Flow Equation Solution for the Weak- to Strong-Coupling Crossover in the Sine-Gordon Model

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A continuous sequence of infinitesimal unitary transformations, combined with an operator product expansion for vertex operators, is used to diagonalize the quantum sine-Gordon model for $\beta^2 \in (2\pi, \infty)$. The leading order of this approximation already gives very accurate results for the single-particle gap in the strong-coupling phase. This approach can be understood as an extension of perturbative scaling theory since it links weak- to strong-coupling behavior in a systematic expansion. The method should also be useful for other strong-coupling problems that can be formulated in terms of vertex operators.

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Perturbative scaling arguments play an important role for analyzing a large variety of physical systems with many degrees of freedom. For strong-coupling problems, however, the perturbative renormalization group (RG) equations lead to divergences in the running coupling constants and the perturbative RG approach becomes invalid. In condensed matter theory the well-known paradigm for this kind of behavior is the Kondo model: The perturbative scaling equations still allow one to identify the low-energy scale of the Kondo model, but by themselves they do not lead to an understanding of the physical behavior associated with this energy scale. Wilson's numerical RG [1] could remedy this problem, but an analytical RG-like approach that links weak- to strong-coupling behavior in an expansion that can be systematically improved would still be desirable for many strong-coupling problems.

In this Letter it will be shown exemplary how Wegner's *flow equations* [2] can provide such an analytical description for a weak- to strong-coupling behavior crossover. In the flow equation approach a continuous sequence of infinitesimal unitary transformations is employed to make a Hamiltonian successively more diagonal. Large energy differences are decoupled before smaller energy differences, which makes the method similar to the conventional RG approach. However, degrees of freedom are not integrated out as in the RG but instead diagonalized. A similar framework that contains Wegner's flow equations as a special case has independently been developed by Głazek and Wilson (*similarity renormalization scheme*) [3].

The model under investigation in this Letter is the $(1 + 1)$D quantum sine-Gordon model [4]

$$H = \int dx \left(\frac{1}{2} \Pi(x)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + u\tau^{-2} \cos[\beta \phi(x)] \right)$$

with the commutator $[\Pi(x), \phi(x')] = -i\delta(x - x')$. Regularization with a UV-momentum cutoff $\Lambda \propto \tau^{-1}$ is implied and $u, \beta > 0$ are dimensionless parameters. The sine-Gordon model is one of the best studied integrable models and it has been solved using the inverse scattering method [5]. This model is therefore a good test case for studying the new approach. We will be interested in the universal low-energy properties ($E \ll \Lambda$) for small coupling constants u. It should be emphasized that the integrable structure underlying the inverse scattering solution will *not* be used in the approximate flow equation solution; the new method can also be used when nonintegrable perturbations are added.

The phase diagram of the sine-Gordon model consists of a gapped phase for $\beta^2 \lesssim 8\pi$ with massive soliton excitations and a gapless phase for $\beta^2 \gtrsim 8\pi$ with massless solitons [4]. The phase transition between these two phases for $\beta^2/8\pi = 1 + O(u)$ is of the Kosterlitz-Thouless type. In the massive phase the perturbative scaling equations [6] lead to an unphysical strong-coupling divergence of the running coupling constant u. The inverse scattering solution [5] furthermore shows the existence of bound soliton states (breathers) in the spectrum for $\beta^2 < 4\pi$ while such bound states are absent for $\beta^2 > 4\pi$. For $\beta^2 = 4\pi$ the sine-Gordon model can be mapped to a noninteracting massive Thirring model [7], which in turn can be diagonalized easily leading to the identification of the massive solitons with the Thirring fermions [8]. The sine-Gordon model is also related to a variety of other models like the spin-1/2 X-Y-Z chain, a 1D Fermi system with backward scattering, and the 2D Coulomb gas with temperature $T = \beta^{-2}$ and fugacity $z \propto u$; the IR-unstable fixed point corresponds to $T = 1/8\pi$ [9].

