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Noise in nonlinear dynamical systems

Volume 2

Theory of noise induced processes in special applications

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5 Discrete dynamics perturbed by weak noise

PETER TALKNER and PETER HÄNGGI

5.1 Perspectives

In Chapter 9 of Volume 1, one of us elaborated on noisy dynamical flows described by the set of first-order differential equations

$$\dot{x}_\alpha = f_\alpha(x, \lambda) + \sum_i q_{\alpha i}(x, \lambda)\xi_i(t). \tag{5.1.1}$$

Such systems exhibit already for a small number of coupled state variables an overwhelming complexity which generally defies an analytic description. It turns out, however, that the complexity of such systems can be extracted from a stroboscope-like discretization in time of a single trajectory $x(t)$. With strong dissipation even a multidimensional flow of a huge number of coupled degrees of freedom can be described in terms of a one-dimensional discrete iterative mapping. That is, one can predict in a causal manner with surprisingly good accuracy, e.g., the maximum of a state variable $x_\alpha(t_{n+1})$ of a complex dynamics if we only know the previous maximum $x_\alpha(t_n)$; i.e. $x(t_{n+1}) = F(x(t_n); \lambda)$, where λ denotes a set of control parameters. Likewise, the set of values $\{x_\alpha(t_0), x_\alpha(t_0+\tau), \ldots, x_\alpha(t_0+n\tau)\}$ of a sequence of observations at constant time intervals, τ, also obeys the same law. $F(x, \lambda)$ often exhibits turning points, or maxima and minima. This stroboscope-like procedure then yields an approximation to the stochastic flow in (5.1.1) which takes on the form of a noisy discrete dynamics

$$x_{n+1} = F(x_n, \lambda) + \xi_n. \tag{5.1.2}$$

Without the noisy term ξ_n, equations such as (5.1.2) have been studied extensively in the context of 'deterministic chaos' (e.g. see Coullet and Tresser, 1978a, b; Feigenbaum, 1978a, b, 1979; Grossman and Thomae, 1977; for a review, see Schuster, 1984). In the following we shall not comment on the many interesting results of deterministic chaos, but rather put our focus on the effects of weak random perturbations ξ_n. The importance of residual noise on the discrete dynamics in (5.1.1) has been recognized in numerical studies (for a review see Crutchfield, Farmer and Huberman, 1982; some recent papers are Linz and Lücke, 1986; B. Morris and F. Moss, 1986, and 1988, unpublished).

Our goal in this study of discrete dynamical systems is the (predominantly analytical) calculation of stationary probabilities, mean first passage times, and related quantities, in the *limit of weak noise* (Talkner, Hänggi, Freidkin and Trautmann, 1987).

5.2 Discrete dynamics driven by white noise

In the following we shall restrict the discussion to discrete systems driven by white noise only. In other words, we shall consider the map dynamics in (5.1.1) with ξ_n being independent and identically distributed noise. It should be stressed, however, that ξ_n generally will depend on the state x_n, or more precisely it will depend on the iterated state $F(x_n)$; i.e.

$$x_{n+1} = F(x_n) + \xi_n(F(x_n)). \tag{5.2.1}$$

$\xi_n(x)$ is specified by its probability density $\rho(\xi, x)$ for finding $\xi_n(x)$ in the interval $(\xi, \xi + d\xi)$

$$P(\xi(x) \in [\xi, \xi + d\xi]) = \rho(\xi, x)\, d\xi. \tag{5.2.2}$$

With these conditions for ξ_n, (5.2.1) defines a Markov process; put more precisely, $\{x_n\}$ in (5.2.1) defines a Markov chain (Feller, 1966).

