing the following identities [Nis86],

$$\int \prod_{pc} [dt_{12}^{pc}] \exp \left(-\frac{1}{2} \sum_{manb} \left(\Pi^{-1} \right)_{ab}^{mn} \operatorname{trg}[t_{12}^{ma} t_{21}^{nb}] \right) = 1 \ , \tag{7.76}$$

and

$$\int \prod_{pc}[dt_{12}^{pc}] \exp\left(-\frac{1}{2}\sum_{manb}\left(\Pi^{-1}\right)_{ab}^{mn} \operatorname{trg}[t_{12}^{ma}t_{21}^{nb}]\right) \operatorname{trg}[t_{21}^{m_1a_1}\Lambda t_{12}^{m_2a_2}\Lambda] = 16\,\Pi_{a_1a_2}^{m_1m_2} \ . \quad (7.77)$$

Collecting everything, we obtain

$$\begin{aligned} \langle m_1a_1|\overline{\rho}|m_2'a_2'\rangle &= \overline{G}_{m_1a_1}\langle m_1a_1|\hat{\rho}(0)|m_2'a_2'\rangle\overline{G^\dagger}_{m_2'a_2'} \\ &+ \sum_{m_1'a_1'}\overline{G}_{m_1a_1}\langle m_1a_1|\hat{\rho}(0)|m_1'a_1'\rangle\overline{G^\dagger}_{m_1'a_1'}\,4\,\Pi_{a_1a_1'}^{m_1m_1'}\,\delta_{m_1m_2'}\delta_{a_1a_2'} \ . \quad (7.78) \end{aligned}$$

Eq. (7.78) has to be compared with the result Eq. (3.54) of the diagrammatic theory. We observe that the disconnected terms are identical in both equations. However, we fail to reproduce the connected part, although the presence of the Kronecker deltas in (7.78) leads to the correct vanishing of the off–diagonal gain terms. It is not completely clear to us whether this is due to the asymptotic expansion itself or, more likely, to the evaluation of the Π matrix in the exponent. Here, the non–diagonal elements of the M matrix play an important role. Although vanishingly small in the limit $1 \ll \Delta \ll N_B$, they are essential in the evaluation of the averaged one–point function.

7.5 Summary

This chapter has been devoted to the rederivation of the ensemble average of the Dyson and von Neumann equations for system plus bath, considering the symmetrized random–matrix coupling. These averaged evolution equations have been obtained previously in Chapter 3 by means of a diagrammatic perturbation expansion. Here we have employed an averaging procedure which is more suitable for further generalization of our random–matrix model for quantum Brownian motion. This chapter can thus be considered as a first step towards an extension to a parameter dependent random–matrix interaction. By introducing a functional integral over both commuting and anticommuting variables (the generating function) we have been able to perform the ensemble average exactly. The averaged one–point and two–point functions have then been evaluated with the help of a saddle–point approximation. The resulting averaged propagator coincides exactly with the one obtained within the diagrammatic theory, both for weak and strong coupling. An agreement between the asymptotic expansion of the averaged density operator and its perturbation expansion expression could so far only be found for the disconnected part.

Chapter 8

Conclusion

The work presented here has been motivated by the question of the dependence of the Markovian description of quantum Brownian motion on model assumptions, and by the desire to go beyond the standard Caldeira–Leggett approach where the system of interest is coupled to a collection of harmonic oscillators. To this end, we have used a random band–matrix model for the system–bath interaction to derive Markovian master equations for one–dimensional quantum systems weakly coupled to a heat bath.

Our random–matrix model for quantum Brownian motion has been presented in Chapter 2. We have distinguished two cases: (I) That part of the interaction which depends on the bath variables is a member of a Gaussian ensemble of random matrices of proper symmetry; (II) the entire interaction is a member of a Gaussian ensemble of random matrices of proper symmetry. The second form of the random–matrix coupling, symmetric in both the variables of the system and the bath, may be considered as an approximation of form (I) and is related to the rotating wave approximation of the oscillator bath model.

