Center for Electronic Correlations and Magnetism
University of Augsburg

Construction of exact ground states for correlated electron models

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Outline:

- Electronic correlations
- Comprehensive strategy for the construction of exact many-electron ground states
- Diamond Hubbard chains
  → e.g., correlation-induced half-metal

  Triangle Hubbard chains/1D periodic Anderson model
  → ferromagnetism in $CeRh_3B_2$

  Pentagon Hubbard chains
  → route to ferromagnetism in organic polymers

In collaboration with

Zsolt Gulacsi (Debrecen, Hungary)
Arno Kampf (Augsburg)

Correlations

Correlation [lat.]: *con* + *relatio* *("with relation")*

Grammar: *either ... or*

Correlations in mathematics, natural sciences:

$$\boxed{\langle AB\rangle \neq \langle A\rangle\langle B\rangle}$$

e.g., densities

$$\boxed{\langle \rho(\mathbf{r})\rho(\mathbf{r}')\rangle \neq \langle \rho(\mathbf{r})\rangle\langle \rho(\mathbf{r}')\rangle}$$

Correlations (I):

Effects beyond factorization approximations (e.g., Hartree-Fock)

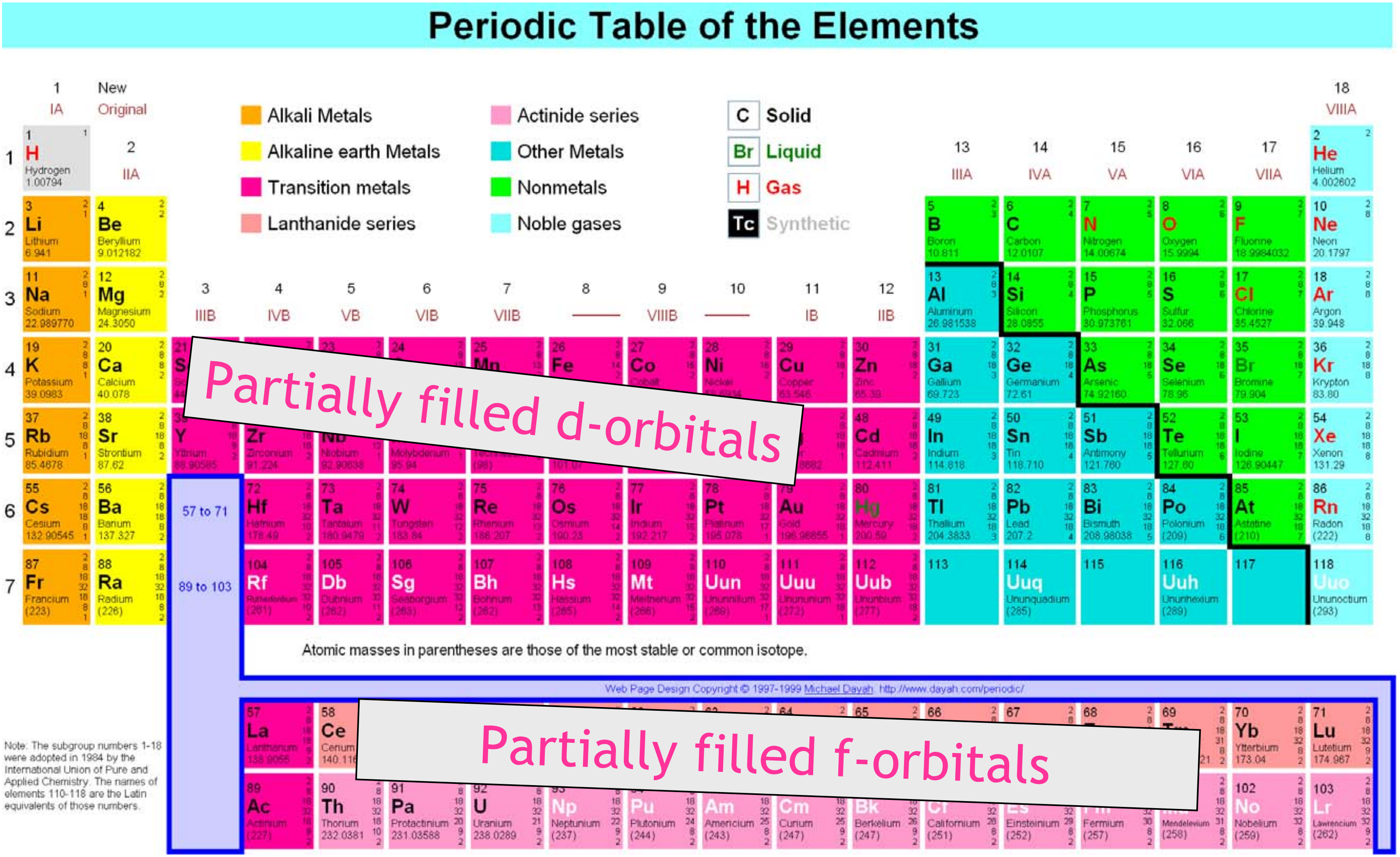
Temporal/spatial correlations in everyday life

Beware: External periodic potential

Temporal/spatial correlations in everyday life

Time/space average insufficient

Electronic Correlations in Solids



Partially filled d-orbitals

Partially filled f-orbitals

Narrow d,f-orbitals/bands → strong electronic correlations

Correlated electron materials

High sensitivity to small changes of microscopic parameters

- large resistivity changes
- huge volume changes
- high T_c superconductivity
- strong thermoelectric response
- colossal magnetoresistance

with

Technological applications:

- sensors, switches
- magnetic storage
- thermoelectrics
- functional materials, ...

Wanted:

Exact results

for correlated electron models

(with experimental relevance)

Construction of exact many-electron ground states

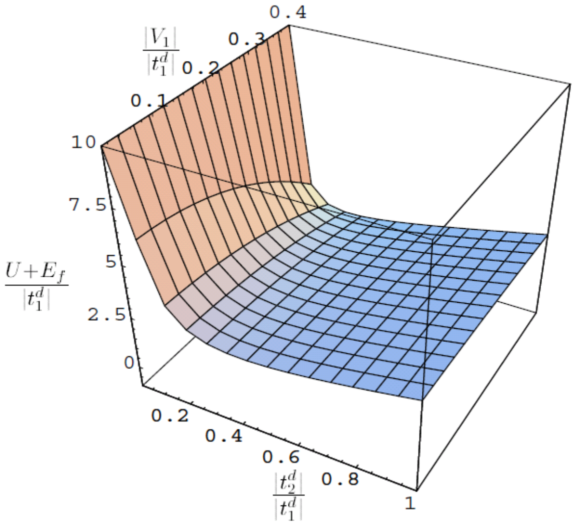
Strategy

Step 1: Transform a many-electron Hamiltonian of your choice on a lattice of your choice into positive semidefinite form

Step 2: Construct a ground state

Step 3: Prove the uniqueness of the ground state

- Works in any dimension
- No integrability required
- Applicable to any Hamiltonian with sufficiently many microscopic parameters



Hypersurface in parameter space

Construction of exact many-electron ground states

Strategy

Step 1: Transform a many-electron Hamiltonian of your choice
on a lattice of your choice into positive semidefinite form

Step 2: Construct a ground state

Hubbard model and PAM on decorated lattices in $d \geq 2$ at $U = \infty$	Brandt, Giesekus (1992)
PAM, Emery model in d=1,2 on regular lattices	Strack (1993)
Ferromagnetism in extended Hubbard models for arbitary d	Strack, DV (1993, 1994, 1995)
	de Boer, Schadschneider (1995)
Superconductivity in the extended Hubbard model	Montorsi, Campbell (1996)
Composite operators defined on the entire unit cell	Orlik, Gulacsi (1998, 2001)
PAM on regular lattices in d=2,3 at $U < \infty$	Gurin, Gulacsi (2001, 2002)
Application to d=1	Sarasua, Continentino (2002)
Optimal ground state of quantum spin models	Ahrens, Schadschneider, Zittartz (2005)
Gossamer Hamiltonian	Laughlin (2002/2006); Bernevig *et al*. 2003, 2006)

Construction of exact many-electron ground states

Strategy

Step 1: Transform a many-electron Hamiltonian of your choice on a lattice of your choice into positive semidefinite form

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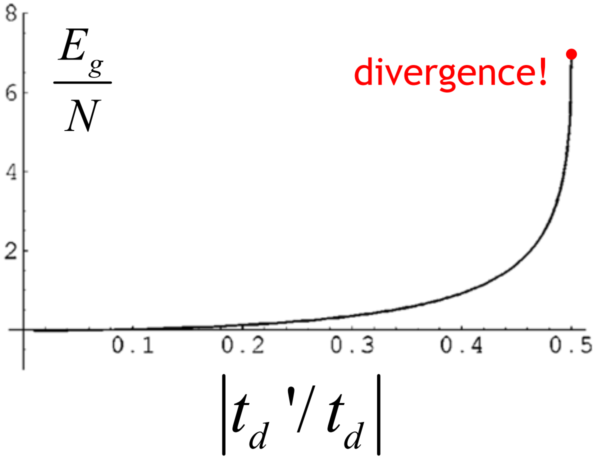
PAM in d=3

- Exact insulating and itinerant (non-Fermi liquid) ground states at ¼ and ¾ filling
- Proof of ferromagnetism in the PAM in d=3



Gulacsi, DV (2003, 2005)

High sensitivity to small changes of microscopic parameters found

Construction of exact many-electron ground states: Details

Step 1: Transform the many-electron Hamiltonian into positive semidefinite form

$$\hat{H} = \hat{H}_0 + \hat{H}_U \overset{!}{=} \sum_n \hat{P}_n + E_g \equiv \hat{H}' + E_g, \quad \hat{P}_n : \text{ positive semidefinite operators}$$

$$\langle \psi | \hat{P}_n | \psi \rangle \geq 0$$

e.g., $\hat{P}_n = \Omega^\dagger \Omega \,, \;\; \Omega \Omega^\dagger$

→ spectrum has a well-defined lower bound

Transformation is feasible (if at all)
- in several different ways
- on a hypersurface in the full parameter space

Construction of exact many-electron ground states: Details

$$\hat{H} = \hat{H}_0 + \hat{H}_U \overset{!}{=} \sum_n \hat{P}_n + E_g \equiv \hat{H}' + E_g, \quad \hat{P}_n : \text{positive semidefinite} \quad \left\langle \psi \middle| \hat{P}_n \middle| \psi \right\rangle \geq 0$$

Step 2: Construct a ground state

$$\hat{P}_n \left| \Psi_g \right\rangle = 0 \Rightarrow \hat{H} \left| \Psi_g \right\rangle = E_g \left| \Psi_g \right\rangle$$

ground state ↑ ground-state energy ↑

Construction of $|\Psi_g>$ depends on the structure of P_n

Construction of exact many-electron ground states: Details

$$\hat{H} = \hat{H}_0 + \hat{H}_U \overset{!}{=} \sum_n \hat{P}_n + E_g \equiv \hat{H}' + E_g, \quad \hat{P}_n :$$ positive semidefinite $\langle \psi | \hat{P}_n | \psi \rangle \geq 0$

Step 3: Prove the uniqueness of the ground state:

$|\Psi_g\rangle$ has to span $\ker(\hat{H}') := \left\{ |\phi\rangle \;\middle|\; \hat{H}'|\phi\rangle = 0 \right\}$

$\hat{H}' = \sum_{n=1}^{L} \hat{P}_n$ → $\ker(\hat{H}') = \bigcap_{n=1}^{L} \ker(\hat{P}_n)$

→ Necessary and sufficient to prove that:

(i) $|\Psi_g\rangle \in \ker(\hat{H}')$

(ii) all states $|\Psi\rangle \in \bigcap_{n=1}^{L} \ker(\hat{P}_n)$ can be written in terms of the constructed ground state $|\Psi_g\rangle$.