It will be shown that the flow equation approach generates a diagonalization of the sine-Gordon Hamiltonian both in the weak-coupling and in the strong-coupling phase in a systematic expansion that can be successively improved: No divergences of the running couplings are encountered in the strong-coupling regime for $\beta^2 > 2\pi$. The soliton mass is found to be in very good agreement with the inverse scattering solution. The crossover from weak- to strong-coupling behavior can be described and using this the soliton dispersion relation, for example, can be analyzed on all energy scales.

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For the purposes of this Letter it will be more convenient not to use a regularization of the sine-Gordon model with an explicit momentum cutoff, but instead to "smear out" the interaction term,

$$H = \int dx \left\{ \frac{1}{2}\,\Pi(x)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{u}{\pi a^2}\cos\left[\beta \int dy\, c(y)\phi(x+y)\right]\right\} \tag{1}$$

with the Lorentzian $c(y) = a/(2\pi y^2 + \pi a^2/2) \xrightarrow{a\to 0} \delta(y)$ [10]. This does not affect the universal properties for small energies $E \ll \Lambda \propto a^{-1}$. We expand the fields in normal modes,

$$\phi(x) = -\frac{i}{\sqrt{4\pi}} \sum_{k\neq 0} \frac{\sqrt{|k|}}{k}\, e^{-ikx}[\sigma_1(k) + \sigma_2(k)],$$

$$\Pi(x) = \frac{1}{\sqrt{4\pi}} \sum_{k\neq 0} \sqrt{|k|}\, e^{-ikx}[\sigma_1(k) - \sigma_2(k)],$$

where $\sum_k \stackrel{\text{def}}{=} \frac{2\pi}{L}\sum_{n=-\infty}^{\infty}$ with $k = 2\pi n/L$. L is the system size. The basic commutators are $(k, k' > 0)$ $[\sigma_1(-k), \sigma_1(k')] = [\sigma_2(k), \sigma_2(-k')] = \delta_{kk'}L/2\pi$; notice $\sigma_i^\dagger(-k) = \sigma_i(k)$. The vacuum is defined by $\sigma_1(-k)\,|\Omega\rangle = \sigma_2(k)\,|\Omega\rangle = 0$ for $k > 0$.

The flow equation approach [2] (see also [11]) generates a family of unitarily equivalent Hamiltonians $H(B)$ as a function of a flow parameter B [with dimension (energy)$^{-2}$], where $H(B=0)$ is the initial Hamiltonian and $H(B=\infty)$ the final diagonal Hamiltonian. This flow is generated by the differential equation

$$\frac{dH(B)}{dB} = [\eta(B), H(B)] \tag{2}$$

with $\eta(B) = -\eta(B)^\dagger$ some anti-Hermitian generator. Generically, Eq. (2) leads to the generation of new interaction terms not contained in the initial Hamiltonian. We therefore write $H(B) = H_0 + H_{\text{int}}(B) + H_{\text{new}}(B)$ with

$$H_0 = \sum_{p>0} p[\sigma_1(p)\sigma_1(-p) + \sigma_2(-p)\sigma_2(p)],$$
$$H_{\text{int}}(B) = \int dx_1\, dx_2\, u(B; x_1 - x_2)[V_1(\alpha; x_1)V_2(-\alpha; x_2) + \text{H.c.}]. \tag{3}$$

Here $V_j(\alpha; x)$ are normal ordered vertex operators with scaling dimension $\alpha(B) \stackrel{\text{def}}{=} \beta(B)/\sqrt{4\pi}$,

$$V_j(\alpha; x) =: \exp\left(\pm\,\alpha \sum_{p\neq 0} \frac{\sqrt{|p|}}{p}\, e^{-(a/2)|p| - ipx}\sigma_j(p)\right):$$

where $+$ (upper sign) for $j = 1$ and $-$ (lower sign) for $j = 2$. To avoid confusion the initial values of the couplings will from now on be denoted by u_0, β_0. $H(B=0)$ is identical to the sine-Gordon Hamiltonian (1) for $H_{\text{new}}(B=0) = 0$, $\alpha(B=0) = \beta_0/\sqrt{4\pi}$, and $u(B=0; x) = u_0\delta(x)(2\pi a/L)^{\alpha(B=0)^2}/2\pi a^2$.

