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Ninth International Conference
on

NOISE IN PHYSICAL SYSTEMS

(including 1/f noise and noise in biological systems and membranes)

Montréal, May 25–29, 1987.

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Université de Montréal, Montréal Québec H3C 3J7

Sponsored by National Science and Engineering Research Council
Ottawa, Canada

World Scientific
Singapore • New Jersey • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.
P.O. Box 128, Farrer Road, Singapore 9128

U.S.A. office; World Scientific Publishing Co., Inc.
687 Hartwell Street, Teaneck NJ 07666, USA

Library of Congress Cataloging-in-Publication data is available.

NINTH INTERNATIONAL CONFERENCE ON NOISE IN PHYSICAL SYSTEMS

ISBN 9971-50-397-2

Printed in Singapore by Utopia Press.

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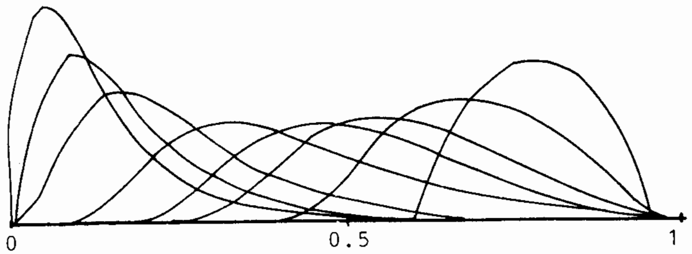
distribution is given as a narrow one around $x \sim 0.7$. As time passes, it becomes once very flat, and sharp again around $x \sim 0$. The final distribution corresponds to the complete extinction.



Numerical studies on small groups, for which fluctuations are to be dominant, are not yet finished, partly because of the complexity of the master equation. It is expected that the singular behavior at x=0 and x=1 will be much stressed. A part of this report have been published.[3)]

References

1) Eigen,M., Naturwiss. 78, 465 (1971).
2) Ebeling,W., et al., Studia biophysica 84, 87 (1981).
3) Ueyama, H., Jour. Phys. Soc. Japan 55, 91 (1986).

DISCRETE DYNAMICS AND METASTABILITY: MEAN FIRST PASSAGE TIMES AND ESCAPE RATES

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It is known since Poincaré, but it has been recognized widely only during the past decade, that a great deal of the complexity of the solutions of coupled nonlinear first order differential equation is already present in discrete maps of lower dimensionality than that of the original differential equations [1)]. These maps may be obtained e.g. from Poincaré sections, or by a stroboscopic observation of the trajectories. Rather often, already a one dimensional map can reflect important features of the continuous time dynamics both qualitatively and quantitatively [2)]. In such a case, a characteristic value x , as e.g. the maximum of an amplitude, assumes at consecutive time steps t_n values x_n , which follow by a simple iterative procedure

$$x_{n+1} = F(x_n) \tag{1}$$

Depending on the problem under consideration, the continuous time dynamics leading to (1) might be disturbed by random forces caused e.g. by thermal or externally applied noise. Then, as a consequence, the map $F(x_n)$ is noisy, too. The influence of noise on maps has been studied mainly in cases where the map already is sensitive to small deterministic perturbations [3-6)]. In these cases apart from a finite resolution noise does not introduce new effects which would not be present either in the original map or in a slightly perturbed deterministic map. Our goal in this paper is to study a genuine weak noise effect, namely the lifetime of a metastable state by means of a mean first passage time [7,8)].

For the sake of definiteness we will study a circle map

$$x_{n+1} = f(x_n) + \varepsilon^{1/2}\xi_n \qquad (2a)$$

$$f(x) = x + a \sin(2\pi x) \qquad (2b)$$

which is perturbed by independent Gaussian distributed noise ξ_n with a small amplitude $\varepsilon^{1/2}$.

$$\rho(\xi \in [\xi, \xi+d\xi]) = (2\pi)^{-1/2}\exp\{-\xi^2/(2\varepsilon)\} \qquad (3)$$

The strength of the sine-force is denoted by $a>0$ which will be assumed to be small, $a<1/(2\pi)$. For $\varepsilon=0$, the deterministic map has unstable fixed points at $x^u = n$, and stable fixed points at $x^s = (2n+1)/2$, $n = 0, \pm 1, \pm 2, \ldots$. A trajectory which starts in the interior of the interval $I = [0,1]$ will be attracted by the stable point $x^s = 1/2$, and never leaves the interval I. For arbitrarily small ε, however, the interval will be left eventually. A quantitative measure for the occurrence of these rare events is the mean first passage time (MFPT), i.e. the mean number of steps after which a random walker starting at $x \in [0,1]$ reaches the exterior of $[0,1]$ for the first time. For a discrete dynamics the MFPT obeys the inhomogeneous backward equation [7,9]

$$t(x)-1=(2\pi\varepsilon)^{-1/2}\int_0^1 t(y)\exp\{-(y-f(x))^2/2\varepsilon\}dy \quad x\in[0,1] \qquad (4.a)$$

$$t(x) = 0 \quad x\in[0,1] \quad . \qquad (4.b)$$

Due to the symmetry of $f(x)$ the MFPT itself is a symmetric function about $x^s = 1/2$ and attains its maximal value T at $x^s = 1/2$. From (4a), one finds

$$t(x) \leqslant T \leqslant \ \mathrm{erfc}(1/(2\sqrt{2\varepsilon})) \qquad (5)$$