The probability density $W_n(x)$ for finding x_n in the interval $[x, x + dx]$ then obeys the master equation

$$W_{n+1}(x) = \int P(x|y)\, W_n(y)\, dy, \tag{5.2.3}$$

where the transition probability $P(x|y)$ is given by

$$P(x|y) = \rho(x - F(y), F(y)). \tag{5.2.4}$$

Often one assumes for ξ_n a *multiplicative* noise structure of the form

$$\xi_n(x) = g(x_n)\zeta_n, \tag{5.2.5}$$

with ζ_n being white noise with an x-independent probability $\varphi(\zeta)$. The noise ξ_n is termed *additive* if the coupling function $g(x)$ is independent of the state variable x. The probability density for $\xi_n(x)$ in (5.2.5) reads (Z: normalization)

$$\rho(\xi, x) = |g(x)|^{-1}\varphi(\xi/g(x))/Z. \tag{5.2.6}$$

In many other cases, the physics dictates a process x_n that is restricted to an *a priori* fixed interval $I = [x^{(1)}, x^{(2)}]$, for all times n. Then the noise ξ_n cannot be additive. With multiplicative noise, $g(x)$ then must vanish at the boundary points $x^{(1)}, x^{(2)}$, in order to prevent an eventual escape out of the interval. This, however, implies that there cannot be any fluctuations at the boundary itself; i.e.

$$\rho(\xi, x^{(B)}) = 0, \quad x^{(B)} = x^{(1)}, x^{(2)}. \tag{5.2.7}$$

Depending on the problem under consideration the condition in (5.2.7) might not be appropriate. As an illustrative example let us take a random number ξ from an ensemble with density $\varphi(y)$; this will be taken as an allowed realization of $\xi_n(x)$ if $(\xi + x_n)$ is contained in the interval. If not true, we must select another random number. This procedure implies a state-dependent fluctuating force with density (Haken and Mayer-Kress, 1981)

$$\rho(\xi, x) = \begin{cases} \varphi(\xi) \Big/ \displaystyle\int_{x^{(1)}}^{x^{(2)}} \varphi(\xi' - x)\, d\xi', & \text{if } (x + \xi) \in I \\ 0, & \text{otherwise.} \end{cases} \tag{5.2.8}$$

Other examples that also cannot be modeled by multiplicative noise are processes driven by additive noise with *periodic* or reflecting boundary conditions (b.c.). In the case of a periodic b.c., $(x + \xi)$ is taken modulo the length of the interval, if not contained in I; likewise, for a reflecting b.c., $(x + \xi)$ is mirrored at the adjacent boundary as many times until the image falls into the interval. Thus, we find for the density ρ_P and ρ_R for periodic b.c. and reflecting b.c., respectively

$$\rho_{P,R}(\xi, x) = \sum_{n=-\infty}^{\infty} \varphi(S_n^{P,R}(\xi + x) - x) \tag{5.2.9a}$$

with

$$S_n^P(x) = 2n(x^{(2)} - x^{(1)}) + x \tag{5.2.9b}$$

and

$$S_n^R(x) = 2n(x^{(2)} - x^{(1)}) + (-1)^n x. \tag{5.2.9.c}$$

Probability densities for the random force in the presence of more general b.c., such as, e.g., a sticking b.c., etc., are constructed analogously. In conclusion, the multiplicative form in (5.2.5) does not present the most general possible noise structure.

5.3 Stationary probability for weak Gaussian noise

Next we shall consider one-dimensional maps which are weakly disturbed (strength measured by the parameter $\varepsilon < 1$) by white Gaussian noise; i.e.

$$x_{n+1} = F(x_n) + \varepsilon^{1/2}\xi_n, \tag{5.3.1}$$

where $\varepsilon > 0$ and ξ_n has a density

$$\rho(\xi) = (2\pi)^{-1/2}\exp(-\tfrac{1}{2}\xi^2). \tag{5.3.2}$$

If

$$F'_{\pm\infty} \equiv \lim_{x \to \pm\infty} F'(x) \quad \text{exists, and if } |F'_{\pm\infty}| < 1,$$

there is a uniquely defined stationary probability $W(x), x \in (-\infty, \infty)$, of the process defined by (5.3.1) and (5.3.2). This invariant probability (solution with eigenvalue 1) obeys the master equation

$$W(x) = (2\pi\varepsilon)^{-1/2} \int_{-\infty}^{\infty} \exp[-(x - F(y))^2/2\varepsilon]\, W(y)\, \mathrm{d}y. \tag{5.3.3}$$

