Noise in nonlinear dynamical systems

Volume 1

Theory of continuous Fokker–Planck systems

Edited by
Frank Moss, *Professor of Physics,*
University of Missouri at St Louis
and
P. V. E. McClintock, *Reader in Physics,*
University of Lancaster

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9 Colored noise in continuous dynamical systems: a functional calculus approach

PETER HÄNGGI

9.1 Introduction

Recent work on dye lasers (Fox and Roy, 1987; Jung and Risken, 1984; Lett, Short and Mandel, 1984; Roy, Yu and Zhu, 1985; Short, Mandel and Roy, 1982) and the optical ring laser gyroscope (Vogel *et al.*, 1987a, b) has emphasized the physically important role of colored noise sources. A well-known classical situation in which strongly colored noise has an impact on the physics is the phenomenon of motional narrowing in magnetic resonance (Kubo, 1962). Kubo has shown that a fluctuating magnetic field with very short noise correlation time (almost white noise) does typically not manifestly affect the motion of spins; on the contrary, if the fluctuations are correlated over a long time scale (colored noise) the motion of the spin becomes greatly modified.

Another area where there has been much recent activity addresses escape problems. These are currently in the limelight both from the theoretical viewpoint (Grote and Hynes, 1980; Hänggi, 1986; Hänggi and Mojtabai, 1982; Hänggi and Riseborough, 1983; Hänggi, Mroczkowski, Moss and McClintock, 1985) as well as from an experimental point of view (Devoret, Martinis, Esteve and Clarke, 1984; Fleming, Courtney and Balk, 1986; Hänggi *et al.*, 1985; Maneke, Schroeder, Troe and Voss, 1985). In this latter case, a frequency-dependent friction, or noise of finite correlation time, can considerably modify the *classical barrier transmission*. Except for two-state noise (Hänggi and Talkner, 1985; Masoliver, Lindenberg and West, 1986; Rodriguez and Pesquera, 1986; Van den Broeck and Hänggi, 1984) there exist no exact analytic methods for truly nonlinear systems, being driven by correlated noise. Generally, the finite correlation time of the noise will affect not only dynamical aspects but also the form of the stationary probability. This fact has been exploited, for example, in recent studies of colored noise-induced transitions (see Chapter 8, Volume 2).

Our main interest in this chapter is the study of dynamical continuous-time systems of the form (with λ a set of control parameters)

$$\dot{x}_\alpha = f_\alpha(x, \lambda) + \sum_i g_{\alpha i}(x, \lambda)\xi_i(t) \tag{9.1.1}$$

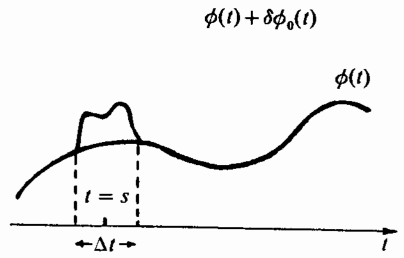


Figure 9.1. A pictorial view of a variation $\delta\phi_0(t)$ of a functional $\phi(t)$.

being driven by noise forces $\{\xi_i(t)\}$ of generally finite correlation time. For the sake of clarity only, we shall primarily restrict in the following the discussion to one-dimensional flows. Without loss of generality we shall confine ourselves to additive noise. Note that a multiplicative noise structure $\xi(t)\to g(x)\xi(t)$ can (in one dimension) be transformed into additive noise if we consider the transform $x\to\tilde{x}=\int^x g^{-1}(y)\mathrm{d}y$, $g(y)\neq 0$, observing that the probabilities transform according to $\tilde{p}_t(\tilde{x})=p_t(x(\tilde{x}))|\mathrm{d}x/\mathrm{d}\tilde{x}|$.

9.2 Functional calculus

The stochastic dynamics in (9.1.1) is suitably described in terms of probability notions. If we now focus on one-dimensional flows of the form

$$\dot{x}=f(x)+\xi(t) \tag{9.2.1}$$

our interest is in evaluation of mean values $\langle x(t)\rangle$, variances, or the probability $p_t(x)=\langle\delta(x(t)-x)\rangle$ itself. The trajectory $x(t)$ depends, of course, on the whole previous history of the random force $\xi(s)$, $t\geqslant s\geqslant t_0=0$; i.e. $x(t)$ is a functional of the noise $\xi(s)$.

If $\rho_t(\xi)$ denotes the (normalized) probability function of the random variable $\xi_t\equiv\xi(t)$ at time t, we obtain for the mean value a functional integral over the realizations of the noise: i.e.

$$\langle x(t)\rangle=\int \mathrm{D}\xi(t)x\{\xi(t)\}\rho_t(\xi). \tag{9.2.2}$$

For the probability $p_t(x)$ itself one obtains

$$p_t(x)=\langle\delta(x(t)-x)\rangle=\int \mathrm{D}\xi(t)\rho_t(\xi)\delta(x(t)-x). \tag{9.2.3}$$

For the rate of change of $p_t(x)$ one finds

$$\dot{p}_t(x)=\int \mathrm{D}\xi(t)\rho_t(\xi)\left\{-\frac{\partial}{\partial x}\delta(x(t)-x)\right\}\dot{x}(t)$$

$$=-\frac{\partial}{\partial x}\langle\delta(x(t)-x)\dot{x}(t)\rangle$$

$$=-\frac{\partial}{\partial x}\{f(x)p_t(x)\}-\frac{\partial}{\partial x}\langle\delta(x(t)-x)\xi(t)\rangle. \tag{9.2.4}$$

This latter relation can be evaluated further if we make explicit use of the statistics of the noise $\xi(t)$. This task is suitably performed in terms of functional derivatives techniques (Hänggi, 1978b). An extended introduction into the techniques of functional derivatives together with its use in the description of noisy dynamical systems has been given by the author elsewhere (Hänggi, 1985). In the following we shall restrict ourselves to essentials only.

9.3 The functional derivative: a poor man's approach

Let us consider the values of a functional $F[\phi]$ for the functions $\phi(t)$ and $(\phi(t)+\delta\phi_0(t))$ with $\delta\phi_0(t)\neq 0$ within a small interval, $s-\Delta t/2\leqslant t\leqslant s+\Delta t/2$. In other words, we vary the function $\phi(t)$ near the position s (see Figure 9.1).