This estimate has been used as an approximation[5] . In spite of the absorbing states outside the interval $[0,1]$, there is a finite return probability from its boundary points to the interior. This implies finite jumps of $t(x)$ at 0 and 1. We find from (4.a)

$$t(0) = CT + 1 \qquad (6)$$

where, in terms of the from function $\tilde{h}(x)$

$$t(x) = T\,\tilde{h}(x) \quad , \quad \tilde{h}(x) \leqslant 1 \qquad (7)$$

the constant C in (6) is given by

$$C = (2\pi\varepsilon)^{-1/2}\int_0^1 \tilde{h}(y)\exp\{-y^2/2\varepsilon\} < 1/2 \qquad (8)$$

It then follows from the behaviour of $\tilde{h}(y)$ at weak noise (see eqs (11), C12) below) that the constant C is of order $O(\varepsilon^0)$. In other words the jump is a finite fraction of the maximum of the MFPT. This maximum T , can be expressed in terms of a stationary probability $W(x)$ and the form function $\tilde{h}(y)$

$$T^{-1} = \frac{2\int_{-\infty}^{0} dx\, W(x)\int_0^1 dy\, \tilde{h}(y)\exp\{-(y-fx))^2/2\varepsilon\}}{(2\pi\varepsilon\ )^{1/2}\int_0^1 W(x)\, dx} \qquad (9)$$

where $W(x)$ obeys the master equation

$$W(x) = (2\pi\varepsilon)^{-1/2}\int_{-\infty}^{+\infty}\exp\{-(x-f(y))^2/2\varepsilon\}W(y)dy \ . \qquad (10)$$

At a weak noise level a trajectory remains for most of the time near the stable fixed point, and only rarely will it make a large excursion. Thus, the MFPT attains a very large value inside the interval $[0,1]$ which deviates only little from its maximal value; i.e. $t(x)\cong T$. Significant deviations from the constant T occur only near the exit boundaries. Thus $h(x)$ defined in (7) deviates from the value $h(x)\cong 1$ only in a small neighbourhood near the boundaries. Near these boundaries, say near $x = 0$, for the scaled boundary layer function

$$h(x) = \tilde{h}\ (\sqrt{2\varepsilon}x) \qquad (11)$$

we obtain from (4a) the following integral equation, valid in leading order in ε

$$h(x) = \pi^{-1/2}\int_0^\infty h(y)\exp\{-(y-Ax)^2\}\, dy \qquad (12)$$

where

$$A = f'(0) = 1 + 2\pi a \ . \qquad (13)$$

A properly normalized solution ($h(x)\to 1$ for $x\to\infty$) of (12) can be obtained numerically[7]. For small amplitudes, a , of the deterministic force the solution of (10) reads [7,8]

$$W(x) = N \exp\{-a/(\pi\varepsilon)\cos 2\pi x\} .$$

This allows the evaluation of the integrals in (9) to yield in leading order in ε

$$T = 1/(4a) \exp(2a/(\pi\varepsilon)) \quad (14)$$

T is determined by an Arrhenius-like exponential leading part, and a prefactor, $1/(4a)$. From the lifetime T one obtains for the rate, λ , at which there occurs an escape from the metastable state at $x = 1/2$ across the boundary $x = 0$ or $x = 1$

$$\lambda = \frac{1}{2} T^{-1} = \lambda^{+} + \lambda^{-} . \quad (15)$$

The factor 1/2 takes into account that, in absence of a capture beyond the unstable fixed points, $x = 0,1$, half of the random walkers would return into the original interval [0,1]. For the individual rates of escape either to the left λ^{-} or to the right λ^{+} , we have

$$\lambda^{+} = \lambda^{-} = \frac{1}{2} = (4T)^{-1} . \quad (16)$$

These rate results (15,16) can also be derived by an alternative method [7] which utilizes Kramers' flux method [10]. In the present situation of a periodical map, the random walker undergoes a noise-induced diffusive motion across periodic barriers with a diffusion constant D

$$\langle (x_n - \langle x_n \rangle)^2 \rangle \to 2Dn \text{ as } n \to \infty . \quad (17)$$

D itself is determined by the forward and backward hopping rates λ^{+} , λ^{-} , and the step size $L = 1$; i.e.

$$D = \frac{1}{2} (\lambda^{+} + \lambda^{-}) L^2 = (4T)^{-1} . \quad (18)$$

References

1. R.H.G. Helleman, in "Fundamental Problems in Statistical Mechanics V", ed. E.G.D. Cohen, North Holland, (1980)

2. H.G. Schuster, Deterministic Chaos, VCH Publishers, Weinheim, (1984)

3. J.P. Eckmann, Rev.Mod.Phys. 53, 643 (1981)

4. J.P. Crutchfield, J.D. Farmer, B.A. Huberman, Phys.Rep. 92, 45 (1982)

5. F.T. Arecchi, R. Badii, A. Politi, Phys.Lett.103A, 3 (1984

6. S.J. Linz, M. Lücke, Phys.Rev. A33, 2694 (1986)

7. P. Talkner, P. Hänggi, E. Freidkin, D. Trautmann, J. Stat. Phys. (1987) (in press)

8. P. Talkner, P. Hänggi, in "Noise in Nonlinear Dynamical Systems". Ed. P.V.E. McClintock, Cambridge Univ. Press Cambridge (1987)

9. H. Haken, A. Wunderlin, Z.Phys. B46, 181 (1982)

10. H.A. Kramers, Physica 7, 284 (1940)