In particular: Exact ground states with flat bands

Condensed matter systems with macroscopic degeneracies
→ very sensitive reaction to perturbations
→ emergent behavior

Examples:

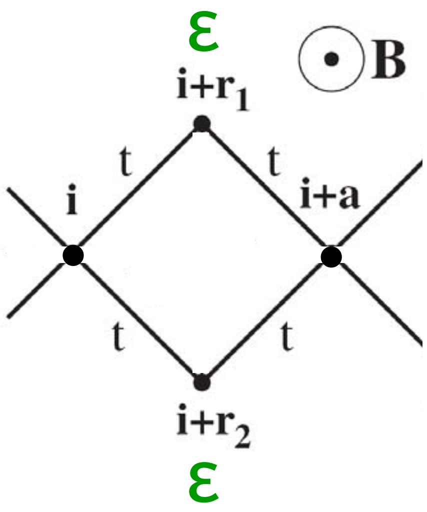
• Electrons in a magnetic field in 2D	Landau (1930)
• Spins on lattices with geometric frustration	Moessner, Ramirez (2006)
• Dispersionless (“flat“) bands in solids	Mielke (1991) Mielke, Tasaki (1993) Arita, Suwa Kuroki, Aoki (2002)

→ First step:
Find exact solutions of correlation models with flat bands

Diamond Hubbard chains:
Correlation induced half-metallic behavior

Flat bands in the single-electron band structure stay flat for U>0

Z. Gulacsi, A. Kampf, DV
Phys. Rev. Lett. **99**, 026404 (2007)

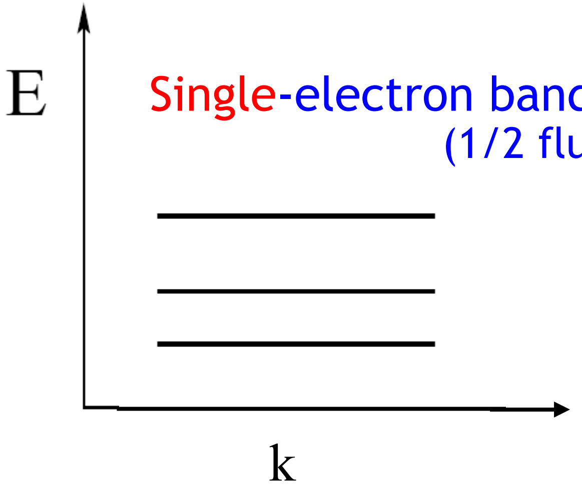


“Aharonov-Bohm cage”

Peierls phase factor

$\delta = 2\pi\Phi/\Phi_0$ $\qquad$ $\mathbf{A} \parallel \mathbf{a}$

$t_{\mathbf{j},\mathbf{j}'}(\mathbf{B}) = t_{\mathbf{j},\mathbf{j}'}(0)\exp[(i2\pi/\Phi_0)\int_{\mathbf{j}}^{\mathbf{j}'}\mathbf{A}\cdot d\mathbf{l}]$


Single-electron bands for $\delta = \pi/2$
(1/2 flux per unit cell)

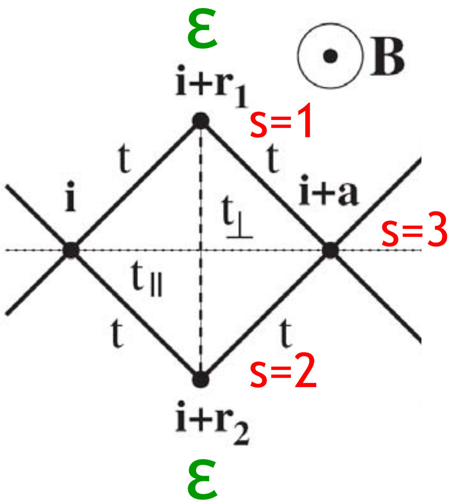
Vidal, Doucot, Mosseri, Butaud (2000):

ε=0, $\delta = \pi/2$, 2 electrons: excited singlet eigenstates

- are localized for U=0
- become delocalized for U>0

→ Interaction *U* is able to induce subtle correlations leading to conducting states

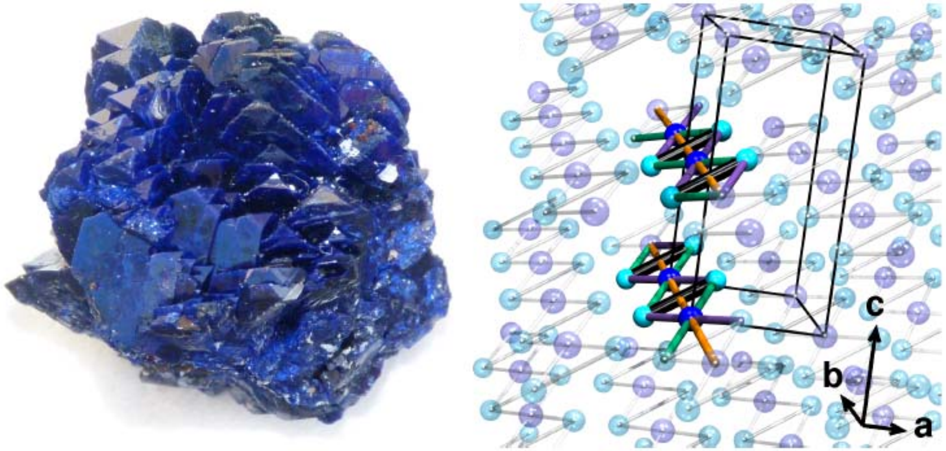
Also valid for finite densities ? √



3 sites per cell → 3 bands

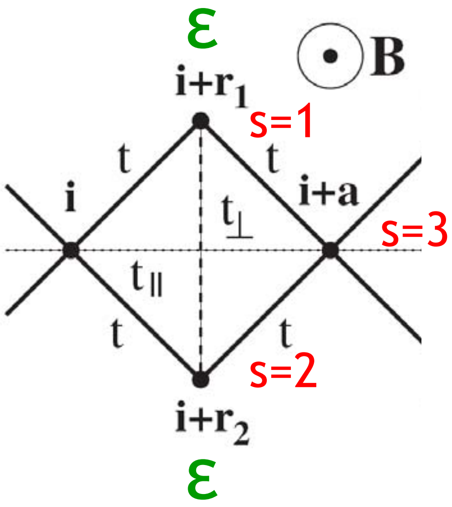
s=1,2,3 sublattice index

N_c = # cells
N = # electrons
$n = \dfrac{N}{3N_c}$ electron density



Jeschke *et al.*,
arXiv:1012.1090

Azurite $Cu_3(CO_3)_2(OH)_2$



3 sites per cell → 3 bands

s=1,2,3 sublattice index

N_c = # cells

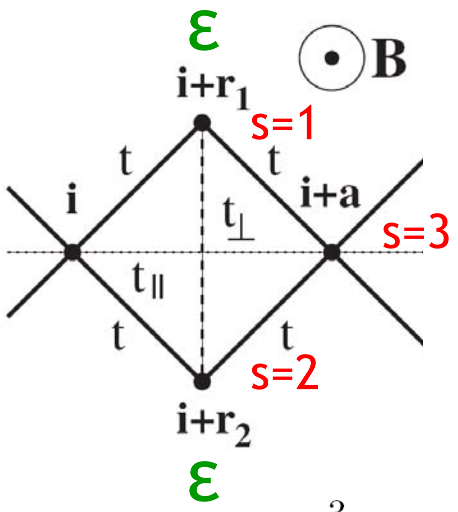
N = # electrons

$n = \dfrac{N}{3N_c}$ electron density

$$\begin{aligned}\hat{H}_0 = \sum_{\sigma}\sum_{\mathbf{i}=1}^{N_c}\{[te^{i\frac{\delta}{2}}(&\hat{c}^{\dagger}_{\mathbf{i}+\mathbf{r}_2,\sigma}\hat{c}_{\mathbf{i},\sigma} + \hat{c}^{\dagger}_{\mathbf{i}+\mathbf{a},\sigma}\hat{c}_{\mathbf{i}+\mathbf{r}_2,\sigma} + \\ &\hat{c}^{\dagger}_{\mathbf{i}+\mathbf{r}_1,\sigma}\hat{c}_{\mathbf{i}+\mathbf{a},\sigma} + \hat{c}^{\dagger}_{\mathbf{i},\sigma}\hat{c}_{\mathbf{i}+\mathbf{r}_1,\sigma}) + t_{\perp}\,\hat{c}^{\dagger}_{\mathbf{i}+\mathbf{r}_2,\sigma}\hat{c}_{\mathbf{i}+\mathbf{r}_1,\sigma} + \\ &t_{\parallel}\,\hat{c}^{\dagger}_{\mathbf{i}+\mathbf{a},\sigma}\hat{c}_{\mathbf{i},\sigma} + H.c.] + \varepsilon\sum_{s=1,2}\hat{n}_{\mathbf{i}+\mathbf{r}_s,\sigma}\}\end{aligned}$$

$$\hat{H}_U = U\sum_{\mathbf{i}=1}^{N_c}\sum_{s=1}^{3}\hat{n}_{\mathbf{i}+\mathbf{r}_s,\uparrow}\hat{n}_{\mathbf{i}+\mathbf{r}_s,\downarrow}$$

$$\boxed{\hat{H} = \hat{H}_0 + \hat{H}_U}$$



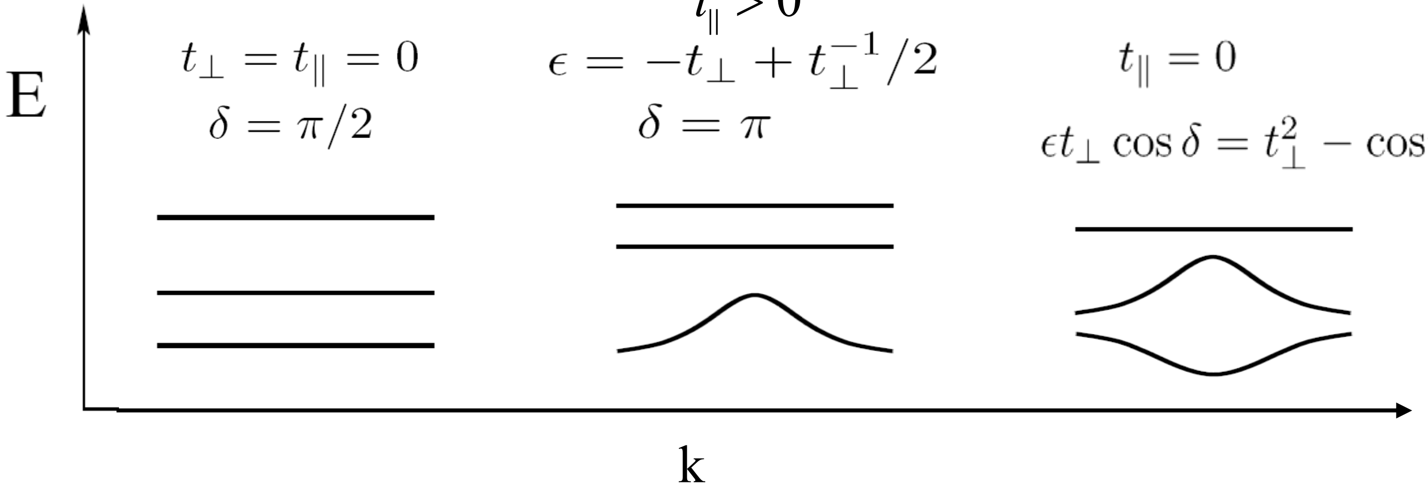
3 sites per cell → 3 bands

s=1,2,3 sublattice index

N_c = # cells
N = # electrons
$n = \frac{N}{3N_c}$ electron density

FT $$\hat{H}_0 = \sum_{\mathbf{k},\sigma} \sum_{s,s'=1}^{3} M_{s,s'}(\mathbf{k})\, \hat{c}^\dagger_{s,\mathbf{k},\sigma} \hat{c}_{s',\mathbf{k},\sigma}$$