The functional derivative is then defined as the limit

$$\frac{\delta F[\phi]}{\delta\phi(s)}\equiv\lim_{\Delta t\to 0}\frac{F[\phi+\delta\phi_0]-F[\phi]}{\int_{\Delta t}\delta\phi_0(t)\,\mathrm{d}t}. \tag{9.3.1}$$

Before we proceed further, let us exercise this limiting procedure with an example of a quadratic functional $F[\phi]$

$$F[\phi]=\iint g(t,s)\phi(t)\phi(s)\,\mathrm{d}t\,\mathrm{d}s.$$

Therefore we have

$$\Delta F[\phi]\equiv F[\phi+\delta\phi_0]-F[\phi]=\iint g(t,s)[\phi(t)\delta\phi_0(s)+\phi(s)\delta\phi_0(t)]\,\mathrm{d}t\,\mathrm{d}s+\mathrm{O}(\delta\phi^2).$$

Using for the second part of the integral a substitution of variables, i.e. $t\to s$, $s\to t$, we obtain

$$\Delta F[\phi]=\int\phi(t)\,\mathrm{d}t\int\mathrm{d}s\{g(t,s)+g(s,t)\}\,\delta\phi_0(s).$$

We now vary the function ϕ around the position $t=s\equiv\tau$. Next we apply the average theorem for integrals and get with $\tau+\Delta t/2\geqslant\bar{s}\geqslant\tau-\Delta t/2$

$$\Delta F[\phi]=\int\phi(t)\{g(t,\bar{s})+g(\bar{s},t)\}\,\mathrm{d}t\int_{\Delta t}\delta\phi_0(s)\,\mathrm{d}s.$$

Observing that

$$A\equiv\iint\{g(t,s)-g(s,t)\}\,\mathrm{d}t\,\mathrm{d}s$$

$$= \int\int \{g(s,t) - g(t,s)\}\,dt\,ds = -A = 0,$$

only the symmetric part of $g(t,s)$, i.e. $g_s(t,s) = \frac{1}{2}(g(t,s) + g(s,t))$ enters the result

$$\frac{\delta F[\phi]}{\delta \phi(\tau)} = 2 \int g_s(t,\tau)\phi(t)\,dt.$$

Inspecting the above example one might wonder if the answer could not have been obtained more simply. We remember that a functional $F[\phi]$ can be looked upon as the continuum limit of a multivariable function. Thus we can perturb the functional at the position $t = \tau$ by varying the 'variable' $\phi_\tau = \phi(\tau)$ by an amount $\delta\phi(\tau) = \lambda\delta(t-\tau)$. We may then express the limit in (9.3.1) as the limit of

$$\frac{\delta F[\phi]}{\delta \phi(\tau)} = \left.\frac{dF[\phi(t) + \lambda\delta(t-\tau)]}{d\lambda}\right|_{\lambda=0} \tag{9.3.2}$$

assuming that both sides (generally only in the sense of a distribution) exist. Clearly with this trick one readily recovers the result of the above example. Analogously one finds the useful relations:

(i) $F[\phi] = f(\phi(t)) \rightarrow \dfrac{\delta F[\phi]}{\delta\phi(\tau)} = \dfrac{\partial f}{\partial \phi}\delta(t-\tau)$

(ii) $F[\phi] = f(g(\phi)) \rightarrow \dfrac{\delta F[\phi]}{\delta\phi(\tau)} = \dfrac{\partial f}{\partial g}\dfrac{\delta g}{\delta\phi(\tau)}$

(iii) $F[\phi] = \displaystyle\int f(\phi,\dot{\phi})\,dt \rightarrow \dfrac{\delta F[\phi]}{\delta\phi(\tau)} = \dfrac{\partial f}{\partial\phi(\tau)} - \dfrac{d}{d\tau}\dfrac{\partial f}{\partial\dot{\phi}(\tau)}.$

Examples

The characteristic functional of a stochastic process $\xi(t)$ is defined by

$$\chi[\phi] = \left\langle \exp i\left(\int \phi(t)\xi(t)\,dt\right)\right\rangle, \quad \chi[\phi = 0] = 1.$$

The *moments* $m_n(t_1, \ldots, t_n) = \langle \xi(t_1)\xi(t_2)\ldots\xi(t_n)\rangle$ are then obtained via the nth order functional derivative of $\chi[\phi]$; i.e.

$$m_n(t_1,\ldots,t_n) = i^{-n}\left.\frac{\delta^n \chi[\phi]}{\delta\phi(t_1)\ldots\delta\phi(t_n)}\right|_{\phi=0}. \tag{9.3.3}$$

The cumulant generating functional $\Psi[\phi]$ is defined by

$$\Psi[\phi] = \ln \chi[\phi],$$

yielding the *cumulants* $C_n(t_1,\ldots,t_n)$

$$C_n(t_1,\ldots,t_n) = i^{-n}\left.\frac{\delta^n \Psi[\phi]}{\delta\phi(t_1)\ldots\delta\phi(t_n)}\right|_{\phi=0}. \tag{9.3.4}$$

9.4 A summary of important correlation formulae

The master equation in (9.2.4) explicitly introduces a correlation between the noise $\xi(t)$ and the functional $F[\xi] = \delta(x(t) - x)$ of the noise $\xi(s), t \geqslant s \geqslant t_0$. The expression can only be disentangled further if we make explicit use of the statistical properties of the random force $\xi(s)$. In this context it should be emphasized that the *initial preparation of the system*, which in turn determines the statistical properties of the noise $\xi(s)$, is of *equal importance as the dynamical law* (Grabert, Hänggi and Talkner, 1980; Hänggi, Marchesoni and Grigolini, 1984; Marchesoni, 1984). It should be stressed that different initial correlations between the dynamical variable $x(t=0)$ and the environment to which the dynamical variable is coupled imply different statistical properties for the noise $\xi(s)$. In particular, the statistical properties of $\xi(s)$ generally cannot be chosen to be independent of the form of initial probability $p_0(x)$ of the (reduced) dynamics $x(s)$. Throughout the rest of the chapter we shall assume that the coarse-grained dynamics $x(s)$ in (9.2.1) has been prepared at initial time $t_0 = 0$ without any memory of the past and without correlations between system and environment (correlation-free preparation); i.e. $p_0(x;$ environment$) = p_0(x)\cdot p_0($environment$)$.