Wegner's idea for constructing a suitable generator η is to choose $\eta(B) \stackrel{\text{def}}{=} [H_0, H_{\text{int}}(B)]$ [2]. This gives

$$\eta = -2i \int dx\, dy\, [\partial_y u(y)] \times [V_1(\alpha; x)V_2(-\alpha; x - y) + \text{H.c.}].$$

Using this generator, matrix elements connecting states with large energy differences are eliminated for small B, while matrix elements coupling more degenerate states are eliminated later for larger values of B.

First we evaluate $[\eta, H_0]$. This leads to $\partial u/\partial B = 4\partial^2 u/\partial x^2$, which makes the interaction increasingly nonlocal along the flow due to the decoupling procedure. In Fourier components $u(B; x) = \sum_p u_p(B)e^{-ipx}$ one finds

$$u_p(B) = \frac{\tilde{u}(B)}{4\pi^2 a^2}\, e^{-4p^2 B}\left(\frac{2\pi a}{L}\right)^{\alpha^2}.$$

$\tilde{u}(B)$ will turn out to be the running coupling constant of the flow equation approach and remains finite also in the strong-coupling phase. $\tilde{u}(B)$ is dimensionless, initially $\tilde{u}(B=0) = u_0$. The term $[\eta, H_{\text{int}}]$ leads to new interactions that have to be truncated to obtain a closed set of equations. The approximation used here is to take only operators with small scaling dimensions into account, i.e., to neglect more irrelevant operators. It is generated by truncating the operator product expansion (OPE) of two vertex operators in the following way:

$$V_j(\alpha; x)V_j(-\alpha; y) = \left(\frac{L}{2\pi}\right)^{\alpha^2} \frac{1}{[a \mp i(x-y)]^{\alpha^2}}\left(1 \mp i\alpha(x-y)\sum_{p\neq 0}\sqrt{|p|}\, e^{-(a/2)|p| - ipx}\sigma_j(p) + \cdots\right). \tag{4}$$

The approximation can be systematically improved by going to higher orders in this OPE. The term $[\eta, H_{\text{int}}]$ contains commutators with the structure

$$[V_1(\alpha; z_1)V_2(-\alpha; z_2), V_2(\alpha; z_2')V_1(-\alpha; z_1')] = -\{V_1(\alpha; z_1), V_1(-\alpha; z_1')\}V_2(\alpha; z_2')V_2(-\alpha; z_2) + V_1(\alpha; z_1)V_1(-\alpha; z_1')\{V_2(\alpha; z_2'), V_2(-\alpha; z_2)\}, \tag{5}$$

and terms where $\alpha \to -\alpha$ in one argument of the commutator (5). After normal ordering, the latter terms lead to

interactions $V_1(2\alpha; z_1)V_2(-2\alpha; z_2)$ with larger scaling dimensions. These will be neglected, but the terms generated by (5) will be included.

For $\alpha = 1$ ($\beta^2 = 4\pi$) the vertex operators describe fermions and the OPE (4) to all orders gives $\{V_j(-1; x), V_j(1; y)\} \xrightarrow{a\to 0} L\delta(x - y)$. Since this is a c-number, no higher order interactions are generated in (5) and the flow equations *close*. The flow equations therefore recover the equivalence of a sine-Gordon model with $\beta_0^2 = 4\pi$ to a massive noninteracting Thirring model [7] and readily diagonalize the latter.