Examples:



$t_\perp = t_\| = 0$
$\delta = \pi/2$

$t_\| > 0$
$\epsilon = -t_\perp + t_\perp^{-1}/2$
$\delta = \pi$

$t_\| = 0$
$\epsilon t_\perp \cos\delta = t_\perp^2 - \cos^2\delta$

Energy unit: $2t$

Single-electron bands

Solution I: Correlated half-metal

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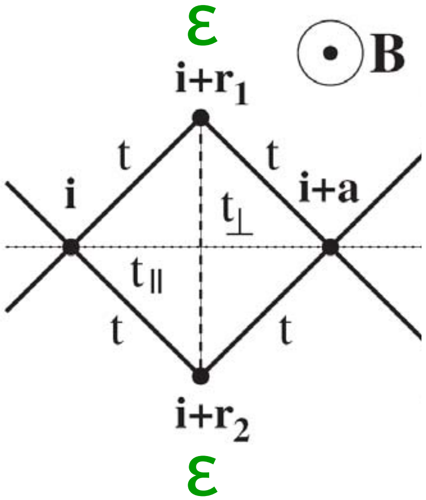
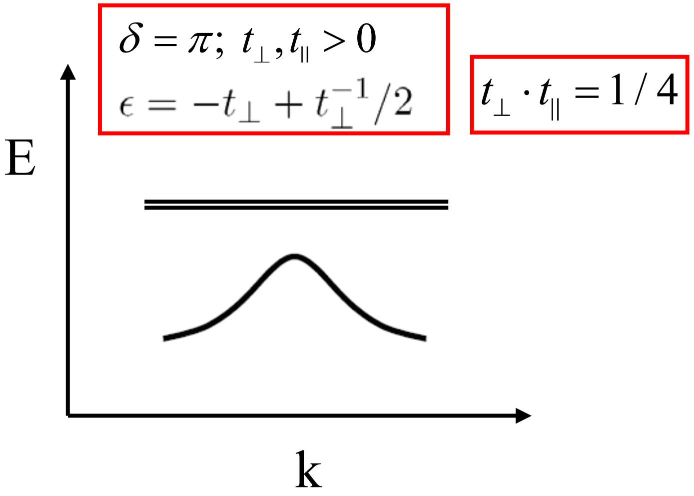


→ Investigate



$\delta = \pi;\ t_\perp, t_\parallel > 0$

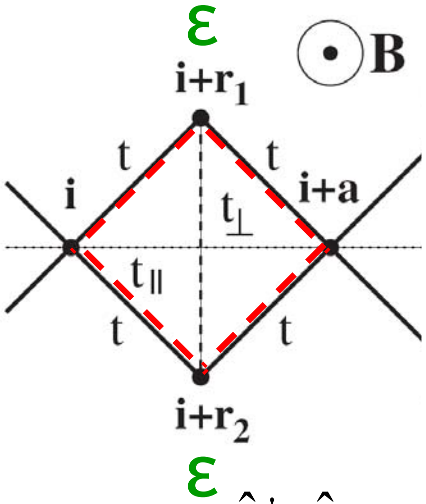
$\epsilon = -t_\perp + t_\perp^{-1}/2$

$t_\perp \cdot t_\parallel = 1/4$

Single-electron bands

Solution I: Correlated half-metal

1. Transformation of the Hamiltonian into positive semidefinite form



$\boxed{\hat{H}_0}$ Define non-canonical fermionic operators:

One composite operator on each plaquette

$$\hat{A}_{i,\sigma} = a_1\hat{c}_{i\sigma} + a_2\hat{c}_{i+\mathbf{r}_2\sigma} + a_3\hat{c}_{i+\mathbf{a}\sigma} + a_4\hat{c}_{i+\mathbf{r}_1\sigma}$$

$$(\hat{A}_{\mathbf{i},\sigma})^2 = 0$$

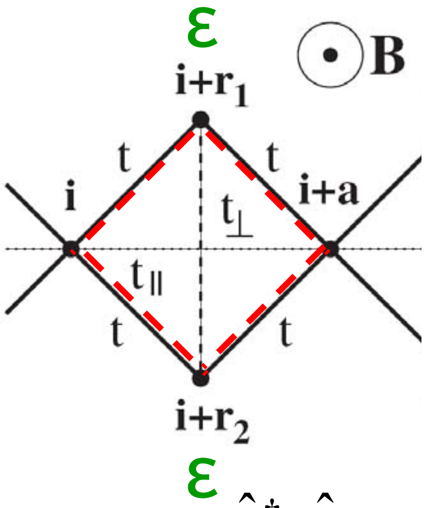
$$\{\hat{A}_{\mathbf{i},\sigma}, \hat{A}^{\dagger}_{\mathbf{j},\sigma}\} \neq \delta_{\mathbf{i},\mathbf{j}}$$

$$\begin{aligned}\Rightarrow \hat{A}^{\dagger}_{i\sigma}\hat{A}_{i\sigma} = {} & (a_2^* a_1 \hat{c}^{\dagger}_{i+\mathbf{r}_2\sigma}\hat{c}_{i\sigma} + a_3^* a_2 \hat{c}^{\dagger}_{i+\mathbf{a}\sigma}\hat{c}_{i+\mathbf{r}_2\sigma} + a_4^* a_3 \hat{c}^{\dagger}_{i+\mathbf{r}_1\sigma}\hat{c}_{i+\mathbf{a}\sigma} + \\ & a_1^* a_4 \hat{c}^{\dagger}_{i\sigma}\hat{c}_{i+\mathbf{r}_1\sigma} + a_2^* a_4 \hat{c}^{\dagger}_{i+\mathbf{r}_2\sigma}\hat{c}_{i+\mathbf{r}_1\sigma} + a_3^* a_1 \hat{c}^{\dagger}_{i+\mathbf{a}\sigma}\hat{c}_{i\sigma} + \text{ H.c.}) + \\ & |a_1|^2 n_{i\sigma} + |a_2|^2 n_{i+\mathbf{r}_2\sigma} + |a_3|^2 n_{i+\mathbf{a}\sigma} + |a_4|^2 n_{i+\mathbf{r}_1\sigma}\end{aligned}$$

$$\begin{aligned}-\sum_{i\sigma}\hat{A}^{\dagger}_{i\sigma}\hat{A}_{i\sigma} \overset{!}{=} \hat{H}_0 = \sum_{\sigma}\sum_{\mathbf{i}=1}^{N_c}\{[te^{i\frac{\delta}{2}}(& \hat{c}^{\dagger}_{\mathbf{i}+\mathbf{r}_2,\sigma}\hat{c}_{\mathbf{i},\sigma} + \hat{c}^{\dagger}_{\mathbf{i}+\mathbf{a},\sigma}\hat{c}_{\mathbf{i}+\mathbf{r}_2,\sigma} + \\ & \hat{c}^{\dagger}_{\mathbf{i}+\mathbf{r}_1,\sigma}\hat{c}_{\mathbf{i}+\mathbf{a},\sigma} + \hat{c}^{\dagger}_{\mathbf{i},\sigma}\hat{c}_{\mathbf{i}+\mathbf{r}_1,\sigma}) + t_{\perp}\hat{c}^{\dagger}_{\mathbf{i}+\mathbf{r}_2,\sigma}\hat{c}_{\mathbf{i}+\mathbf{r}_1,\sigma} + \\ & t_{\parallel}\hat{c}^{\dagger}_{\mathbf{i}+\mathbf{a},\sigma}\hat{c}_{\mathbf{i},\sigma} + H.c.] + \varepsilon\sum_{s=1,2}\hat{n}_{\mathbf{i}+\mathbf{r}_s,\sigma}\}\end{aligned}$$

Solution I: Correlated half-metal

1. Transformation of the Hamiltonian into positive semidefinite form



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$$(\hat{A}_{\mathbf{i},\sigma})^2 = 0$$

$$\{\hat{A}_{\mathbf{i},\sigma}, \hat{A}^\dagger_{\mathbf{j},\sigma}\} \neq \delta_{\mathbf{i},\mathbf{j}}$$

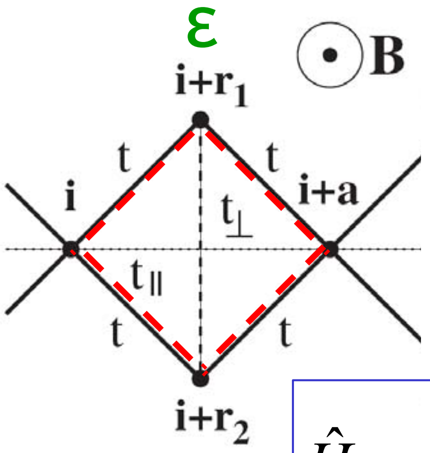
$$\Rightarrow \hat{A}^\dagger_{i\sigma}\hat{A}_{i\sigma} = (a_2^* a_1\hat{c}^\dagger_{i+\mathbf{r}_2\sigma}\hat{c}_{i\sigma} + a_3^* a_2\hat{c}^\dagger_{i+\mathbf{a}\sigma}\hat{c}_{i+\mathbf{r}_2\sigma} + a_4^* a_3\hat{c}^\dagger_{i+\mathbf{r}_1\sigma}\hat{c}_{i+\mathbf{a}\sigma} + a_1^* a_4\hat{c}^\dagger_{i\sigma}\hat{c}_{i+\mathbf{r}_1\sigma} + a_2^* a_4\hat{c}^\dagger_{i+\mathbf{r}_2\sigma}\hat{c}_{i+\mathbf{r}_1\sigma} + a_3^* a_1\hat{c}^\dagger_{i+\mathbf{a}\sigma}\hat{c}_{i\sigma} + \text{H.c.}) + |a_1|^2 n_{i\sigma} + |a_2|^2 n_{i+\mathbf{r}_2\sigma} + |a_3|^2 n_{i+\mathbf{a}\sigma} + |a_4|^2 n_{i+\mathbf{r}_1\sigma}$$

$$-\sum_{i\sigma}\hat{A}^\dagger_{i\sigma}\hat{A}_{i\sigma} \overset{!}{=} \hat{H}_0 \Rightarrow \boxed{\begin{aligned} &a_2^* a_1 = a_3^* a_2 = a_4^* a_3 = a_1^* a_4 = -te^{i\delta/2} \\ &a_2^* a_4 = -t_\perp \\ &a_3^* a_1 = -t_\parallel \\ &|a_1|^2 + |a_3|^2 = \varepsilon + |a_2|^2 = \varepsilon + |a_4|^2 \end{aligned}} \Rightarrow$$

$$\boxed{\begin{aligned} \hat{A}_{\mathbf{i},\sigma} = \sqrt{t_\parallel}[&\hat{c}_{\mathbf{i},\sigma} - \hat{c}_{\mathbf{i+a},\sigma} \\ &- 2t_\perp e^{i\delta/2}(\hat{c}_{\mathbf{i+r_1},\sigma} - \hat{c}_{\mathbf{i+r_2},\sigma})] \end{aligned}}$$