Below we now list without proof some important relations. For the explicit derivation the interested reader is referred to the original papers (Hänggi, 1978b; Hänggi, 1985). The correlation between a functional $g[\xi]$ and the noise $\xi(t)$ is expressed in terms of the cumulants C_n of the noise $\xi(t)$ as

(i)
$$\langle \xi(t) g[\xi]\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^t \cdots \int_0^t dt_1 \ldots dt_n \times C_{n+1}(t, t_1, t_2, \ldots, t_n)\left\langle \frac{\delta^n g[\xi]}{\delta\xi(t_1)\ldots\delta\xi(t_n)}\right\rangle. \tag{9.4.1}$$

This relation can be generalized to a correlation between two functionals $f[\xi]$ and $g[\xi]$ (Hänggi, 1985) to give

(ii)
$$\begin{aligned}\langle f[\xi] g[\xi]\rangle &= \langle f[\xi]\rangle\langle g[\xi]\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\sum_{m_i=1}^{\infty}\sum_{k_i=1}^{\infty}\frac{1}{m_i!k_i!} \\ &\times \int_0^t \underset{m_i+k_i}{\cdots}\int_0^t dt_1^{(i)}\ldots dt_{m_i}^{(i)}\ldots ds_1^{(i)}\ldots ds_{k_i}^{(i)} \\ &\times C_{m_i+k_i}(t_1^{(i)},\ldots,t_{m_i}^{(i)}; s_1^{(i)},\ldots,s_{k_i}^{(i)})\left\langle\frac{\delta^{m_1+\cdots+m_n} f[\xi]}{\delta\xi(t_1^{(1)})\ldots\delta\xi(t_{m_n}^{(n)})}\right\rangle \\ &\times \left\langle \frac{\delta^{k_1+\cdots+k_n} g[\xi]}{\delta\xi(s_1^{(1)})\cdots\delta\xi(s_{k_n}^{(n)})}\right\rangle.\end{aligned} \tag{9.4.2}$$

For a *Gaussian* process there exist only two comulants $C_1(t) = \langle\xi(t)\rangle$ and

$C_2(t,s) = \langle \xi(t)\xi(s)\rangle - \langle \xi(t)\rangle\langle \xi(s)\rangle$. Then the above relations simplify considerably to give (Dubkov and Malakhov, 1975; Furutsu, 1963; Hänggi, 1985; Novikov, 1965)

(iii) $$\langle \xi(t) g[\xi]\rangle = \langle \xi(t)\rangle\langle g[\xi]\rangle + \int_0^t C_2(t,s)\left\langle \frac{\delta g[\xi]}{\delta \xi(s)}\right\rangle \mathrm{d}s \qquad (9.4.3)$$

and (Dubkov and Malakhov, 1975; Hänggi, 1985)

(iv) $$\langle f[\xi] g[\xi]\rangle = \langle f[\xi]\rangle\langle g[\xi]\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}\int_0^t \overset{2n}{\cdots}\int_0^t \left\langle \frac{\delta^n f[\xi]}{\delta\xi(t_1)\ldots\delta\xi(t_n)}\right\rangle \times \left\langle \frac{\delta^n g[\xi]}{\delta\xi(s_1)\ldots\delta\xi(s_n)}\right\rangle \prod_{i=1}^{n} C_2(t_i,s_i)\,\mathrm{d}t_i\,\mathrm{d}s_i. \qquad (9.4.4)$$

The set of above formulae remains true also for white noise (i.e. δ-correlated random forces) if the latest time, s (e.g. $\xi(t)\to\xi(s)$ with $s<t$ in (i)) of the random force occurring in the functionals is less than the final observation time t. Some care is needed for white random forces if functionals of the noise up to the final observation time $s \leqslant t$ are considered; in this latter case it is advantageous to consider curtailed generating functionals (for details see Hänggi, 1978b).

9.5 Functional derivatives for dynamical flows

On inspecting (9.2.4), we find that the master equation in (9.2.4) can be evaluated further by use of relations such as (9.4.1) or (9.4.3). This procedure thus involves the functional derivative

$$\frac{\delta}{\delta\xi(s)}\delta(x(t)-x) = -\frac{\partial}{\partial x}\delta(x(t)-x)\frac{\delta x(t)}{\delta \xi(s)}. \qquad (9.5.1)$$

A dynamical flow with *multiplicative* noise, i.e.

$$\dot{x} = f(x) + g(x)\xi(t) \qquad (9.5.2)$$

then yields via its integral

$$x(t) = \int_0^t \{f(x(s)) + g(x(s))\xi(s)\}\,\mathrm{d}s + x(0) \qquad (9.5.3)$$

the following integral equation (Hänggi, 1978b):

$$\frac{\delta x(t)}{\delta\xi(s)} = \theta(t-s)\left\{g(x(s)) + \int_s^t \mathrm{d}u\,\frac{\partial \dot{x}(u)}{\partial x(u)}\frac{\delta x(u)}{\delta\xi(s)}\right\}, \qquad (9.5.4)$$

where $\theta(t)$ is the step function expressing causality. This integral equation is

readily solved to yield the formal expression

$$\frac{\delta x(t)}{\delta\xi(s)} = \theta(t-s)g(x(s))\exp\int_s^t \frac{\partial \dot{x}(u)}{\partial x(u)}\mathrm{d}u \qquad (9.5.5a)$$

$$= \theta(t-s)g(x(s)) \times \exp\int_s^t\left\{\frac{\partial f(x(u))}{\partial x(u)} + \frac{\partial g(x(u))}{\partial x(u)}\xi(u)\right\}\mathrm{d}u. \qquad (9.5.5b)$$

Equation (9.5.5b) can be recast in alternative form if we make use of a trick due to Fox (1986b). Observing

$$\dot{g} = g'\dot{x} = \frac{g'}{g}(f + g\xi)g$$

one finds the formal solution

$$g(x(s)) = g(x(t))\exp\int_t^s \mathrm{d}u\,\frac{g'}{g}(f+g\xi).$$

Thus, the result in (9.5.5b) is rewritten as

$$\frac{\delta x(t)}{\delta\xi(s)} = \theta(t-s)g(x(t)) \times \exp\int_s^t\left\{f'(x(u)) - f(x(u))\frac{g'(x(u))}{g(x(u))}\right\}\mathrm{d}u, \qquad (9.5.5c)$$

where the prime indicates differentiation with respect to $x(u)$. This latter form is preferred to (9.5.5b) because the multiplicative coupling enters at the final time t. Before we proceed further we now present a few useful examples.

Examples

(1) Consider (9.1.1) and show that in this multidimensional case one obtains for (9.5.4)

$$\frac{\delta x_\alpha(t)}{\delta\xi_i(s)} = \theta(t-s)\left\{\int_s^t \mathrm{d}u\left[\frac{\partial f_\alpha}{\partial x_n} + \frac{\partial g_{\alpha m}}{\partial x_n}\xi_m\right]\frac{\delta x_n(u)}{\delta\xi_i(s)} + g_{\alpha i}(x(s))\right\}, \qquad (9.5.6)$$

where a summation convention over equal indices is implied.

(2) Show that for linear flow $\dot{x} = a(t)x(t) + \xi(t)$ one finds the explicit answer

$$\frac{\delta x(t)}{\delta\xi(s)} = \theta(t-s)\exp\int_s^t a(u)\,\mathrm{d}u. \qquad (9.5.7)$$

(3) For a drift-free flow $\dot{x} = a(t)g(x(t))\xi(t)$ one finds (Hänggi, 1981)

$$\frac{\delta x(t)}{\delta\xi(s)} = \theta(t-s)a(s)g(x(t)). \qquad (9.5.8)$$

(4) For a Mori-type flow

$$\dot{x}(t) = -\int_0^t \gamma(t-s)\,x(s)\,ds + \xi(t)$$

we obtain in terms of the Green's function χ, obeying

$$\dot{\chi}(t) = -\int_0^t \gamma(t-s)\,\chi(s)\,ds, \quad \chi(0) = 1$$

for the functional derivative the result

$$\frac{\delta x(t)}{\delta \xi(s)} = \theta(t-s)\,\chi(t-s). \tag{9.5.9}$$

Generally, it will not be possible to obtain a closed answer for the functional derivative $\delta x(t)/\delta \xi(s)$ taken at different times $s < t$. At this stage we have now collected enough material to evaluate the colored noise master equation in (9.2.4) in terms of the statistics of the random force.