In order to solve (5.3.3) for weak noise we use a WKB-type ansatz

$$W(x) = Z(x) \exp(-\Phi(x)/\varepsilon), \tag{5.3.4}$$

wherein both $\Phi(x)$ and $Z(x)$ shall not depend on ε. This procedure is well-known in the study of continuous-time Markov processes (Graham, 1981; Kubo, Matsuo and Kitahara, 1973; Ludwig, 1975; Matsuo, 1977). Inserting this ansatz into (5.3.3), one finds

$$Z(x) = (2\pi\varepsilon)^{-1/2} \int_{-\infty}^{\infty} Z(y) \exp(-Y^2(x,y)/2\varepsilon)\, \mathrm{d}y, \tag{5.3.5}$$

where we have defined

$$Y^2(x,y) = 2[\Phi(y) - \Phi(x)] + [x - F(y)]^2. \tag{5.3.6}$$

Our goal is to determine Φ in such a way that

$$Y^2(x,y) \geqslant 0, \tag{5.3.7}$$

as indicated already by the notation.

In the following we shall work out a seemingly strange chain of arguments involving the quantity $Y^2(x,y)$. This reasoning, however, will result in (5.3.13) which provides us the key to find the potential function $\Phi(x)$ occurring in (5.3.4).

First we note that the root of Y^2, i.e. $Y(x,y)$, defines for each x an invertible transformation of the old coordinate, y, to the new one, Y. Thus, it is sufficient and necessary that the derivative of $Y(x,y)$ with respect to y is not vanishing; i.e.

$$\frac{\partial Y(x,y)}{\partial y} \neq 0. \tag{5.3.8}$$

For the moment, let us assume that we know a function Φ, defining a function $Y^2(x,y)$, (see(5.3.6)), obeying the properties (5.3.7) and (5.3.8). Then we can perform in the integral (5.3.5) a change of coordinates from y to Y; i.e.

$$Z(x) = (2\pi\varepsilon)^{-1/2} \int_{-\infty}^{\infty} Z\{y(Y,x)\} \times \exp(-Y^2/2\varepsilon) \frac{\mathrm{d}Y}{|\partial Y(x,y)/\partial y|_{y=y(Y,x)}}. \tag{5.3.9}$$

Here, $y(Y,x)$ is the unique solution of (5.3.6) for y. Further, if we assume that the quantity

$$Z\{y(Y,x)\}/|\partial Y/\partial y|_{y=y(Y,x)}$$

is a smoothly varying function of Y in a sufficiently large neighborhood around $Y=0$, say $Y < 10\,\varepsilon^{1/2}$, we can evaluate the integral (5.3.9) for small ε. We find up to order $O(\varepsilon^0)$

$$Z(x) = Z\{y(Y=0,x)\}/|\partial Y/\partial y|_{y=y(Y=0,x)}. \tag{5.3.10}$$

This constitutes a linear functional equation in the unknown function $Z(x)$; it can be solved iteratively.

The positivity condition on Y^2, (see(5.3.7)), has still another important consequence. Setting $x = F(y)$ in (5.3.6), we observe that $\Phi(x)$ *decreases* along the deterministic trajectory $x_{n+1} = F(x_n)$; i.e.

$$\Phi(F(y)) - \Phi(y) = -\tfrac{1}{2} Y^2(x,y) \leqslant 0. \tag{5.3.11}$$

To put it differently, $\Phi(x)$ is a *Lyapunov function* of the deterministic dynamics $x_{n+1} = F(x_n)$. Now we shall establish an equation for $\Phi(x)$ itself. In order for (5.3.8) to hold true at each pair of points (x,y) at which $\partial Y^2(x,y)/\partial y = 0$, $Y^2(x,y)$ must vanish too. First we obtain from (5.3.6) that for every y with $F'(y) \neq 0$ (prime indicates differentiation) there exists an x-value such that $\partial Y^2/\partial y$ vanishes; i.e.