Solution I: Correlated half-metal

1. Transformation of the Hamiltonian into positive semidefinite form



$$\boxed{\hat H_0 = -\sum_{i\sigma} \hat A^{\dagger}_{i\sigma} \hat A_{i\sigma} \overset{!}{=} +\sum_{i\sigma} \hat A_{i\sigma} \hat A^{\dagger}_{i\sigma} - 2N_c \sum_{m=1}^{4} |a_m|^2}$$

Positive semidefinite

$$\boxed{\hat H_U}$$

$$\hat H_U = U \sum_{\mathbf{i}}^{3N_c} \hat n_{\mathbf{i}\uparrow} \hat n_{\mathbf{i}\downarrow} = U\hat P + U\hat N - UN_c$$

N_c: # unit cells

$$\hat P = \sum_{\mathbf{i}} \hat P_{\mathbf{i}} \,,\quad \hat P_{\mathbf{i}} = (\hat n_{\mathbf{i}\uparrow} - 1)(\hat n_{\mathbf{i}\downarrow} - 1) = \begin{cases} 1, & \text{unoccupied site} \\ 0, & \text{at least one electron} \end{cases}$$

$$\Rightarrow \boxed{\hat H = \sum_{\mathbf{i},\sigma} \hat A_{\mathbf{i},\sigma} \hat A^{\dagger}_{\mathbf{i},\sigma} + U\hat P + E_g^I}$$

positive semidefinite

$$E_g^I = (\epsilon + U + t_\perp)N - N_c[3U + 4t_\perp + 1/t_\perp]$$

Solution I: Correlated half-metal

2. Construction of the ground state

$$\boxed{\hat H = \sum_{\mathbf i,\sigma} \hat A_{\mathbf i,\sigma}\hat A^\dagger_{\mathbf i,\sigma} + U\hat P + E^I_g}$$

positive semidefinite

$$\hat P = \sum_{\mathbf i} \hat P_{\mathbf i}\ ,\ \ \hat P_{\mathbf i} = (\hat n_{\mathbf i\uparrow}-1)(\hat n_{\mathbf i\downarrow}-1) = \begin{cases} 1, & \text{unoccupied site} \\ 0, & \text{at least one electron} \end{cases}$$

Ground state for U>0: $\hat A^\dagger_{i\sigma}\left|\Psi_g\right\rangle = 0$ and $\hat P\left|\Psi_g\right\rangle = 0$ $\Rightarrow \hat H\left|\Psi_g\right\rangle = E_g\left|\Psi_g\right\rangle$

$$\Rightarrow \left|\Psi^I_g(4N_c)\right\rangle \propto \prod_{\mathbf i} \hat A^\dagger_{\mathbf i,-\sigma}\hat A^\dagger_{\mathbf i,\sigma}|0\rangle$$

Creates one σ and one -σ electron in each unit cell

At least one electron required at each site

$$\hat F^\dagger_\sigma = \prod_{\mathbf i}[\hat c^\dagger_{\mathbf i+\mathbf r_{s_{\mathbf i,1}},\sigma}\hat c^\dagger_{\mathbf i+\mathbf r_{s_{\mathbf i,2}},\sigma}]$$

Creates two more electrons with fixed spin σ on arbitrary sites of each unit cell

Ground state for

$\delta = \pi;\ t_\perp, t_\parallel > 0$

$\epsilon = -t_\perp + t_\perp^{-1}/2$

$t_\perp \cdot t_\parallel = 1/4$

$$\boxed{|\Psi^I_g(4N_c)\rangle = c\,[\prod_{\mathbf i}\hat A^\dagger_{\mathbf i,-\sigma}\hat A^\dagger_{\mathbf i,\sigma}]\,\hat F^\dagger_\sigma\,|0\rangle}$$

$$\boxed{\begin{gathered} N = 4N_c \Leftrightarrow n = 4/3 \\ n_\sigma = 1,\ n_{-\sigma} = 1/3 \end{gathered}}$$

Solution I: Correlated half-metal

3. Proof of the uniqueness of the ground state $\left|\Psi_g^I(4N_c)\right\rangle$

Prove $\boxed{\left|\Psi_g^I(4N_c)\right\rangle \text{ spans } \ker(\hat{H}') := \left\{ |\phi\rangle \;\middle|\; \hat{H}'|\phi\rangle = 0 \right\},\ \hat{H}' \equiv \hat{H} - E_g}$

$\Leftrightarrow$ a) $\left|\Psi_g^I(4N_c)\right\rangle \in \ker(\hat{H}')$

b) all states $|\psi\rangle \in \ker(\hat{H}')$ can be written in the form

$$|\Psi_g^I(4N_c)\rangle = c\,[\prod_{\mathbf{i}} \hat{A}^\dagger_{\mathbf{i},-\sigma}\hat{A}^\dagger_{\mathbf{i},\sigma}]\,\hat{F}^\dagger_\sigma\,|0\rangle$$

$$\hat{H}' = \sum_{n=1}^{L} \hat{P}_n \Rightarrow \ker(\hat{H}') = \bigcap_{n=1}^{L} \ker(\hat{P}_n)$$

Here: $\hat{H}' = \sum_{\sigma}\sum_{\mathbf{i}=1}^{N_c} \hat{A}_{\mathbf{i},\sigma}\hat{A}^\dagger_{\mathbf{i},\sigma} + U\hat{P}$

$$\Rightarrow \boxed{\ker(\hat{H}') = \bigcap_{\sigma=\uparrow,\downarrow}\bigcap_{\mathbf{i}=1}^{N_c} \ker(\hat{A}_{i\sigma}\hat{A}^\dagger_{i\sigma}) \cap \ker(\hat{P})}$$

Proof of the uniqueness of the ground state

Theorem 1: ker($\hat{A}_{i\sigma}\hat{A}^{\dagger}_{i\sigma}$) is spanned by vectors of the form $|\Psi\rangle = \hat{A}^{\dagger}_{i\sigma}\hat{W}|0\rangle$,

where $\hat{W}$ is an arbitrary operator, as long as $\langle\Psi|\Psi\rangle \neq 0$.

Proof:

a) $\hat{A}_{i\sigma}\hat{A}^{\dagger}_{i\sigma}|\Psi\rangle \overset{\left(\hat{A}^{\dagger}_{i\sigma}\right)^2=0}{=} 0 \Rightarrow |\Psi\rangle \in$ ker($\hat{A}_{i\sigma}\hat{A}^{\dagger}_{i\sigma}$) ✓

b) To show that all vectors $|\Psi\rangle \in$ ker($\hat{A}_{i\sigma}\hat{A}^{\dagger}_{i\sigma}$) can be written

in the form $|\Psi\rangle = \hat{A}^{\dagger}_{i\sigma}\hat{W}|0\rangle$ we assume $|\Phi\rangle = \hat{Y}|0\rangle \in$ ker($\hat{A}_{i\sigma}\hat{A}^{\dagger}_{i\sigma}$), i.e., $\hat{A}_{i\sigma}\hat{A}^{\dagger}_{i\sigma}\hat{Y}|0\rangle$=0.

$$\Rightarrow |\Phi\rangle = \hat{Y}|0\rangle \overset{\left\{\hat{A}_{i\sigma},\hat{A}^{\dagger}_{i\sigma}\right\}=a_{i\sigma}=const}{=} \frac{1}{a_{i\sigma}}(\cancel{\hat{A}_{i\sigma}\hat{A}^{\dagger}_{i\sigma}} + \hat{A}^{\dagger}_{i\sigma}\hat{A}_{i\sigma})\hat{Y}|0\rangle = \hat{A}^{\dagger}_{i\sigma}\underbrace{(\frac{1}{a_{i\sigma}}\hat{A}_{i\sigma}\hat{Y})}_{\hat{W}}|0\rangle = \hat{A}^{\dagger}_{i\sigma}\hat{W}|0\rangle$$

q.e.d.

Theorem 2: ker($\sum_{\sigma}\sum_{i=1}^{N_c}\hat{A}_{i\sigma}\hat{A}^{\dagger}_{i\sigma}$) is spanned by vectors of the form $|\Psi\rangle = [\prod_{\sigma}\prod_{i=1}^{N_c}\hat{A}^{\dagger}_{i\sigma}]\hat{W}|0\rangle$,

where $\hat{W}$ is an arbitrary operator, as long as $\langle\Psi|\Psi\rangle \neq 0$.

Proof: simple (since $\hat{A}_{i\sigma}$ are linearly independent) q.e.d.

Proof of the uniqueness of the ground state

$$\ker(\hat H') = \bigcap_{\sigma=\uparrow,\downarrow}\bigcap_{\mathbf{i}=1}^{N_c}\ker(\hat A_{i\sigma}\hat A_{i\sigma}^\dagger)\cap\ker(\hat P)$$

Spanned by states which have
at least one electron per site

$\Rightarrow |\Psi\rangle = [\prod_{\sigma}\prod_{i=1}^{N_c}\hat A_{i\sigma}^\dagger]\hat W|0\rangle$ spans $\ker(\hat H)$,

provided the form of $\hat W$ is compatible with $\ker(\hat P)$.

Theorem 3: For N=4N$_c$, $\hat W = \hat F_\sigma^\dagger = \prod_{\mathbf{i}}[\hat c^\dagger_{\mathbf{i}+\mathbf{r}_{s_{\mathbf{i},1}},\sigma}\hat c^\dagger_{\mathbf{i}+\mathbf{r}_{s_{\mathbf{i},2}},\sigma}]$

Creates two electrons with fixed spin σ
on arbitrary sites of each unit cell

Proof:

(I) $\hat W = \hat F_\sigma^\dagger$ is a possible choice (rather simple)

(II) $\hat W = \hat F_\sigma^\dagger$ is the unique choice (a little more difficult)   q.e.d.