9.6 The colored noise master equation (Hänggi, 1978b)

The result in (9.2.4) can be evaluated further if we invoke the correlation formula in (9.4.1) with $g[\xi] = \delta(x(t) - x)$. For non-Gaussian noise with vanishing mean, $\langle \xi(t) \rangle = C_1(t) = 0$, we find from (9.2.4) and (9.5.2)

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}\{f(x)p_t(x)\} - \frac{\partial}{\partial x} g(x) \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^t \cdots \int_0^t dt_1 \ldots dt_n$$
$$\times\, C_{n+1}(t, t_1, \ldots, t_n) \left\langle \frac{\delta^n}{\delta\xi(t_1)\ldots\delta\xi(t_n)} \delta(x(t) - x) \right\rangle. \tag{9.6.1}$$

Note here the typical non-Markovian character, i.e. the dependence on the initial time $t_0 = 0$ of preparation, yielding for the initial rate of change of probability:

$$\dot{p}_{t_0=0}(x) = -\frac{\partial}{\partial x}\{f(x)p_{t_0=0}(x)\}.$$

The last term in (9.6.1) can now be written out more explicitly by observing the chain rule:

$$\frac{\delta}{\delta\xi(s)}\delta(x(t) - x) = -\frac{\partial}{\partial x}\delta(x(t) - x)\frac{\delta x(t)}{\delta\xi(s)}.$$

In the remainder of this chapter we restrict ourselves to *Gaussian noise* only. Then we obtain from (9.4.3) the colored noise master equation

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}\{f(x)p_t(x)\}$$

$$+\frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} \int_0^t ds\, C_2(t,s) \left\langle \delta(x(t) - x) \frac{\delta x(t)}{\delta \xi(s)} \right\rangle. \tag{9.6.2a}$$

For *additive* noise, (9.2.1), we find with $g(x) \equiv 1$ from (9.5.5b)

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}\{f(x)p_t(x)\}$$
$$+\frac{\partial^2}{\partial x^2} \int_0^t ds\, C_2(t,s) \left\langle \delta(x(t) - x) \exp \int_s^t \frac{\partial f(u)}{\partial x(u)} du \right\rangle. \tag{9.6.2b}$$

Note that for the *multiplicative* stochastic flow in (9.5.2) the corresponding colored noise master equation reads with (9.5.5c)

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}\{f(x)p_t(x)\} + \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) \int_0^t ds\, C_2(t,s)$$
$$\times \left\langle \delta(x(t) - x) \exp \int_s^t [f' - (fg'/g)]\, du \right\rangle. \tag{9.6.2c}$$

At this stage we generally cannot simplify the master equation any further. In view of the δ-function $\delta(x(t) - x)$ a closed expression is possible only if the functional derivative $\delta x(t)/\delta\xi(s)$ either does not involve the process $x(s)$ itself, or depends on the process $x(s)$ solely on the 'Markovian' end-point $s = t$. In the examples discussed in (9.5.7)–(9.5.9) we obtain thus a *closed colored noise master equation*, which for Gaussian noise $\xi(t)$ is of Fokker–Planck type. We leave it as an exercise to the reader to verify the following results.

(i) A linear flow $\dot{x} = a(t)x(t) + \xi(t)$ yields with (9.5.7) the exact result (Hänggi, 1978b)

$$\dot{p}_t(x) = -a(t)\frac{\partial}{\partial x}(x p_t(x))$$
$$+\left(\int_0^t ds\, C_2(t,s) \exp\left[\int_s^t a(u)\, du\right]\right) \frac{\partial^2}{\partial x^2} p_t(x). \tag{9.6.3}$$

More generally, (9.6.3) holds for a nonlinear process y, which up to a nonlinear transformation, $y \to x = f(y;t)$, coincides with a Gaussian process (linear flow) (see Hänggi, 1978a).
(ii) For a drift-free flow one finds in view of (9.5.8)

$$\dot{p}_t(x) = \left(a(t) \int_0^t C_2(t,s)\, a(s)\, ds\right) \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x)\, p_t(x). \tag{9.6.4a}$$

Note the typical non-Markovian character in (9.6.3) and (9.6.4a). The Fokker–Planck type equation depends – in contrast to the Markovian case – on the initial time point $t_0 = 0$ of preparation. By use of the time scale

$$\hat{t} = \int^t ds\, a(s) \int_0^s C_2(s,r)\, a(r)\, dr$$

we can recast (9.6.4a) into the 'Markovian form'

$$\dot{p}_{\hat{t}}(x) = \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) p_{\hat{t}}(x), \tag{9.6.4b}$$

with all non-Markovian features transformed away. Nevertheless the process $x(\hat{t})$ is, of course, still non-Markovian. Its single event evolution just happens to coincide on the time scale, $t \to \hat{t}$, with the single-event evolution of a corresponding time-homogeneous Markov process with Fokker–Planck operator (9.6.4b).

(iii) For a Mori-type flow

$$\dot{x} = -\Omega^2 x - \int_0^t \mathrm{d}s\, \hat{\gamma}(t-s)\, x(s) + \xi(t)$$

$$= -\int_0^t \gamma(t-s)\, x(s)\, \mathrm{d}s + \xi(t),$$

$\gamma(t-s) = \hat{\gamma}(t-s) + 2\Omega^2 \delta(t-s)$ one finds, in view of (9.5.9) and the explicit solution

$$x(t) = \chi(t) x_0 + \int_0^t \chi(t-u)\, \xi(u)\, \mathrm{d}u$$

for the colored noise master equation, the explicit result (Hänggi, 1985)

$$\begin{aligned}\dot{p}_t(x) = &\left(\int_0^t \mathrm{d}s\, C_2(t,s)\, \chi(t-s)\right) \frac{\partial^2}{\partial x^2} p_t(x) \\ &+ \left(\langle x(0) \rangle \int_0^t \gamma(t-s)\, \chi(s)\, \mathrm{d}s\right) \frac{\partial}{\partial x} p_t(x) \\ &- \left\{ \int_0^t \mathrm{d}s\, \gamma(t-s) \int_0^s \int_0^t \mathrm{d}t_1\, \mathrm{d}t_2\, \chi(s-t_1) \right. \\ &\left. \times \chi(t-t_2) C_2(t_1,t_2) \right\} \frac{\partial^2}{\partial x^2} p_t(x).\end{aligned} \tag{9.6.5}$$