$$x = (\Phi'(y)/F'(y)) + F(y). \tag{5.3.12}$$

Now, however, in order that $\partial Y/\partial y \neq 0$ holds, (see (5.3.8)), for all x and y, $Y^2(x,y)$ must vanish for those x given by (5.3.12). With (5.3.6) this yields the desired equation for $\Phi(x)$ itself

$$\tfrac{1}{2}(\Phi'(y)/F'(y))^2 + \Phi(y) - \Phi\left(\frac{\Phi'(y)}{F'(y)} + F(y)\right) = 0. \tag{5.3.13}$$

This is a nonlinear functional differential equation for the potential function Φ. At first sight, this equation looks even more complicated than the linear integral equation (5.3.3), which was our starting point. Equation (5.3.13), however, can be related to a two-dimensional Hamiltonian system with discrete time, which can be solved iteratively (P. Talkner, manuscript in preparation).

5.3.1 *Examples*

Rather than developing the general theory for the solution of (5.3.13) we shall now discuss two examples. First, let us check the theory for a linear map with a stable fixed point at $x = 0$; i.e.

Example 1

$$F(x) = Ax, \quad |A| < 1. \tag{5.3.14}$$

From (5.3.13) we find

$$(\Phi'(y))^2 + 2A^2\Phi(y) - 2A^2\Phi\left(Ay + \frac{\Phi'(y)}{A}\right) = 0. \tag{5.3.15}$$

Making the ansatz $\Phi(y) = By^2$ we find from (5.3.15) three values for B, namely

$$B_1 = \tfrac{1}{2}(1 - A^2), \quad B_2 = -\tfrac{1}{2}A^2, \quad B_3 = 0. \tag{5.3.16}$$

The solution B_1 yields with (5.3.6) a positive Y^2:

$$Y^2(x, y) = (y - Ax)^2. \tag{5.3.17}$$

Equation (5.3.10) for $Z(x)$ becomes

$$Z(x) = Z(Ax),$$

which implies a constant; i.e. $Z(x) = \text{const}$. The solution B_2 yields via (5.3.6) a negative $Y^2(x, y)$ for some (x, y)-pairs. Thus, it must be excluded. The solution B_3 implies the trivial solution $\Phi(x) = 0$; therefore it must be excluded, too. Hence, the invariant density for a linear map in the presence of Gaussian noise reads

$$W(x) = \left(\frac{1 - A^2}{2\pi\varepsilon}\right)^{1/2} \exp[-(1 - A^2)x^2/2\varepsilon]. \tag{5.3.18}$$

Actually, this is the exact solution for (5.3.3) and (5.3.14); it has been obtained previously by other means (see, e.g., Haken and Wunderlin, 1982).

Example 2

In our example we consider weakly nonlinear maps of the form

$$F(x) = x - aU'(x), \quad a > 0, \tag{5.3.19}$$

where a is a small positive parameter, and $U(x)$ is a smooth potential. In this case (5.3.13) reads

$$\tfrac{1}{2}(\Phi'(y))^2 + (1 - aU''(y))^2\Phi(y) - (1 - aU''(y))^2\Phi\left(y - aU'(y) + \frac{\Phi'(y)}{1 - aU''(y)}\right) = 0.$$

If we set $\Phi(y) = a\varphi(y)$ we find in leading order in a the following a-independent equation for the scaled potential φ:

$$-\tfrac{1}{2}(\varphi'(y))^2 + U'(y)\varphi'(y) = 0.$$

Again we disregard the trivial solution $\varphi'(y) = 0$ and obtain up to an arbitrary constant

$$\varphi(y) = 2U(y). \tag{5.3.20}$$

From (5.3.6) and (5.3.20) we obtain a positive $Y^2(x, y)$, yielding to leading order in a

$$Y(x, y) = y - x - a[U'(y) - 2(U(y) - U(x))/(y - x)].$$

At $Y = 0$, we find

$$y(Y = 0, x) = x - aU'(x) + O(a^2).$$

and

$$\left.\frac{\partial Y(y, x)}{\partial y}\right|_{y = y(Y = 0, x)} = 1 + O(a^2).$$

Equation (5.3.10) thus gives a prefactor

$$Z(x) = Z(x - aU'(x))(1 + O(a^2)). \tag{5.3.21}$$

Hence, up to corrections of order $O(a^2)$, $Z(x)$ is a constant.