Solution I: Correlated half-metal

For $\delta = \pi;\ t_\perp, t_\parallel > 0$

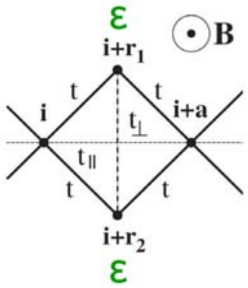
$\epsilon = -t_\perp + t_\perp^{-1}/2$

$t_\perp \cdot t_\parallel = 1/4$

$n = 4/3:\ n_\sigma = 1,\ n_{-\sigma} = 1/3$

$$|\Psi_g^{I}(4N_c)\rangle = c\,[\prod_{\mathbf{i}} \hat{A}^\dagger_{\mathbf{i},-\sigma}\hat{A}^\dagger_{\mathbf{i},\sigma}]\,\hat{F}^\dagger_\sigma\,|0\rangle$$

is the unique ground state



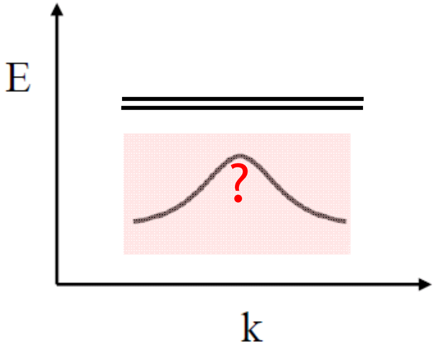
Physical properties:

Effective band structure

$$\hat{H}_{kin} = -\sum_{i\sigma} \hat{A}^\dagger_{i\sigma}\hat{A}_{i\sigma} + \text{const}$$

quadratic in the original c-operators

→ Interaction dependent band structure



Solution I: Correlated half-metal

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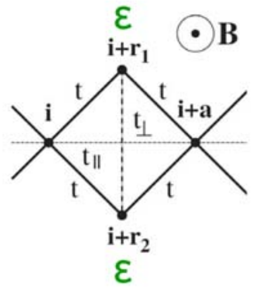
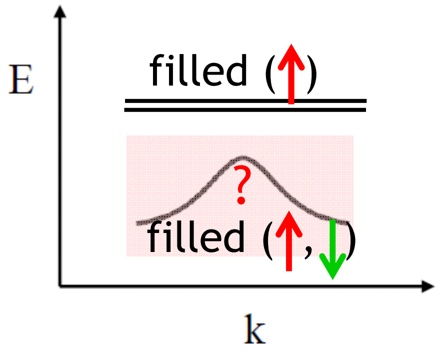
is the unique ground state



Physical properties:



One σ electron on every lattice site → localized

-σ electrons:

Expectation value of hopping term: $\Gamma_{\mathbf{r},-\sigma} = \langle \hat{c}^{\dagger}_{\mathbf{j},-\sigma} \hat{c}_{\mathbf{j}+\mathbf{r},-\sigma} + H.c. \rangle$

$$\Gamma_{m,-\sigma} \overset{N_c \to \infty}{=} \frac{(-1)^m}{\sqrt{1 + 1/t_{\perp}}} e^{-m/\xi_{-\sigma}} \qquad , r/a = m$$

→ -σ electrons: spatially extended but localized for $N_c \to \infty$

N_c: # unit cells

Solution I: Correlated half-metal

Add ΔN many $-\sigma$ electrons → $n_\sigma = 1,\ n_{-\sigma} = \frac{1}{3} + \frac{\Delta N}{3N_c}$

N_c: # unit cells

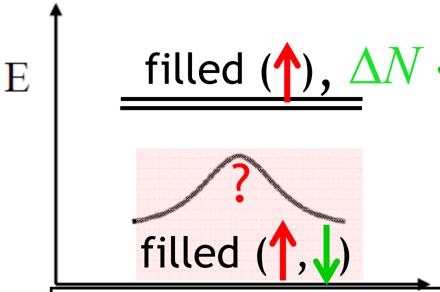
Ground state

$$|\Psi_g^I(4N_c + \Delta N)\rangle = \prod_{\alpha=1}^{\Delta N} \hat{c}^\dagger_{n_\alpha,\mathbf{k}_\alpha,-\sigma} |\Psi_g^I(4N_c)\rangle$$

$n_\alpha:\quad s = 1,2,3$

plane wave-type states due to $-\sigma$ electrons

Physical properties:



- charge gap $\Delta\mu = E_g(N) - 2E_g(N-1) + E_g(N-2) = 0$
- $\Gamma_{\mathrm{r},\,-\sigma} \propto$ plane wave results

→ ΔN many $-\sigma$ electrons itinerant

Ground state for
$4/3 < n < 5/3$

- $3N_c$ immobile σ electrons
- N_c $-\sigma$ electrons confined to localized Wannier function
  + ΔN itinerant $-\sigma$ electrons

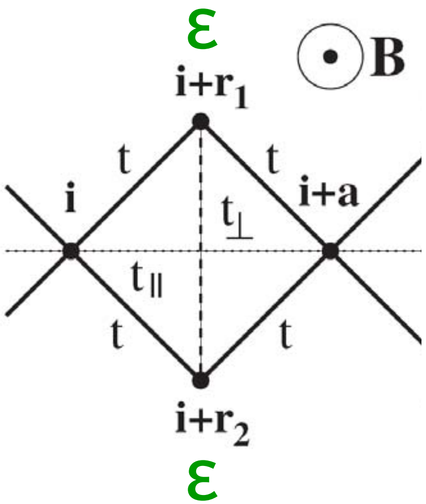
- Magnetization $M \propto (1 - \Delta N/N_c)$

→ Low carrier-density metal due to $-\sigma$ electrons

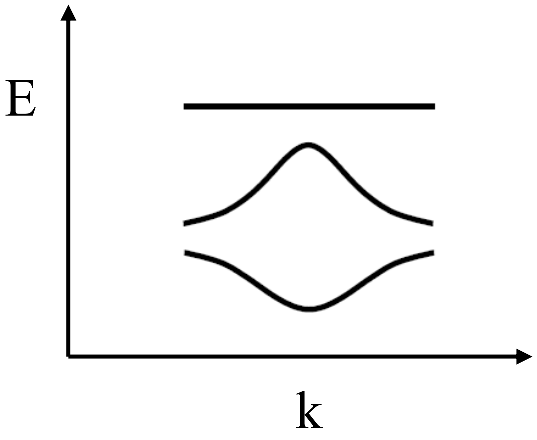
U=0: dispersionless, localized electrons
U>0: correlation induced half-metal

Solution II:
Exact ground states for general magnetic flux

Solution II: Exact ground states for general magnetic flux



$$\delta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$t_\parallel = 0,\ t_\perp < 0$$
$$b \equiv -\cos\delta / t_\perp, \quad \varepsilon = b - b^{-1}$$



Single-electron bands

Ground states for $n \geq 5/3$

$B = 0:$ localized non-magnetic ground state for $n \geq 5/3$

$\xrightarrow{B \neq 0}$

Non-saturated ferromagnet

- insulating for n=5/3
- metallic for n>5/3

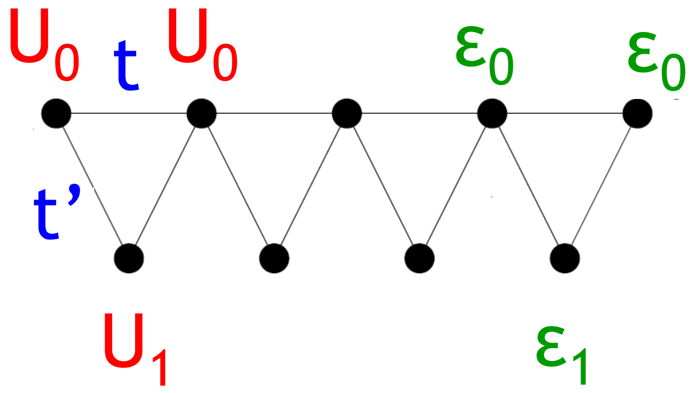
Conclusion

- Diamond Hubbard chain has remarkably complex properties
- Switch between different ground states by variation of B, ε, n

Triangle Hubbard chains:
Explanation of the strange ferromagnetism in $CeRh_3B_2$

Flat bands in the single-electron band structure stay flat for U>0

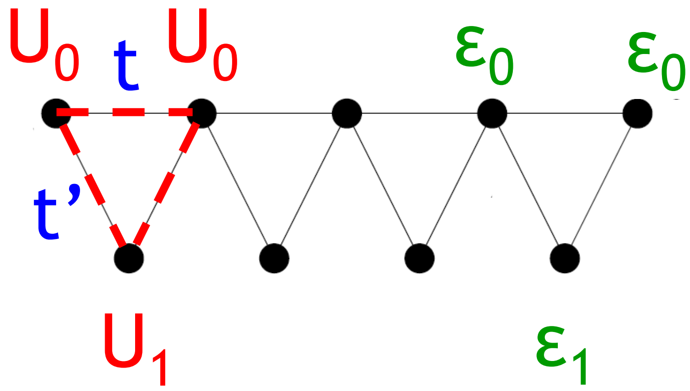
Z. Gulacsi, A. Kampf, DV
Prog. Theor. Phys. Suppl. **176**, 1 (2008)



2 sites per cell → 2 bands

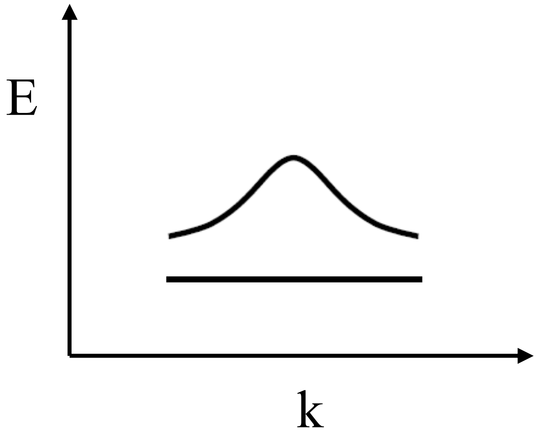
N_c = # cells
N = # electrons
$n = \frac{N}{2N_c}$ electron density

Müller-Hartmann (1995)
Penc, Shiba, Mila, Tsukagoshi (1996)
Fazekas (1997)
Derzho, Honecker, Richter (2007)
Derzho, Honecker, Richter, Maksymenko, Moessner (2010)



$$\frac{(t')^2}{t} = \epsilon_1 - \epsilon_0 + 2t\,, \quad t > 0$$

$$\epsilon_1 - \epsilon_0 > -2t$$



Single-electron bands

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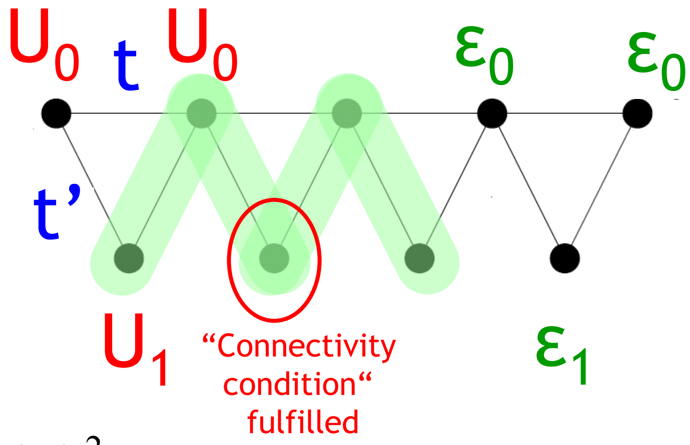