In general we evaluate (5) using (4). The dominating contributions decaying most slowly with B can be identified in closed form [12]. Two structurally different interaction terms are generated: One term contributes to H_{new} and is discussed below [see Eq. (10)]. The other term has the structure $\sigma_1(k)\sigma_2(-k)$. Integrating it from B to $B + dB$, one generates $\sum_k w_k|k|\sigma_1(k)\sigma_2(-k)$ with infinitesimal coefficients w_k. This new interaction can be removed by a further infinitesimal unitary transformation with the structure $e^{-U}H(B)e^U$, where $U = \frac{1}{2}\sum_{k>0} w_k[\sigma_1(k)\sigma_2(-k) - \text{H.c.}]$. This new infinitesimal unitary transformation yields a modification of the scaling dimension of the vertex operators in H_{int} and a flow of the coupling constant $\tilde{u}(B)$. In terms of a logarithmic dimensionless flow parameter $\ell = \frac{1}{2}\ln(32Ba^{-2})$ one derives [12]

$$\frac{d\beta^2(\ell)}{d\ell} = -u_0^2 \frac{\beta^4(\ell)}{4\pi\Gamma[-1 + \beta^2(\ell)/4\pi]} \times \exp\left(4\ell - \frac{1}{2\pi}\int_0^\ell d\ell'\,\beta^2(\ell')\right). \quad (6)$$

The running coupling constant flows according to $\tilde{u}(\ell) = u_0 \exp[F(\ell)]$, where

$$F(\ell) = \frac{1}{4\pi}\left(\ell\beta^2(\ell) - \int_0^\ell d\ell'\,\beta^2(\ell')\right). \quad (7)$$

In the strong-coupling phase the flow terminates at $\beta^2(\infty) = 4\pi$ due to the divergent Γ-function in (6). In our approach $\beta^2 = 4\pi$ is therefore an *attractive* strong-coupling fixed point. For comparison with the RG equations [6] one can introduce $u(\ell) \stackrel{\text{def}}{=} u_0 \exp[2\ell - \frac{1}{4\pi}\int_0^\ell d\ell'\,\beta^2(\ell')]$ and rewrite (6) as two coupled differential equations:

$$\frac{d\beta^{-2}(\ell)}{d\ell} = \frac{1}{4\pi\Gamma[-1 + \beta^2(\ell)/4\pi]}\,u^2(\ell), \qquad \frac{du(\ell)}{d\ell} = \left(2 - \frac{\beta^2(\ell)}{4\pi}\right)u(\ell), \quad (8)$$

with $\beta(\ell = 0) = \beta_0$, $u(\ell = 0) = u_0$. For $\beta_0^2 = 8\pi$ Eq. (8) coincides with the two loop scaling equations [6]: Depending on the value of u_0, the sine-Gordon model for $\beta_0^2 > 8\pi$ flows to either $\beta^2(\infty) = 4\pi$ (*strong coupling*) or $\beta^2(\infty) \geq 8\pi$ (*weak coupling*). Equation (8) therefore reproduces the Kosterlitz-Thouless phase diagram. Also the hidden SU(2) symmetry in (1) for $\beta_0^2 = 8\pi(1 \pm u_0)$, $u_0 \ll 1$, is recovered although our approximation scheme does not manifestly respect this symmetry [13].

In the strong-coupling phase a gap opens in the spectrum and the low-energy excitations are fermionic [5]. In our approach this follows most easily by approximating (3) for large B [such that $|\beta^2(B) - 4\pi| \ll 1$] as

$$H_{\text{int}}(B) = \int dx_1\,dx_2\,\frac{\tilde{u}(B)}{\pi a^2}\,\frac{1}{\sqrt{16\pi B}}\exp\left(-\frac{(x_1 - x_2)^2}{16B}\right) \times [\psi_1^\dagger(x_1)\psi_2(x_2) + \psi_2^\dagger(x_2)\psi_1(x_1)] \quad (9)$$