Combining (5.3.4), (5.3.20) and (5.3.21) the stationary probability at weak noise (small ε) thus reads (note that a/ε may be large)

$$W(x) = N \exp(-2aU(x)/\varepsilon). \tag{5.3.22}$$

This result agrees with a previous treatment (Talkner *et al.*, 1987) of the same class of map functions specified in (5.3.19). Corrections to the leading order result (5.3.22) will be discussed elsewhere (P. Talkner, manuscript in preparation).

5.4 Circle map: lifetime of metastable states

In this section we shall elaborate on the noise-induced escape in a periodically continued map (circle map) $F(n + x) = n + F(x)$. Specifically we take the climbing sine map

$$x_{n+1} = x_n + a\sin(2\pi x_n) + \varepsilon^{1/2}\xi_n, \tag{5.4.1}$$

with ξ_n being independent, Gaussian distributed noise (5.3.2). The strength of the sine-force is denoted by $a > 0$, which will be assumed to be small, $a < (2\pi)^{-1}$. For $\varepsilon = 0$, the deterministic map has unstable fixed points at $x^u = n$, and stable fixed points at $x_n^s = (2n + 1)/2$, $n = 0, \pm 1, \pm 2, \ldots$. A trajectory which starts in the interior of the interval $I = [0, 1]$ will be attracted by the stable point $x^s = \frac{1}{2}$, and never leaves the interval I. For arbitrarily small ε, however, the interval will be left eventually. A quantitative measure for the occurrence of these rare events is the mean first passage time (MFPT), i.e. the mean number of steps after which a random walker starting at $x \in [0, 1]$ reaches the exterior of $[0, 1]$ for the first time. For a discrete dynamics the MFPT obeys the inhomogeneous backward equation (Haken and Wunderlin, 1982; Talkner *et al.*, 1987)

$$t(x) - 1 = \int_0^1 P(y|x)t(y)\,dy, \quad x \text{ in } [0, 1]$$
$$t(x) = 0, \quad x \text{ outside } [0, 1]. \tag{5.4.2}$$

With (5.2.4) and (5.3.2) we find

$$t(x) - 1 = (2\pi\varepsilon)^{-1/2}\int_0^1 t(y)\exp[-(y - F(x))^2/2\varepsilon]\,dy,$$
$$x \in I = [0, 1]. \tag{5.4.3}$$

For physical reasons this equation has a unique, bounded solution for all $\varepsilon > 0$. Due to the symmetry of the corresponding potential $U(x) = \cos(2\pi x)/2\pi$ in (5.4.1) (see (5.3.19)) around $x_s = \frac{1}{2}$, $t(x)$ itself becomes a symmetric function about the stable fixed point $x_s = \frac{1}{2}$, and attains its maximal value at $x = x_s = \frac{1}{2}$. Setting $t(\frac{1}{2}) = T$, it follows from (5.4.3) that

$$\begin{aligned} T - 1 &= (2\pi\varepsilon)^{-1/2} \int_0^1 t(y) \exp(-(y - \tfrac{1}{2})^2/2\varepsilon)\, dy \\ &\leqslant T(2\pi\varepsilon)^{-1/2} \int_0^1 \exp(-(y-\tfrac{1}{2})^2/2\varepsilon)\, dy. \end{aligned} \tag{5.4.4}$$