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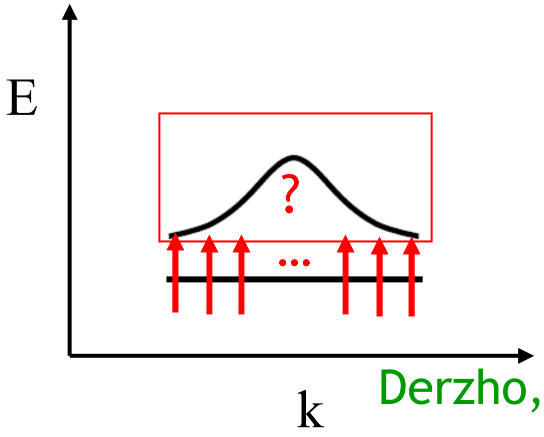


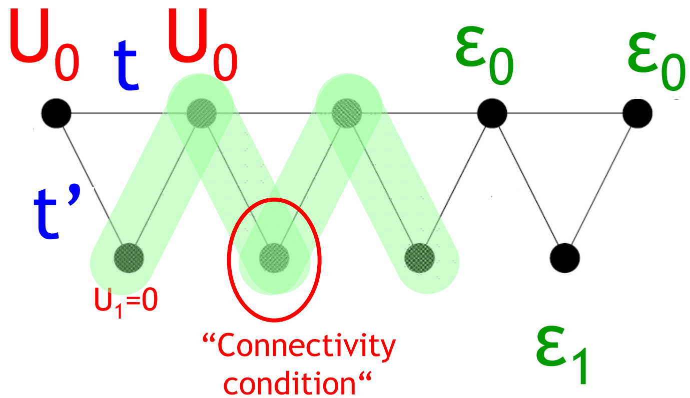
Solution I: $U_0, U_1 > 0$

n<1/2 : ferromagnetic clusters
n=1/2 : fully saturated ferromagnet

Mielke, Tasaki (1993)
Derzho, Honecker, Richter (2007)
Derzho, Honecker, Richter, Maksymenko, Moessner (2010)

→ Flat-band ferromagnetism: Realizes ideas of Gutzwiller and Kanamori from 1963 about the origin of itinerant ferromagnetism

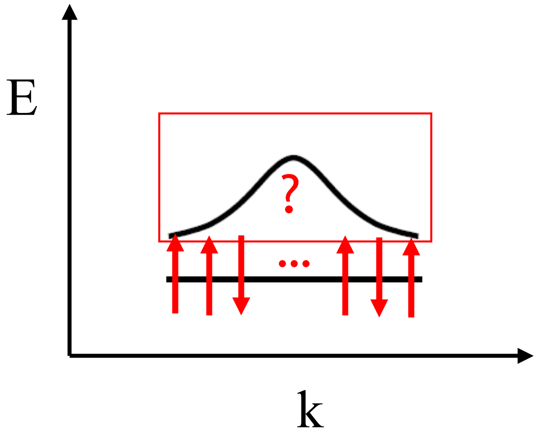
Not related to Stoner theory: $\chi(\omega=0, \mathbf{q}=0) = \infty$ → $UN(E_F)=1$



“Connectivity condition“ fulfilled

$$\frac{(t')^2}{t} = \epsilon_1 - \epsilon_0 + 2t\,, \quad t > 0$$

$$\epsilon_1 - \epsilon_0 > -2t$$



2 sites per cell → 2 bands

N_c = # cells

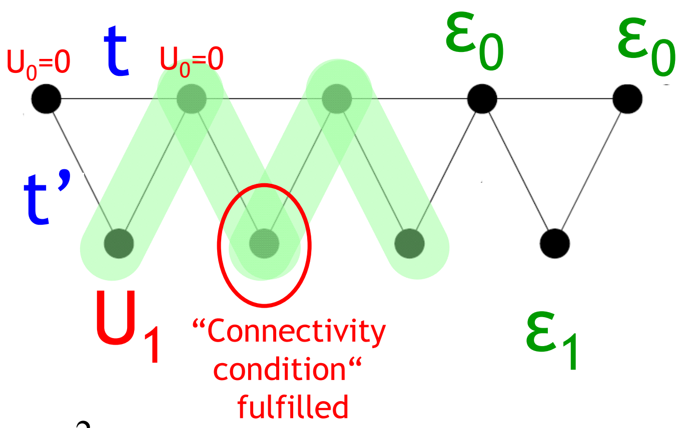
N = # electrons

$n = \frac{N}{2N_c}$ electron density

Solution II: $U_0 > 0,\ U_1 = 0$

n=1/2 : non-magnetic

U_1=0: electrons uncorrelated on sites where Wannier functions connect

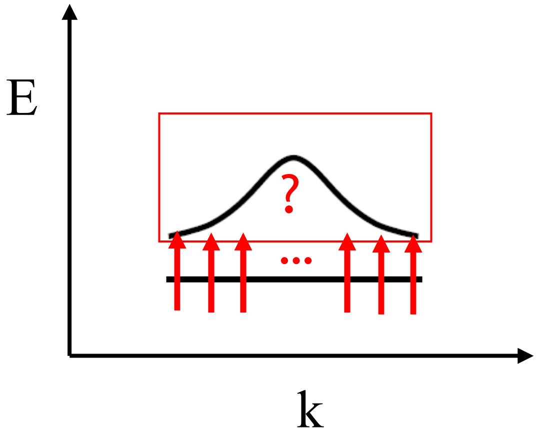


2 sites per cell → 2 bands

N_c = # cells
N = # electrons
$n = \frac{N}{2N_c}$ electron density

$$\frac{(t')^2}{t} = \epsilon_1 - \epsilon_0 + 2t\,, \quad t > 0$$

$$\epsilon_1 - \epsilon_0 > -2t$$

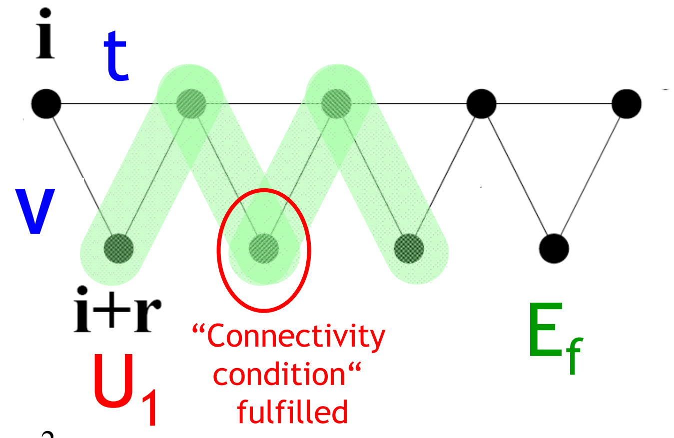


Solution III: $U_0 = 0,\ U_1 > 0$

n=1/2 : fully saturated ferromagnet

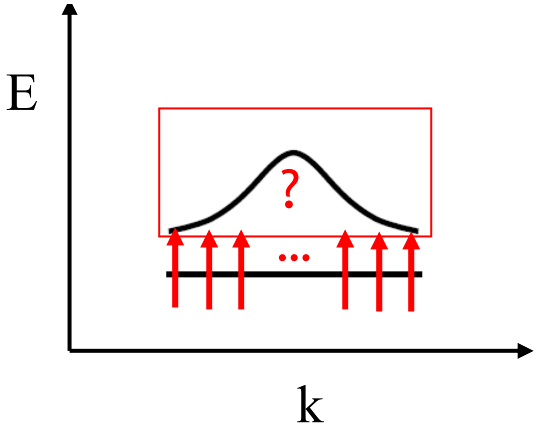
Change of notation: $\boxed{\hat{d}_{\mathbf{i},\sigma}} \equiv \hat{c}_{\mathbf{i},\sigma}\,,$

$\boxed{\hat{f}_{\mathbf{i},\sigma}} \equiv \hat{c}_{\mathbf{i}+\mathbf{r},\sigma}\,, \boxed{V} \equiv t'\,, \boxed{E_f} \equiv \epsilon_1\,,\ \epsilon_0 = 0$



$$\frac{V^2}{t} = E_f + 2t, \quad t > 0$$

$$E_f > -2t$$



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$\boxed{\hat{f}_{\mathbf{i},\sigma}} \equiv \hat{c}_{\mathbf{i}+\mathbf{r},\sigma}, \boxed{V} \equiv t', \boxed{E_f} \equiv \epsilon_1, \; \epsilon_0 = 0$

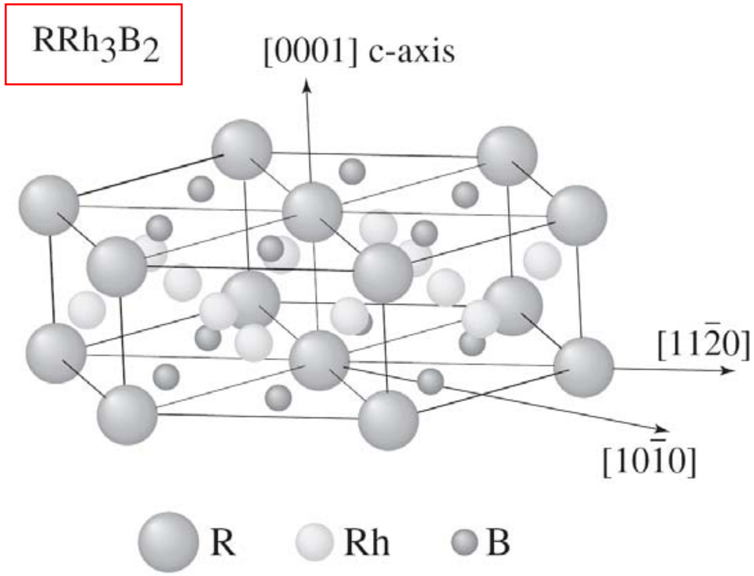
1D periodic Anderson model

Application of the 1D periodic Anderson model to $CeRh_3B_2$

$CeRh_3B_2$ is an interesting 4*f*-system because:

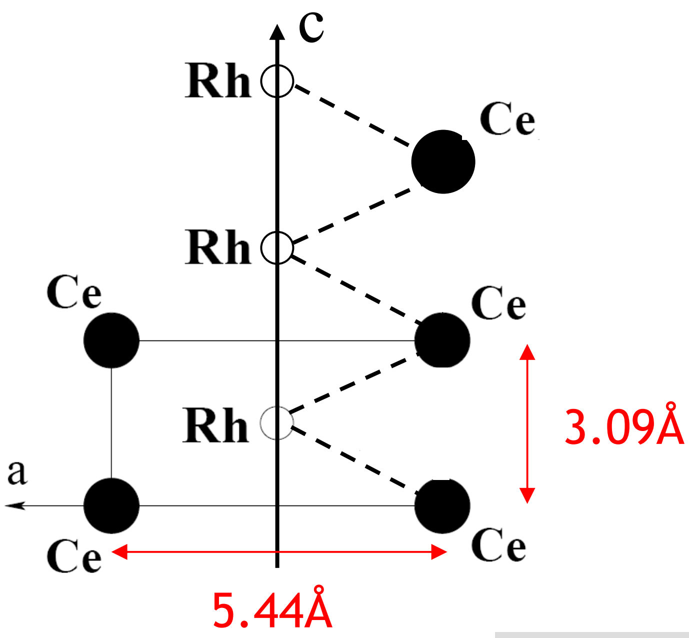
- RKKY interaction cannot explain ferromagnetism
- Small *f*-moment 0.45 μ_B (free Ce^{3+} ion: 2.14 μ_B)