In Chapter 3, the average over the random–matrix ensemble of both the time–evolution operator $U(t)$ and the density operator $\hat{\rho}(t)$ for system plus bath have been calculated in the limit of weak–coupling. For the approximated coupling (II), a diagrammatic perturbation expansion in powers of the interaction has been employed. By summing up the resulting Born series, the averaged Dyson equation for $\overline{U}(t)$ could be solved in closed form. In contrast, in the case of the non–symmetrized form (I), a closed solution of the averaged Dyson equation is not available. However, by using a variant of the diagrammatic method, valid only in the limit of weak coupling, an expression for the time derivative of $\overline{U}(t)$ could be found.

The averaged von Neumann equation for $\overline{\rho}(t)$ combined with the solution of the averaged Dyson equation have then been utilized to derive Markovian master equations for the reduced density operator of the system in Chapter 4. We have considered case (I) and case (II) separately and the form of the corresponding master equations differs in both cases. Although both equations show the typical gain–loss structure, there are no off–diagonal gain terms in the averaged master equation obtained with the symmetrized coupling (II). The physical consequences of the absence of these terms have been examined in Chapter 6.

In the following chapter, we have applied the averaged master equations to two simple quantum systems, the damped harmonic oscillator and the dissipative two–level system. The coupling to the heat bath has been taken linear in the position coordinate of the system. For the damped oscillator and random–matrix coupling (I), we have obtained the same equation as Agarwal who considered the oscillator bath model. In the high–

temperature limit, this equation coincides with the master equation derived by Caldeira and Leggett. For case (II), our master equation for the diagonal elements of the reduced density operator is identical to the Agarwal master equation evaluated in the rotating wave approximation (RWA). This is because in case (II) we impose conditions upon the interaction matrix elements of the position coordinate of the quantum system. These conditions are tantamount to the RWA. For the non–diagonal elements of the reduced density operator, these same conditions imply the vanishing of the gain terms. In this point our result differs from that obtained by Agarwal. For the dissipative two–level system and case (I), our master equation reduces, at zero temperature, to the Agarwal equation for spontaneous emission of a two–level atom. For case (II), as for the damped harmonic oscillator, only the equation for the diagonal elements agrees with the rotating wave approximation equation. This demonstrates that Markovian master equations for quantum Brownian motion derived in the weak–coupling limit possess universal validity: These equations are independent of the specific microscopic model used for their derivation. This is not true of approximations like the rotating wave approximation. Typically, such approximations violate certain invariance requirements (translational invariance in the case of the RWA) and, thereby, lose universal validity.

In Chapter 6, we have made use of the quantum Langevin approach to quantum dissipation to investigate the consequences of the lack of off–diagonal gain terms in the master equation for the damped oscillator in case (II). First, we have found that decoherence takes place in energy representation, in contrast to the RWA case where the reduced density operator becomes diagonal in the coherent–state basis. Second, the absence of the non–diagonal gain terms leads to a non–stationary stochastic process: The first moment of the noise operator is explicitly time–dependent. However, in the zero–temperature limit, this mean value is seen to vanish and the dynamics of our damped oscillator reduces to that of the RWA oscillator. Further, by expressing the Langevin equation in terms of position and momentum of the system and taking the classical limit with the help of the Weyl transform, we have found that translational invariance is violated, as in the RWA case. This is due to the symmetrization of the random interaction which treats the deterministic position of the system as a random quantity.

Finally, in Chapter 7, we have used the supersymmetry method to calculate the ensemble average of the evolution equations for system plus bath, considering the symmetrized random–coupling (II). This non–perturbative averaging procedure is most suitable for handling parameter dependent random–matrix interactions. The results obtained in this chapter therefore constitute a first step towards an extension of our random–matrix model to situations where the environment is strongly influenced by the system.

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