Except for times $\{\bar{t}\}$ satisfying $\chi(\bar{t}) = 0$, the Mori-flow can be recast into time-convolutionless form

$$\dot{x} = \left[\frac{\dot{\chi}(t)}{\chi(t)}\right] x(t) + \chi(t) \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{\chi(t-s)}{\chi(t)}\, \xi(s)\, \mathrm{d}s$$

yielding (Hänggi, 1978b)

$$\begin{aligned}\dot{p}_t(x) = &\frac{-\dot{\chi}(t)}{\chi(t)} \frac{\partial}{\partial x}(x p_t(x)) - \frac{\dot{\chi}(t)}{\chi(t)} \left(\int_0^t \mathrm{d}s\, \chi(t-s) \right. \\ &\left. \times \int_0^t \mathrm{d}u\, \chi(t-u)\, C_2(s,u) \right) \frac{\partial^2}{\partial x^2} p_t(x) \\ &+ \left(\int_0^t \mathrm{d}s\, \dot{\chi}(t-s) \int_0^t \mathrm{d}u\, \chi(t-u)\, C_2(s,u) \right) \frac{\partial^2}{\partial x^2} p_t(x) \\ &+ \left(\int_0^t \mathrm{d}u\, \chi(t-u)\, C_2(t,u) \right) \frac{\partial^2}{\partial x^2} p_t(x).\end{aligned} \tag{9.6.6}$$

Example

Consider the colored noise Brownian motion of a particle with mass m

$$\dot{x} = u$$

$$\dot{u} = -\omega^2 x - \int_0^t \gamma(t-s) u(s)\, \mathrm{d}s + \xi(t)$$

obeying the fluctuation–dissipation relation

$$\langle \xi(t) \xi(s) \rangle = \frac{kT}{m} \gamma(t-s). \tag{9.6.7}$$

With

$$\chi(t) = L^{-1}\left[\frac{1}{z + \tilde{\gamma}(z)}\right],$$

where L^{-1} is the inverse Laplace transform being denoted by $f(t) \to \tilde{f}(z)$, we obtain

$$\begin{aligned}u(t) = &\chi(t) u(0) + \int_0^t \chi(t-s) \xi(s)\, \mathrm{d}s \\ &- \omega^2 \int_0^t \chi(t-s) x(s)\, \mathrm{d}s.\end{aligned} \tag{9.6.8a}$$

For $x(t)$ we find with

$$\eta(t) = L^{-1}\left[\frac{1}{z + \omega^2 \tilde{\chi}(z)}\right]$$

$$\begin{aligned}x(t) = &\eta(t) x(0) + \int_0^t \mathrm{d}s\, \eta(t-s) \int_0^s \chi(t-u) \xi(u) \mathrm{d}u + u(0) \\ &\times \int_0^t \eta(t-s) \chi(s)\, \mathrm{d}s.\end{aligned} \tag{9.6.8b}$$

Using the time-convolutionless form for $x(t)$ and observing (9.6.7) one arrives from (9.6.6) at the result

$$\begin{aligned}\dot{p}_t(x) = &\frac{\partial}{\partial x}\left[-\frac{\dot{\eta}(t)}{\eta(t)} x p_t(x) + \frac{u(0)}{\omega^2} \eta(t) \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\dot{\eta}(t)}{\eta(t)} \right) p_t(x) \right] \\ &+ \frac{\partial^2}{\partial x^2}\left(\left\{ \frac{-kT \dot{\eta}(t)}{m\omega^2\, \eta(t)} - \frac{kT}{2 m\omega^4} \eta^2(t) \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\dot{\eta}(t)}{\eta(t)} \right)^2 \right] \right\} p_t(x) \right).\end{aligned} \tag{9.6.9}$$

For an equilibrium Maxwell distribution of the initial probability $u(0)$ we obtain from (9.6.8a)

$$\dot{x}(t) = u(t) = -\omega^2 \int_0^t ds\,\chi(t-s)x(s) + \hat{\xi}(t),$$

where

$$\langle \hat{\xi}(t)\hat{\xi}(s) \rangle = \frac{kT}{m}\chi(t-s). \tag{9.6.10a}$$

This then yields the simple answer (see Hänggi, 1978b; San Miguel and Sancho, 1980)

$$\dot{p}_t(x) = -\frac{\dot{\eta}(t)}{\eta(t)}\left(\frac{\partial}{\partial x}(xp_t(x)) + \frac{kT}{m\omega^2}\frac{\partial^2}{\partial x^2}p_t(x)\right). \tag{9.6.10b}$$

For the non-Markovian master equation for the joint probability $p_t(x, u)$ of the colored noise Brownian motion

$$\ddot{x} = \pm\omega^2 x - \int_0^t \gamma(t-s)u(s)\,ds + \xi(t), \quad \dot{x}(s) = u(s)$$

we refer the reader to the original papers (Adelman, 1976; Hänggi and Mojtabai, 1982).

These selected examples of exactly solvable cases make it clear that in general the functional derivative cannot be evaluated explicitly. Thus, in most cases of physical interest we are stuck with (9.6.2b) or (9.6.2c). Therefore one needs approximative schemes which are tailored to the specific noise parameters under consideration, such as the noise correlation time τ and/or the noise strength D, i.e.

$$D = \int_0^\infty \langle \xi(t)\xi(0) \rangle\, dt.$$

Generally, these approximative schemes become useful in practice only if they reduce to a Fokker–Planck form.

9.7 Approximation schemes

In the following we shall report, extend and interpret various approximative schemes for colored noise driven nonlinear stochastic flows. First we start with the widely used *small correlation time expansion.*

9.7.1 Small correlation time expansion

If the noise $\xi(t)$ is close to the white noise limit (i.e. colored noise with very small correlation time) it seems appropriate to expand the functional derivative around its Markovian value which it attains for δ-correlated noise. Thus, we expand $\delta x(t)/\delta\xi(s)$ into a Taylor series around the Markovian end point t, i.e.

$$\frac{\delta x(t)}{\delta \xi(s)} = \frac{\delta x(t)}{\delta \xi(t)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\left[\frac{d^n}{ds^n}\frac{\delta x(t)}{\delta \xi(s)}\right]_{s}(t-s)^n. \tag{9.7.1}$$

For the multiplicative stochastic flow in (9.5.2) we obtain from (9.5.5a–c)

$$\frac{\delta x(t)}{\delta \xi(s)} = \theta(t-s)\Bigg[g(x(t)) + \{g'(x(t))f(x(t)) - g(x(t))f'(x(t))\}(t-s)$$
$$+ \left\{f^2\left[f\left(\frac{g}{f}\right)'\right]' - g^2\left[g\left(\frac{f}{g}\right)'\right]'\xi(t)\right\}(t-s)^2 + \cdots \tag{9.7.2}$$