Application of the 1D periodic Anderson model to CeRh$_3$B$_2$



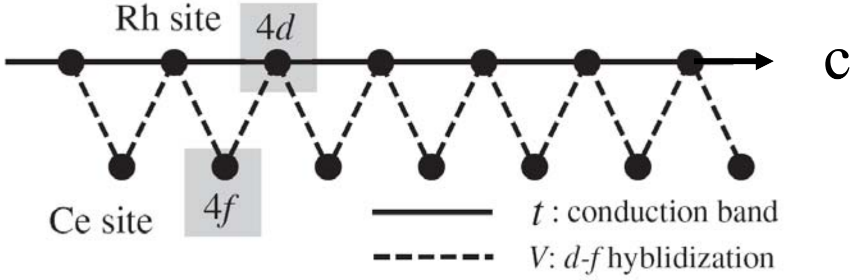
Rare Earth

Yamada *et al*. (2004)



(a,c) plane

Quasi-1D structure



Kono, Kuramoto (2006)

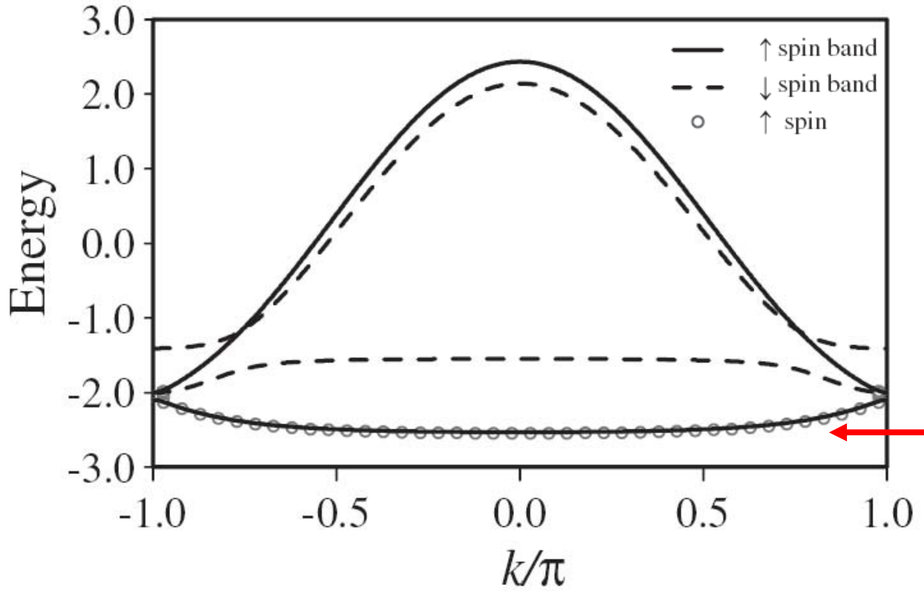
Mechanism for *f*-electron ferromagnetism in $CeRh_3B_2$?

Gutzwiller projected variational wave function $\boxed{|\Psi\rangle = P|\Phi\rangle}$

Evaluations by variational Monte Carlo (VMC)   Kono, Kuramoto (2006)

f-electron ferromagnetism in $CeRh_3B_2$

VMC



Kono, Kuramoto (2006)

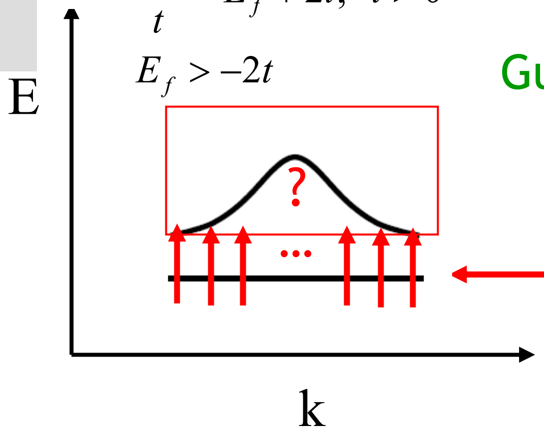
almost flat band

$$t = 0.34 \text{ eV},\ V = 0.24 \text{ eV},\ E_f = -0.714 \text{ eV},\ U = 7 \text{ eV}, n = 0.55$$

Exact ground state (Solution III)

$$\frac{V^2}{t} = E_f + 2t,\quad t > 0$$

$$E_f > -2t$$



Gulacsi, Kampf, DV (2008)

saturated ferromagnetism,
bare flat band unchanged by U

e.g., $t = 0.34 \text{ eV},\ V = 0.23 \text{ eV},\ E_f = -0.52 \text{ eV},\ U > 0 \text{ arbitrary}, n = 0.5$

f-electron ferromagnetism in $CeRh_3B_2$: Magnetic moments

Free Ce^{3+} ion $m_f = 2.14\ \mu_B$

Experiment: $m_f = 0.45\ \mu_B$ Galatanu *et al.* (2003)

VMC $t = 0.34$ eV, $V = 0.24$ eV, $E_f = -0.714$ eV, $U = 7$ eV, $n = 0.55$

$m_f = 0.94\ \mu_B$ Kono, Kuramoto (2006)

Exact ground state $\frac{V^2}{t} = E_f + 2t,\ \ t > 0,\ E_f > -2t$

$$\boxed{m_f = g\mu_B \frac{1}{2N_c} \sum_i < (\hat{n}^f_{i,\uparrow} - \hat{n}^f_{i,\downarrow}) >}$$

$t = 0.34$ eV, $V = 0.23$ eV, $E_f = -0.52$ eV, $U > 0$ arbitrary, $n = 0.5$

$m_f = 0.68\ \mu_B$ Gulacsi, Kampf, DV (2008)

Pentagon Hubbard chains:
Ferromagnetism in polymers

Exact dispersive band structure can be tuned to become flat

Z. Gulacsi, A. Kampf, DV
Phys. Rev.Lett. **105**, 266403 (2010)

Conducting polymers: wide range of applications in

Heeger, Kivelson, Schrieffer, Su (1988)

- Nanoelectronics
- Nanooptics
- Medicine

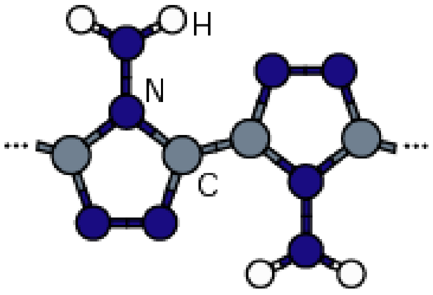
Search for plastic ferromagnets and
ferromagnetism in systems with non-magnetic elements

Candidate: Flat-band ferromagnetism in organic polymers

Suwa, Arita, Kuroki, Aoki (2003, arXiv:0907.2477)

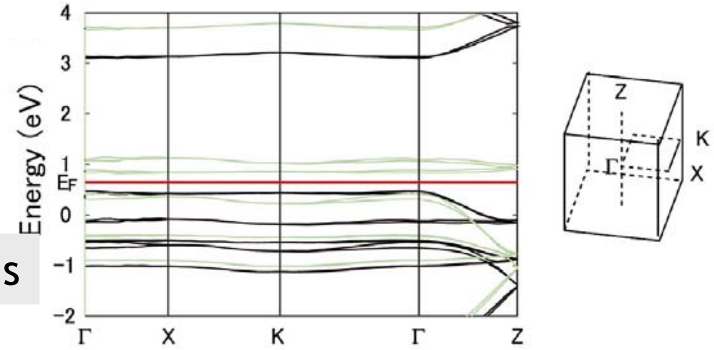
Arita, Suwa, Kuroki, Aoki (2002, 2003)

Polyaminotriazole



Spin-DFT

Single-particle bands

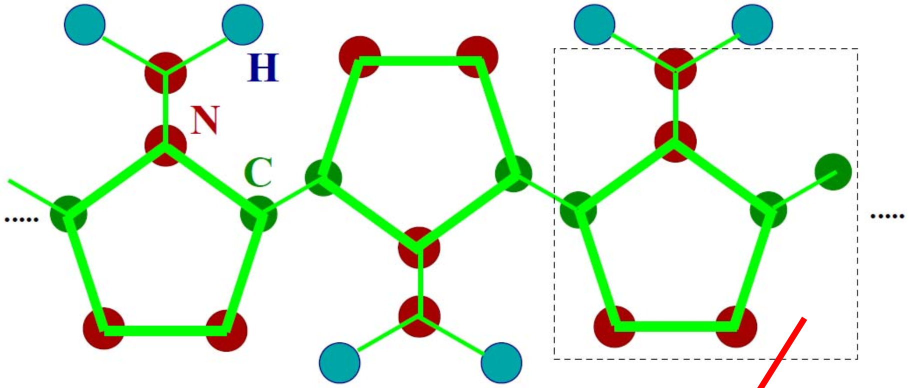


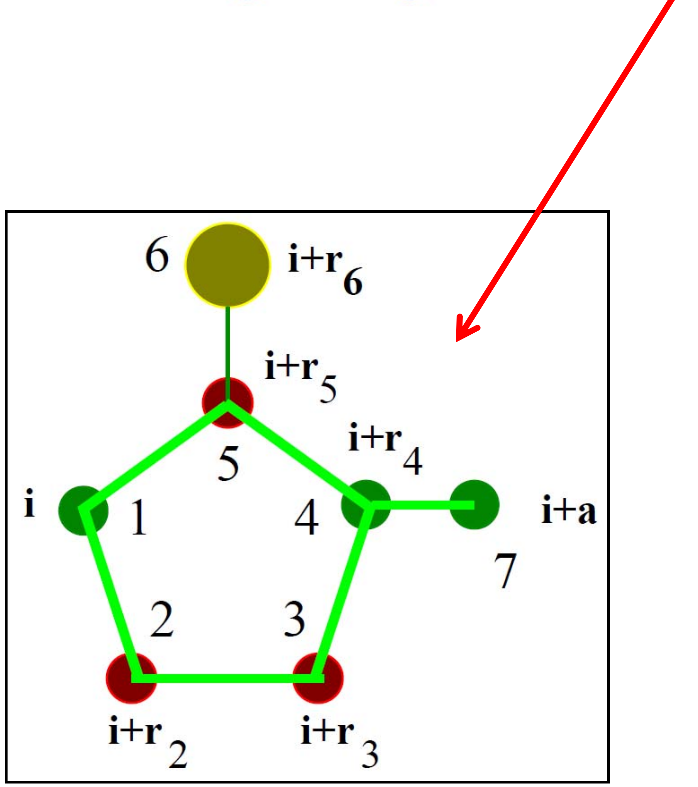
- Strong correlations in acene and thiophene organic molecular crystals
- stabilization of magnetic phases at high electron densities