Hence, the MFPT does obey the inequality

$$t(x) \leqslant T \leqslant \left[\operatorname{erfc}\left(\frac{1}{2(2\varepsilon)^{1/2}} \right) \right]^{-1}, \tag{5.4.5}$$

with erfc (x) denoting the complementary error function. This estimate just happens to agree with the approximation by Arecchi, Badii and Politi (1985). In contrast to the case of Markovian Fokker–Planck processes we shall see that $t(x)$ possesses jumps at the exit boundaries $x = 0$ and $x = I^*$. If we start at a boundary, say $x = 0$, the trajectory can return into the interior of the interval $[0, 1]$ with finite probability

$$p = (2\pi\varepsilon)^{-1/2} \int_0^1 \exp(-y^2/2\varepsilon)\, dy;$$

For weak noise, $\varepsilon \ll 1$, p almost equals 1/2. This finite return probability p thus implies a non-zero jump $t(0) > 0$. A more precise estimate for the jump can be devised from (5.4.3); i.e

$$t(0) - 1 = (2\pi\varepsilon)^{-1/2} \int_0^1 t(y) \exp(-y^2/2\varepsilon)\, dy, \tag{5.4.6}$$

or

$$t(0) = CT + 1. \tag{5.4.7}$$

In terms of the form function $\tilde{h}(x)$

$$t(x) = T\tilde{h}(x), \quad \tilde{h}(x) \leqslant 1, \tag{5.4.8}$$

the constant C in (5.4.7) is given by

$$C = (2\pi\varepsilon)^{-1/2} \int_0^1 \tilde{h}(y) \exp(-y^2/2\varepsilon)\, dy < \tfrac{1}{2}. \tag{5.4.9}$$

It then follows from the behavior of $h(y)$ at weak noise (see (5.4.11) and (5.4.12)

* Similar jumps for the MFPT occur for continuous time processes driven by colored noise (Hänggi and Talkner, 1985) or white non-Gaussian noise sources (Knessl, Matkowsky, Schuss and Tier, 1986; Troe, 1977; Weiss and Szabo, 1983).

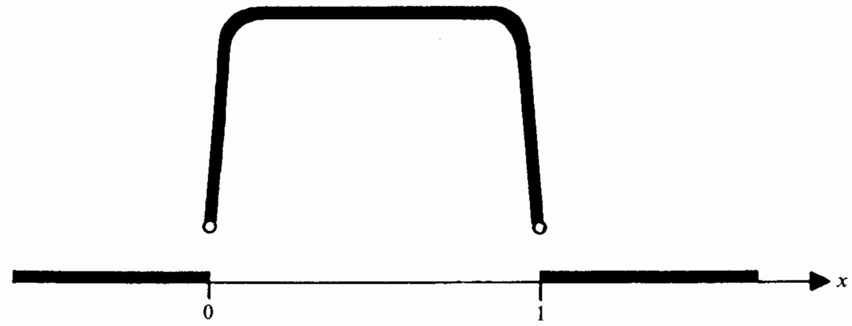

Figure 5.1. Qualitative sketch of the MFPT $t(x)$ versus x. Illustrated are the absorbing lines outside the domain of attraction, as well as the characteristic boundary jumps at the exit points.

below) that the constant C is of the order $O(\varepsilon^0)$; in other words the jump does not approach zero as $\varepsilon \to 0$. In Figure 5.1 we depict the qualitative behavior of the MFPT $t(x)$.

5.4.1 *Weak noise analysis of the MFPT*

At a weak noise level the trajectory remains for most of the time near the stable fixed point, and only rarely will it make a large excursion. Thus, the MFPT attains a very large value inside the internal $[0, 1]$ which deviates only little from its maximal value; i.e. $t(x) \simeq T$. Significant deviations from the constant T occur only near the exit boundaries. Thus $\tilde{h}(x)$ defined in (5.4.8) becomes a boundary layer function which deviates from the value $\tilde{h}(x) \simeq 1$ only in a small neighborhood near the boundaries. The width of the boundary layer function will turn out to be of order $O(\varepsilon^{1/2})$.
We now insert the ansatz (5.4.8) into (5.4.3) to find

$$h(x) - T^{-1} = (2\pi\varepsilon)^{-1/2} \int_0^1 h(y) \exp(-(y - F(x))^2/2\varepsilon)\, dy. \tag{5.4.10}$$