Note that this expansion involves already in second order again the noise $\xi(t)$. This leads to new correlations, which in turn must be expanded again into functional derivatives, and so on, yielding never-ending sums of series of series. Most commonly, one truncates the series at the first order. Using exponentially correlated Gaussian noise of vanishing mean, i.e.

$$C_2(t-s) = \frac{D}{\tau}\exp - \frac{|t-s|}{\tau}, \tag{9.7.3}$$

we find from (9.6.2a), by neglecting transients (i.e. we extend the time integration to infinity) the *small correlation time result*

$$\dot{p}_t(x;\tau) = -\frac{\partial}{\partial x}\{f(x)p_t(x;\tau)\}$$
$$+ D\frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}g(x)\left[1 + \tau g(x)\left(\frac{f}{g}\right)'\right]p_t(x;\tau). \tag{9.7.4}$$

This very result has been repeatedly derived in the literature by a variety of different, but equivalent methods (Dekker, 1982; Fox 1983; Horsthemke and Lefever, 1980; Kaneko, 1981; Lax, 1966; Lindenberg and West, 1983; Lugiato and Harowicz, 1985; Sancho and San Miguel, 1980; Sancho, San Miguel, Katz and Gunton, 1982; Schenzle and Tél, 1985; Stratonovich, 1963; Suzuki, 1980; Van Kampen, 1976; Zoller, Alber and Salvador, 1981). This same small correlation time approximation is also inherent within the large body of work performed on the problem of elimination of the velocity variable in nonlinear Brownian motion (for extended reviews on this body of work see Grigolini and Marchesoni, 1985; Marchesoni, 1985; Van Kampen, 1985). Unfortunately this approximation is of limited use (even for a linear flow) (Hänggi, Marchesoni and Grigolini, 1984); it does not converge uniformly in x, and the diffusion coefficient in (9.7.4) exhibits generally negative values for sufficiently large values of x (and/or τ), thereby introducing unphysical singularities (boundaries) into the problem. In other words (9.7.4) does not constitute a truly Markovian process with a well-defined corresponding Langevin equation driven by white Gaussian noise.

Some authors attempted to 'improve' the Fokker–Planck approximation in (9.7.4) by evaluating additional Fokker–Planck contributions proportional to $D\tau^n$ (Lindenberg and West, 1983; Sancho *et al.*, 1982; Van Kampen, 1976) while at the same time neglecting noise-dependent contributions in (9.7.2) which yield additional Fokker–Planck terms together with non-Fokker–Planck terms. This procedure, however, does not cure the short-comings already present in (9.7.4). As pointed out previously (Hänggi, Marchesoni and Grigolini, 1984; see also Marchesoni, 1988) such a formal ordering of the τ-expansion according to powers of D and τ is fictitious: $D\tau^n$ is not a systematic expansion coefficient; it is the action of the operator with coefficient $D\tau^n$, $n = 1, 2, \ldots$, acting on $p_t(x;\tau)$ which must be compared with the terms neglected (note that $p_t(x;\tau)$ itself depends on D and τ). It then turns out that the regime of validity of (9.7.4) is given by: $\tau/D \ll 1$, $\tau \ll 1$ (in dimensionless units). An approximation which overcomes these difficulties (at least within a certain class of stochastic flows) will be presented in the following.

9.7.2 *Unified colored noise approximation*

There is certainly a need (see the introduction to this chapter) to consider not only very short correlation times τ but also moderate-to-large noise correlations τ. With this viewpoint in mind, there have been some recent advances in the theory. The result in (9.7.4) becomes of course exact in the limit $\tau \to 0$. In recent work (Jung and Hänggi, 1987) *we have been guided to seek an approximation which becomes exact both for $\tau \to 0$ and $\tau \to \infty$. For intermediate τ-values such a scheme will then hopefully give a useful approximation.* In the following we restrict ourselves to additive noise only; a multiplicative noise source can, as mentioned previously, be transformed into additive noise via $x \to \tilde{x} = \int^x g^{-1}(y)\,dy$, $g(y) \neq 0$. Starting with (9.2.1), i.e.

$$\dot{x} = f(x) + \xi(t),$$

with $\xi(t)$ Gaussian, exponentially correlated noise (see (9.7.3)) we can recast the one-dimensional flow as a two-dimensional Markovian flow of the form

$$\left.\begin{aligned} \dot{x} &= f(x) + \xi \\ \dot{\xi} &= -\frac{1}{\tau}\xi + \frac{D^{1/2}}{\tau}Q(t), \end{aligned}\right\} \tag{9.7.5}$$

$\langle Q(t)Q(s)\rangle = 2\delta(t-s)$. Upon elimination of ξ, i.e.

$$\ddot{x} + \dot{x}\left(\frac{1}{\tau} - f'(x)\right) - \frac{1}{\tau}f(x) = \frac{D^{1/2}}{\tau}Q(t)$$

and the new time scale, $\hat{t} = \tau^{-1/2}t$, we find the nonlinearly damped random oscillator motion

$$\ddot{x} + \dot{x}(\tau^{-1/2} + \tau^{1/2}[-f'(x)]) - f(x) = D^{1/2}Q(\tau^{1/2}\hat{t}). \tag{9.7.6}$$

The damping $\gamma(x,\tau) = (\tau^{-1/2} + \tau^{1/2}[-f'(x)])$ is positive in regions of local stability and approaches infinity both for $\tau \to 0$ and $\tau \to \infty$. If in addition $\gamma(x,\tau)$ attains a large positive value in some x-region it is justifiable to eliminate the fast velocity variable $\dot{x}$ by setting $\ddot{x} = 0$. Setting the change of velocity equal to zero implies in turn that the change of the force field over the characteristic length, $l = D^{1/2}/\gamma(x,\tau)$, is small. Therefore, the Smoluchowski dynamics for this very same x-region becomes a *truly Markovian Fokker–Planck process*

$$\dot{x} = \frac{f(x)}{\gamma(x,\tau)} + \frac{D^{1/2}\tau^{-1/4}}{\gamma(x,\tau)}Q(\hat{t}), \tag{9.7.7a}$$

with $\langle Q(\hat{t})Q(\hat{s})\rangle = 2\delta(\hat{t}-\hat{s})$ being Gaussian δ-correlated (white) noise. Equation (9.7.7a) is valid on *time scales* $\hat{t} \gg \gamma^{-1}(x,\tau)$; i.e. $t \gg \tau/(1-\tau f')$ and in *space regions* x obeying $l|\bar{f}'| \ll |\bar{f}|$, i.e. $\gamma(x,\tau) \gg D^{1/2}(|\bar{f}'(x)/\bar{f}(x)|)$ (the bar indicates the characteristic value of the quantity f'/f within the length l).