Brocks, van den Brink, Morpurgo (2004)

Polyaminotriazole

Gulacsi, Kampf, DV (2010)





6 sites per cell → 6 bands

N_c = # cells
N = # electrons
$n = \frac{N}{6N_c}$ electron density

Arbitrary

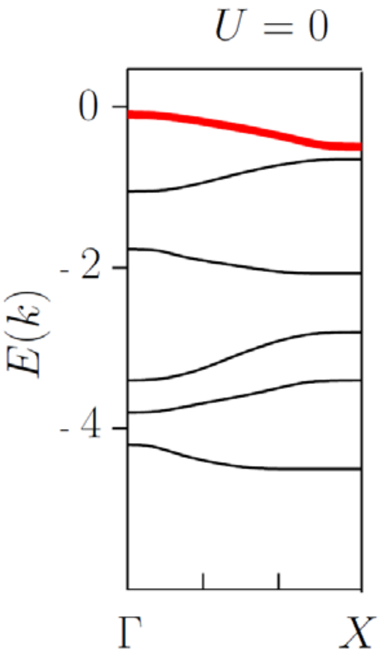
- local interactions $U_n > 0$
- on-site potentials ε_n
- hopping amplitudes $t_{n,n'}$

Single-particle bands

Gulacsi, Kampf, DV (2010)

Band dispersion

$\epsilon = E_\nu(k),\ \nu \leq 6 \qquad k = \mathbf{k}\cdot\mathbf{a}$



$$t_c \equiv t_{4,7} = 0.5,\ t_h \equiv t_{3,2} = -1.1,\ t_f \equiv t_{5,6} = 1.2$$

$$\epsilon_1 = \epsilon_4 = -2.5,\ \epsilon_2 = \epsilon_3 = -2.0,\ \epsilon_5 = -2.1,\ \epsilon_6 = -2.1$$

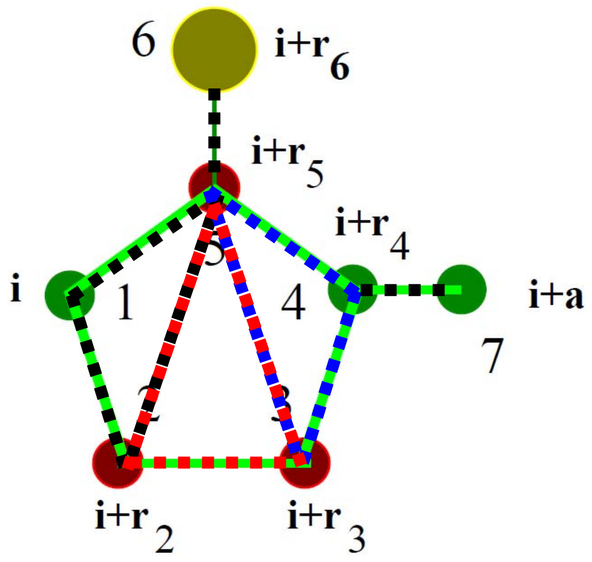
Transform Hamiltonian in positive semidefinite form

Define operators acting on blocks $\mathcal{B}_{\mathbf{i}}$

$$\boxed{\hat{G}^{\dagger}_{\alpha,\mathbf{i},\sigma} = \sum\nolimits_{\ell\in\mathcal{B}_{\mathbf{i},\alpha}} a_{\alpha,\ell}\hat{c}^{\dagger}_{\mathbf{i}+\mathbf{r}_{\ell},\sigma}} \quad \alpha=1,\ldots,5$$

3 three-site blocks

2 two-site blocks



Original form

$$\hat{H} = \hat{H}_0 + \hat{H}_U$$

$$\begin{aligned}\hat{H}_0 &= \sum_{\sigma,\mathbf{i}} \sum_{n,n'(n>n')} (t_{n,n'}\hat{c}^{\dagger}_{\mathbf{i}+\mathbf{r}_n,\sigma}\hat{c}_{\mathbf{i}+\mathbf{r}_{n'},\sigma} + H.c.) + \\ &+ \sum_{\sigma,\mathbf{i}} \sum_{n=1}^{6} \epsilon_n \hat{n}_{\mathbf{i}+\mathbf{r}_n,\sigma}, \\ \hat{H}_U &= \sum_{\mathbf{i}} \sum_{n=1}^{6} U_n \hat{n}_{\mathbf{i}+\mathbf{r}_n,\uparrow}\hat{n}_{\mathbf{i}+\mathbf{r}_n,\downarrow}.\end{aligned}$$

Positive semidefinite form

$$\hat{H} - C_g = \hat{H}_G + \hat{H}_P$$

with $\hat{H}_G = \sum_{\mathbf{i},\sigma} \sum_{\alpha=1}^{5} \hat{G}_{\alpha,\mathbf{i},\sigma}\hat{G}^{\dagger}_{\alpha,\mathbf{i},\sigma}, \ \hat{H}_P = \sum_{n=1}^{6} U_n \sum_{\mathbf{i}} \hat{P}_{\mathbf{i}+\mathbf{r}_n}$

→ Matching conditions

free parameters in block operators < # matching conditions

Parameter space for which the transformation holds

$$t_h < 0, \quad Z = (q_U - Q_3^2) > \epsilon_6,$$

$$W = q_U - [(q_U - U_2 - \epsilon_2)^2 - t_h^2]/|t_h| > \epsilon_5,$$

$$W - \epsilon_5 > U_5 > 0, \quad U_6 = Z - \epsilon_6.$$

No strong restrictions on parameters

Effective band structure

$$\hat{H}_{kin} = -\sum_{\mathbf{i},\sigma}\sum_{\alpha=1}^{5} \hat{G}^{\dagger}_{\alpha,\mathbf{i},\sigma}\hat{G}_{\alpha,\mathbf{i},\sigma} + \text{const}$$

↑

quadratic in the original c-operators

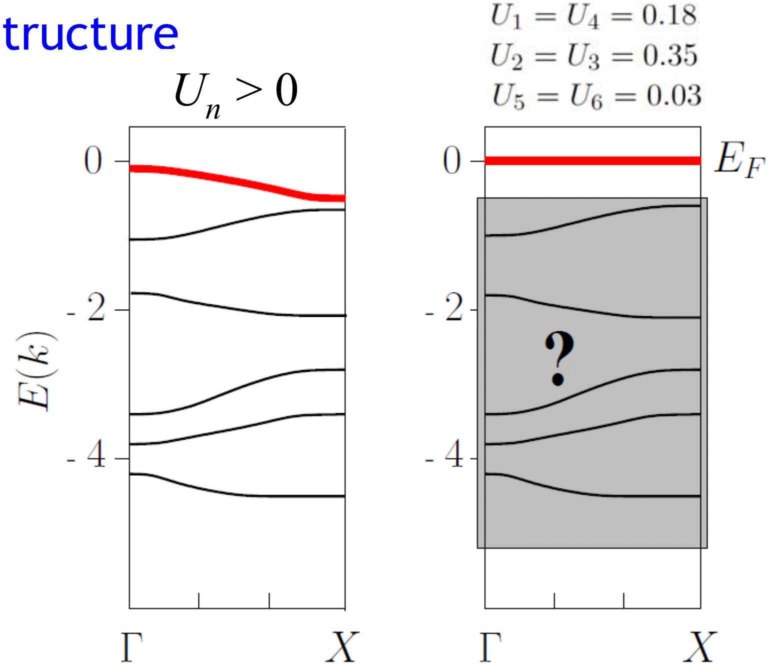
→ Effective, interaction dependent band structure

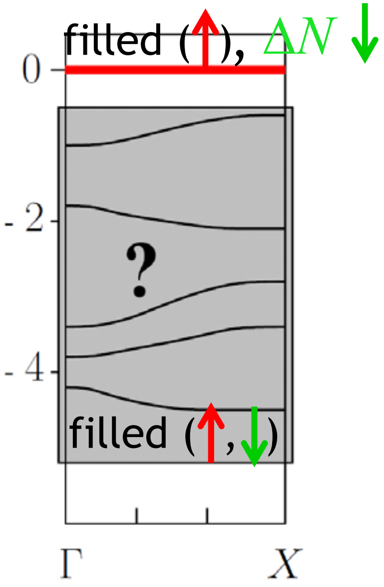
same as for H_0 but with renormalization of local potentials

$$\boxed{\epsilon_n \rightarrow \epsilon_n^R = \epsilon_n + U_n - q_U}$$

→ exact effective band structure is in general dispersive

Upper band can be tuned to become flat due to interactions





Upper flat band half filled for N=11N_c → density n=11/6

Ground state for n=11/6:

$$\boxed{\left|\Psi_g(11N_c)\right\rangle = \prod_{\mathbf{i}} [(\prod_{n=1}^{6} \hat{c}^\dagger_{\mathbf{i}+\mathbf{r}_n,\sigma})(\prod_{\alpha=1}^{5} \hat{G}^\dagger_{\alpha,\mathbf{i},-\sigma})]|0\rangle}$$

creates 6N_c
σ electrons
→ localized

creates 5N_c
-σ electrons

Physical properties of -σ electrons

$$\Gamma_{\mathbf{i}}(\mathbf{r}) = \langle\Psi_g(N^*)|(\hat{c}^\dagger_{\mathbf{i}+\mathbf{r}_n,-\sigma}\hat{c}_{\mathbf{i}+\mathbf{r}_n+\mathbf{r},-\sigma} + H.c.)|\Psi_g(N^*)\rangle \xrightarrow{N_c\to\infty} e^{-r/\xi}$$

→ Ground state for n = 11/6: localized ferromagnet

Ground state for n > 11/6 ($S_{z,max}$ sector) : add ΔN many -σ electrons

- charge gap $\Delta\mu = E_g(N) - 2E_g(N-1) + E_g(N-2) = 0$
- localization length $\xi = \infty$

→ itinerant ferromagnet (half-metal)

Matching conditions are valid for

$$\boxed{n \geq 11/6, \quad W/t = 0.15, \text{ with } 0.2 \leq U_n/W \leq 2.3}$$

Fulfills experimental conditions:

- Strong correlations in acene and thiophene organic molecular crystals
- stabilization of magnetic phases at high electron densities

Brocks, van den Brink, Morpurgo (2004)

Conclusion 1:
Strategy for the construction of exact many-electron ground states

Step 1: Transform a many-electron Hamiltonian of your choice
for a given lattice of your choice into positive semidefinite form

Step 2: Construct many-electron ground state

Step 3: Prove uniqueness of ground state

- Works in any dimension
- No integrability required
- Applicable to any Hamiltonian with sufficiently many microscopic parameters

Conclusion 2:
Exact many-electron ground states on Hubbard chains

Hubbard chains have remarkably complex properties, e.g.,

Square Hubbard chain:

- Lowest flat-band ferromagnetism (general property)
- Correlated half-metal behavior
- Metal-insulator transitions

- Tune between different ground states by varying B, ε, n, U, t

Pentagon chain polymers:

- Tune dispersive band structure by the interaction
  → design of flat bands with ferromagnetic or half-metallic ground states