Because the integral kernel is sharply peaked around $y = F(x)$ we can for x near zero linearize $F(x)$ around $x = 0$. In terms of the scaled boundary layer function $h(x)$

$$h(x) = \tilde{h}((2\varepsilon)^{1/2} x) \tag{5.4.11}$$

we thus obtain the following integral equation:

$$h(x) = \pi^{-1/2} \int_0^\infty h(y) \exp(-(y - Ax)^2)\, dy, \tag{5.4.12}$$

where

$$A = F'(0) = 1 + 2\pi a. \tag{5.4.13}$$

In arriving at (5.4.12) we have neglected the small inhomogeneity, T^{-1}, and approximated the upper limit of integration, $\varepsilon^{-1/2}$, by infinity. The solution of (5.4.12) must be normalized to an asymptotic behavior, $h(x) \to 1$ as $x \to \infty$. The results of a numerical solution of $h(x)$ for various A-values, (5.4.13), are shown in Figure 5.2.

An analytic expression for the constant large lifetime T can be obtained as follows: multiply (5.4.3) by the stationary probability $W(x)$, and integrate over all x in [0,1]. Then use the invariant property of $W(x)$, (5.3.3), to obtain

$$\int_0^1 t(x)W(x)\,\mathrm{d}x - \int_0^1 W(x)\,\mathrm{d}x$$
$$= \int_0^1 t(y)W(y)\,\mathrm{d}y - (2\pi\varepsilon)^{-1/2}\left\{\int_{-\infty}^0 + \int_1^\infty\right\}\mathrm{d}x$$
$$\times\left\{\int_0^1 t(y)\exp[-(y-F(x))^2/2\varepsilon]\,\mathrm{d}y\right\}W(x). \tag{5.4.14}$$

Utilizing (5.4.8) we thus find the central result

$$T^{-1} = 2(2\pi\varepsilon)^{-1/2}\left[\frac{\int_{-\infty}^0 \mathrm{d}x\, W(x)\int_0^1 \mathrm{d}y\,\tilde{h}(y)\exp(-(y-F(x))^2/2\varepsilon)}{\int_0^1 W(x)\,\mathrm{d}x}\right]. \tag{5.4.15}$$

Hereby we made use of the symmetry about $x^s = \frac{1}{2}$. The result in (5.4.15) is an *exact* expression for $t(\frac{1}{2}) = T$. At weak noise it can be simplified further: the invariant probability, $W(x) = N\exp(-\Phi(x)/\varepsilon)$, is sharply peaked at $x^s = \frac{1}{2}$. For small nonlinearity, a, we can now use the results of Example 2 in Section 5.3; i.e. with (5.3.19), (5.3.22) and (5.4.1) we have

$$\Phi(x) = \frac{a}{\pi}\cos(2\pi x). \tag{5.4.16}$$

With a steepest descent approximation the denominator of (5.4.15) thus becomes

$$\int_0^1 W(x)\,\mathrm{d}x \simeq N\frac{1}{2}\left(\frac{\varepsilon}{a}\right)^{1/2}\exp\left(\frac{a}{\pi\varepsilon}\right).$$

The numerator of (5.4.15) simplifies for weak noise as well. The map $F(x)$ can be linearized around $x^u = 0$, and the invariant probability can be approximated around $x^u = 0$ by