The reduced dynamics in (9.7.7a) implies the (Stratonovich) Fokker–Planck dynamics on the time scale $\hat{t}$ (Jung and Hänggi, 1987)

$$\dot{p}_{\hat{t}}(x;\tau) = -\frac{\partial}{\partial x}\left(\gamma^{-1}(x,\tau)\left\{f(x) - D\tau^{-1/2}\frac{\partial}{\partial x}\gamma^{-1}(x,\tau)\right\}p_{\hat{t}}(x;\tau)\right), \tag{9.7.7b}$$

with stationary probability*

$$p_{st}(x;\tau) = N^{-1}|1 - \tau f'(x)|\exp\left[-\frac{1}{2D}\tau f^2(x)\right] \times \exp\left[\frac{1}{D}\int^x f(y)\,dy\right]. \tag{9.7.7c}$$

In contrast to the previous approximation in (9.7.4) the diffusion coefficient in (9.7.7b) is strictly positive. Most importantly, the dynamics in (9.7.7b) does not restrict the τ-value to very small correlation times only. In regions of local stability; i.e. $-f'(x) > 0$, the theory is exact for $\tau = 0$ and again becomes exact with τ approaching infinity. The theory in (9.7.7a, b) clearly supersedes the conventional small correlation time approach in Section 9.7.1: For sufficiently (very) small τ-values, i.e. τ so small that it becomes justified to replace factors like $(1 - \tau f'(x)) \to (1 + \tau f'(x))^{-1}$ (which are formally divergent) the solution (9.7.7c) approaches the solution of (9.7.4).

We conclude this section on the unified colored noise approximation by

* For multiplicative noise, $\dot{x} = f(x) + g(x)\xi(t)$, one finds $\gamma(x,\tau) = [\tau^{-1/2} - \tau^{1/2}(f' - (g'/g)f)]$, and

$$p_{st}(x;\tau) = Z^{-1}|[1 - \tau(f' - (g'/g)f)]/g| \times \exp\left\{\int^x dy\, f(y)[1 - \tau(f' - (g'/g)f)]/(Dg^2)\right\}.$$

pointing out that the stationary probability in (9.7.7c) *precisely coincides* with the stationary probability of a recently improved small correlation time theory due to Fox (Fox, 1986a, b); the Fokker–Planck dynamics of the two theories, however, *differ*! In the small correlation time theory of Fox the diffusion is still plagued by possible negative values; nevertheless the two Fokker–Planck approximations possess identical stationary probabilities. Also it should be remarked that the approximative dynamics in (9.7.7b) is close to the equilibrium dynamics (more precisely, it corresponds approximately to the stationary preparation class; for details see Grabert, Hänggi and Talkner, 1980). Thus we also expect that (9.7.7b) provides *a reasonable* approximation for quantities such as equilibrium correlation functions.

9.7.3 *Decoupling approximation*

Although the unified colored noise approximation of Section 9.7.2 works generally for small-to-moderate-to-large noise correlation times τ; it is, however, still restricted to a positive damping $\gamma(x, \tau)$. Unfortunately, this novel approximation with $f'(x) > 0$ does not cover all situations such as the case of multistability at moderate-to-strong-noise color; nor can it describe exponentially large (or small) asymptotic statistical quantities such as escape times or rates at weak noise intensities D. On inspecting the colored noise master equation in (9.6.2c) a different approximation scheme would simply involve a repeatedly applied decoupling between the functional $\delta(x(t) - x)$ and the residual functional stemming from the functional derivative. Such a decoupling is particularly suitable if the probabilities possess small widths. In general, this latter condition implies small noise intensities D – a reasonable condition present in most applications – but otherwise does not restrict the value of the noise correlation time τ. For exponentially correlated Gaussian noise (9.7.3) this quasi-stationary decoupling approximation, put forward originally by Hänggi (Hänggi *et al.*, 1985) thus reads

$$\dot{p}_t(x;\tau) = -\frac{\partial}{\partial x}\{f(x)p_t(x;\tau)\} + \left\{\frac{D}{1-\tau(\langle f'\rangle - \langle fg'/g\rangle)}\right\} \times \frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}g(x)p_t(x;\tau), \tag{9.7.8}$$

where the average ($\langle\cdots\rangle$) is over the stationary probability. Comparing this equation to the corresponding Fokker–Planck dynamics for white noise (i. e. $\tau = 0$), we observe that the influence of colored noise is contained in a renormalized diffusion coefficient. Due to the neglect of transients (i.e. $t \to \infty$ in (9.6.2c)), the diffusion coefficient can be evaluated self-consistently via the stationary probability. It should also be noted that the averages themselves depend on τ and D. In addition, the dynamical law which determines the

stationary probability guarantees that the diffusion stays positive.

The approximation in (9.7.8) can actually be improved systematically if we invoke the correlation formula between two functionals in (9.4.4): The approximation in (9.7.8) simply refers to the zeroth order approximation of the correlation in (9.6.2c); higher orders involve the cumulant(s) of the noise and yield, via the functional derivatives of $\delta(x(t) - x)$, higher order non-Fokker–Planck terms.

In conclusion we emphasize that the approximative schemes in Sections 9.7.2 and 9.7.3 do *not* rely on an expansion in the correlation time τ.

9.7.4 *A case study: bistability driven by colored Gaussian noise*

The archetype of a colored bistable flow is given by the overdamped motion in a double-well potential driven by correlated Gaussian noise of zero mean (9.7.3), i.e.

$$\dot{x} = x - x^3 + \xi(t). \tag{9.7.9}$$

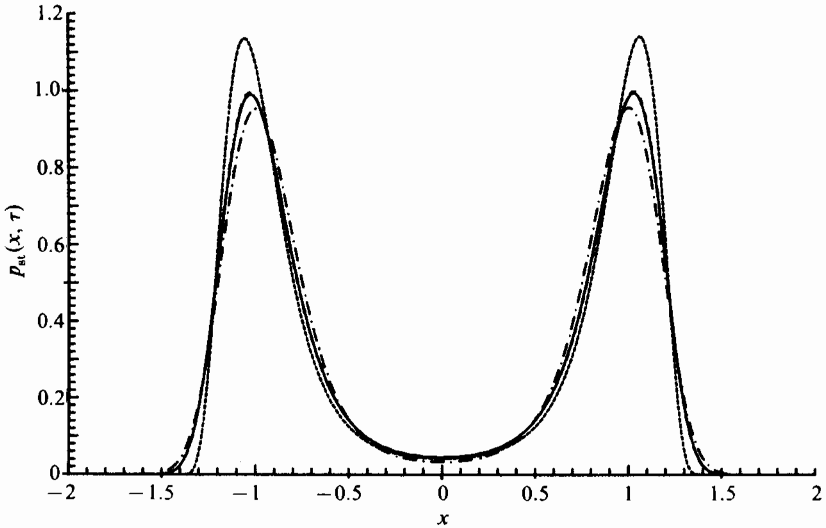
This example has been used as a basis for testing the quality of different colored noise approximation schemes (Hänggi, Marchesoni and Grigolini, 1984). This latter work has started a flurry of activity around this specific colored noise driven dynamics (Fox, 1986a, b, 1988; Fronzoni *et al.*, 1986; Hänggi *et al.*, 1985; Jung and Hänggi, 1988; Jung and Risken, 1985; Leiber, Marchesoni and Risken, 1987; Malchow and Schimansky–Geier, 1985; Marchesoni, 1987; Masoliver, West and Lindenberg, 1987; Moss and McClintock, 1985; Moss, Hänggi, Mannella and McClintock, 1986; Sancho, Sagués and San Miguel, 1986).