with fermions $\psi_j(x) \stackrel{\text{def}}{=} V_j(-1; x)$. The running coupling constant $\tilde{u}(B)$ approaches a *finite* value in this limit [14]. The asymptotic value $\tilde{u}(\infty)$ can therefore be interpreted as the renormalized coupling constant parametrizing a quadratic Hamiltonian $H_0 + H_{\text{int}}(B)$ that describes the low-energy behavior of the initial sine-Gordon model. The RG strong-coupling divergence of the running coupling constant is avoided since the interaction (9) in this effective low-energy Hamiltonian becomes increasingly nonlocal for $B \to \infty$. The gap $\Delta = 2m$ in the spectrum is obtained in terms of the renormalized coupling constant $\tilde{u}(\infty)$ that sets the mass of the effective low-energy Thirring fermions with $m = \tilde{u}(\infty)/a$.

The soliton mass derived from Eqs. (7) and (8) can be compared with the predictions of perturbative scaling: For example, for $\beta_0^2 = 8\pi(1 - u_0)$ [corresponding to the hidden SU(2) symmetry] one can show analytically $m \propto u_0^\tau \exp(-1/2u_0)/a$ for $u_0 \downarrow 0$ with $\tau = (1 + \gamma)/3 \approx 0.526$ ($\gamma \approx 0.577$ is Euler's constant). This agrees well with the third order RG result $\tau_{\text{RG}} = 1/2$ and the same exponential dependence [6,15]. For smaller values of β_0 one finds the power law behavior

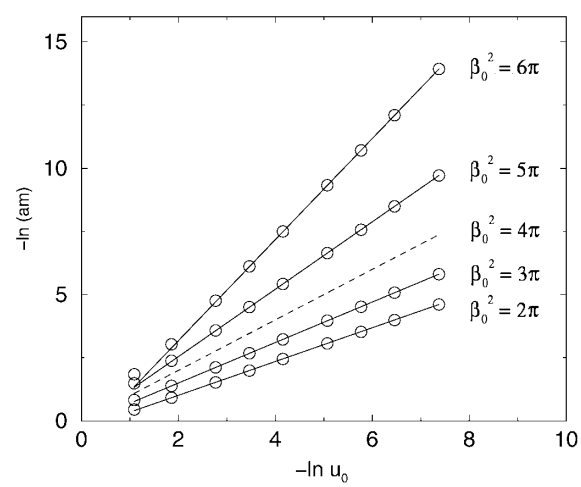

FIG. 1. Soliton mass as a function of the coupling constant for various values of β_0^2: The full lines are constrained fits of the power law behavior $am \propto u_0^{1/(2-\beta_0^2/4\pi)}$ [4] to the flow equation results (open circles) with the proportionality constant being fitted. The dashed line is the case $\beta_0^2 = 4\pi$ where the flow equation approach agrees trivially (see text).

$m \propto u_0^{1/(2-\beta_0^2/4\pi)}/a$ known from the inverse scattering solution [4]: This is very well confirmed by the numerical integration of the flow equations (Fig. 1). The proportionality constant depends only on β_0 for $u_0 \downarrow 0$, in particular $m = u_0/a$ for $\beta_0^2 = 4\pi$ as known from the noninteracting Thirring model [7].

The flow equation solution not only provides the excitation gap, but one can also obtain information about the crossover, e.g., the full dispersion relation: The Hamiltonian for $B = \infty$ takes the form $H(\infty) = H_0 + H_{\text{new}}(\infty)$ since the interaction term $H_{\text{int}}(B)$ is eliminated. One can verify that $H_{\text{new}}(B)$ does not modify the flow of $\beta(B)$ and $\tilde{u}(B)$ as derived above [12]. The new terms generated during the flow in $H_{\text{new}}(\infty)$ can be split up as $H_{\text{new}}(\infty) = H_{\text{diag}}(\infty) + H_{\text{res}}(\infty)$, where $H_{\text{diag}}(\infty)$ contains the terms that follow in leading order of the OPE from $[\eta, H_{\text{int}}]$ in (5), while $H_{\text{res}}(\infty)$ formally contains everything not taken into account in the present order of the OPE. Integration of the flow equations gives [12]