$$W(x) \simeq N\exp\left(-\frac{a}{\pi\varepsilon}\right)\exp(4\pi a x^2).$$

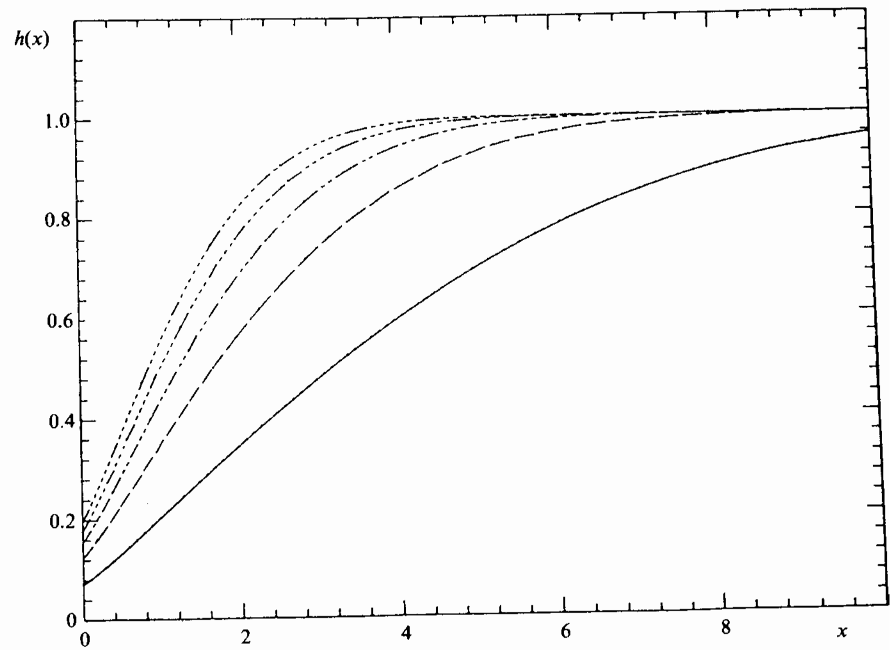

Figure 5.2. Numerical solution of the scaled boundary layer function $h(x)$ for the following A-values: $A = 1.09$; $A = 1.07$; $A = 1.05$; $A = 1.03$; $A = 1.01$.

Collecting everything thus yields for (5.4.15)

$$T = \frac{1}{2a^{1/2}R(A)}\exp\left(\frac{2a}{\pi\varepsilon}\right), \tag{5.4.17a}$$

where ($A = 1 + 2\pi a$, see (5.4.13))

$$R(A) = \int_0^\infty \mathrm{erfc}(Ay)h(y)\exp[(A^2-1)y^2]\,\mathrm{d}y. \tag{5.4.17b}$$

The quantity $R(A)$ is an ε-*independent* function which can be evaluated numerically. For small positive a, one finds within numerical accuracy

$$R(A) = \left(\frac{A^2-1}{\pi}\right)^{1/2} + O(A^2-1). \tag{5.4.18}$$

Combining (5.4.18) with (5.4.17a) we find for the lifetime T at weak noise the result

$$T = \frac{1}{4a}\exp\left(\frac{2a}{\pi\varepsilon}\right). \tag{5.4.19}$$

T is determined by an Arrhenius-like exponential leading part and a prefactor, $(4a)^{-1}$.

From the lifetime T one obtains for the rate, λ, at which there occurs an escape from the metastable state at $x = \frac{1}{2}$ across the boundary $x = 0$ or $x = 1$

$$\lambda = \tfrac{1}{2}T^{-1} = \lambda^{+} + \lambda^{-}. \tag{5.4.20}$$

The factor of $(\frac{1}{2})$ takes into account that, in the absence of a capture beyond the unstable fixed points, $x = 0, 1$, half of the number of random walkers would return into the original interval $[0, 1]$ (Matkowsky and Schuss, 1979). For the individual rates of escape either to the left, λ^{-}, or to the right, λ^{+}, respectively, we have

$$\lambda^{+} = \lambda^{-} = \tfrac{1}{2}\lambda = (4T)^{-1}. \tag{5.4.21}$$

These rate results (5.4.20) and (5.4.21) can also be derived by an alternative method (Talkner *et al.*, 1987) which utilizes ideas underlying Kramers' flux method (Hänggi, 1986; Kramers, 1940); i.e. one evaluates the rates as the ratio of a nonvanishing, stationary probability current across the exit points and the population inside the interval.

In our present situation of a periodically continued map function, the random walker undergoes a noise-induced diffusive motion across periodic barriers with a diffusion constant D

$$\langle (x_n - \langle x_n \rangle)^2 \rangle \to 2Dn \quad \text{as} \quad n \to \infty. \tag{5.4.22}$$

D itself is determined by the forward and backward hopping rates λ^{+}, λ^{-}, and the step size $L = 1$; i.e.

$$D = \tfrac{1}{2}(\lambda^{+} + \lambda^{-})L^2 = (4T)^{-1}. \tag{5.4.23}$$

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