$$H_{\text{diag}}(\infty) = \sum_{p>0} \omega_p(\infty)[P_1^\dagger(p)P_1(p) + P_1(-p)P_1^\dagger(-p) + P_2^\dagger(-p)P_2(-p) + P_2(p)P_2^\dagger(p)] \qquad (10)$$

with certain coefficients $\omega_p(\infty)$. Here $P_j(p) = \int dx\, e^{-ipx} V_j(-\alpha(B_p); x)$ and $B_p \stackrel{\text{def}}{=} 1/4p^2$. The spectrum can be analyzed easily since $[H_0, H_{\text{diag}}(\infty)] = 0$: In leading order the single-particle (soliton) excitations of $H_0 + H_{\text{diag}}(\infty)$ are $P_1^\dagger(k)|\Omega\rangle$ for $k > 0$ and $P_2^\dagger(k)|\Omega\rangle$ for $k < 0$. The single-hole (antisoliton) excitations are $P_1(k)|\Omega\rangle$ for $k < 0$ and $P_2(k)|\Omega\rangle$ for $k > 0$. In the strong-coupling phase for $\beta_0^2 \geq 4\pi$ the respective excitation energies are very accurately (but not exactly) described by $E_k^2 = k^2 + [\tilde{u}(\infty)/a]^2$ in the small coupling limit: There are β_0-dependent universal corrections in the crossover region $k = O(m)$ that vanish for $\beta_0^2 \to 4\pi$ and reach at most of the order of 2% for $\beta_0^2 = 8\pi$. The character of the excitations varies from scaling dimension $\alpha(B = 0)$ to the low-energy Thirring fermions with $\alpha(B = \infty) = 1$. In the weak-coupling phase the spectrum remains gapless and $E_k = |k|$ for $k \to 0$. Notice that the elementary excitations are expressed with respect to a transformed basis since $H(\infty)$ and $H(0)$ are related by a complicated unitary transformation.

For $\beta_0^2 < 2\pi$ the differential equations for $\omega_p(B)$ lead to divergences since then the $\cos[\beta_0\phi(x)]$ perturbation is too relevant, limiting the approach to $\beta_0^2 > 2\pi$. For $2\pi < \beta_0^2 < 4\pi$ the single-particle spectrum is still well described by (10); our approximations become better as $\beta_0^2 \uparrow 4\pi$. Higher orders in the OPE are nevertheless required to study the formation of bound states for $\beta_0^2 < 4\pi$ [5] due to residual interactions in $H_{\text{res}}(\infty)$.

Summing up, we have applied a continuous sequence of infinitesimal unitary transformations, combined with an operator product expansion for vertex operators, to the quantum sine-Gordon model with $\beta^2 \in (2\pi, \infty)$. The approximations are systematic since more terms in the OPE can successively be taken into account and will not endanger the stability of the strong-coupling fixed point. The results for the soliton mass in the strong-coupling phase agree with two loop scaling predictions for $\beta^2 \approx 8\pi$ (approximate agreement was even found to three loop order) and exact methods [4] applicable for smaller β^2. The full dispersion relation could be obtained and the crossover from weak- to strong-coupling behavior described. The method also allows one to study correlation functions [16] and other strong-coupling problems that can be formulated in terms of vertex operators.

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[12] Details of this calculation will be presented elsewhere.
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[14] In the strong-coupling phase $u(\ell)$ diverges according to Eq. (8). But notice that $u(\ell)$ is *not* the expansion parameter of our approach and has only been defined for rewriting Eq. (6): The real expansion parameter is $\tilde{u}(\ell)$.
[15] Higher orders in the OPE can modify τ, therefore the already small difference to τ_{RG} is remarkable.
[16] See, e.g., S. Kehrein and A. Mielke, Ann. Phys. (Leipzig) **6**, 90 (1997) for the spin-boson model.