In Figure 9.2 we compare results of different approximation schemes (Jung and Hänggi, 1988) for the stationary probability $p_{st}(x;\tau)$ for a noise intensity $D = 0.1$ and various τ-values. The small correlation time approximation (Section 9.7.1) starts to break down at τ-values $\tau \geqslant 0.2$. Actually this small correlation time approximation *loses* its bistable character (i.e. two maxima and one minimum) for

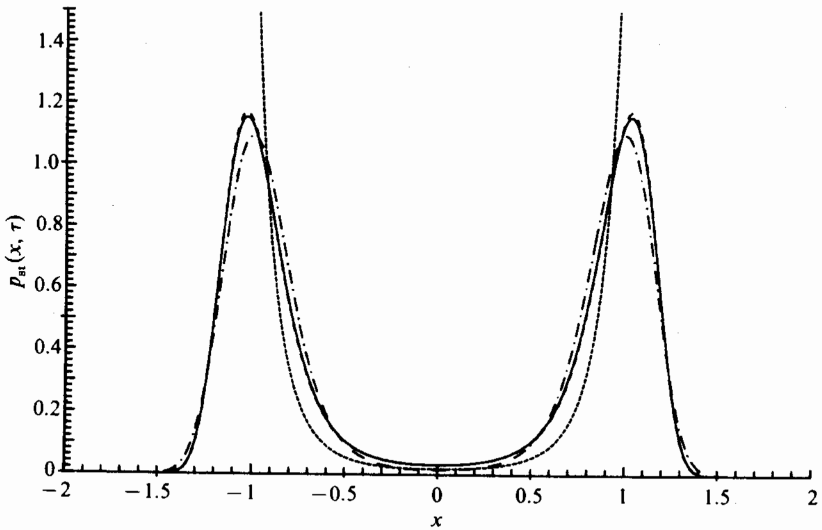
$$p_{st}(x;\tau) \quad \text{at} \quad \tau \geqslant \{-1 + (1 + 18D)^{1/2}\}/(18D).$$

For $D = 0.1$ this occurs at $\tau \geqslant 0.3740\ldots$. Notwithstanding claims to the contrary (Masoliver, West and Lindenberg, 1987), this theory can thus not describe mean sojourn times, or escape rates at weak-to-moderate noise color $0.3 \leqslant \tau \leqslant 1.5$.

The unified colored noise approximation with $\gamma(x = 0, \tau) \leqslant 0$ for $\tau \geqslant 1$ cannot describe the instable behavior around $x \approx 0$. This behavior is in contrast to the decoupling theory, which works qualitatively for all τ and increases in accuracy for probabilities of small widths, i.e. for very small D and/or large τ. Unfortunately, the continued fraction method used to evaluate $p_{st}(x;\tau)$ numerically (Jung and Risken, 1985) encounters convergence pro-

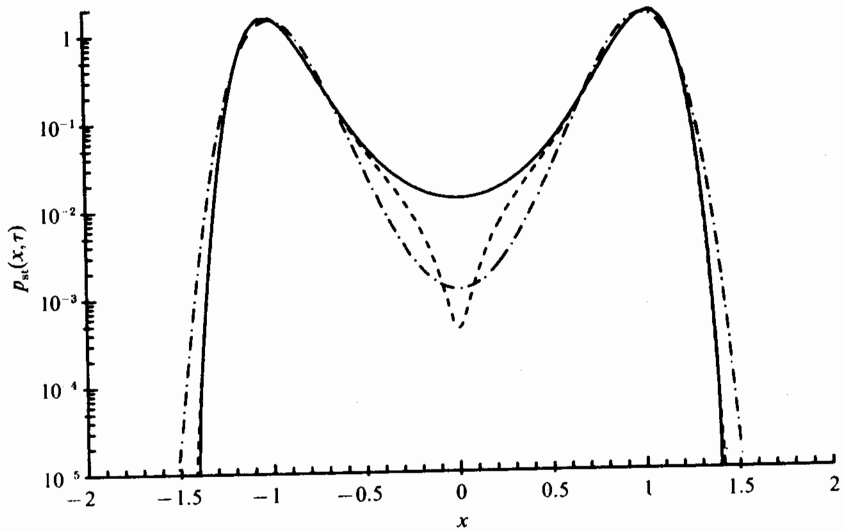


(a)

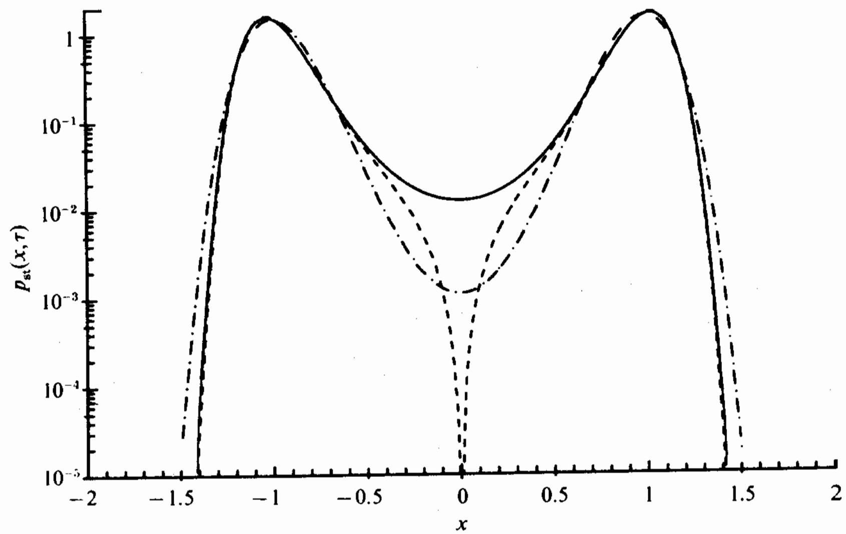


(b)

blems for small D-values, $D < 0.1$ (and also for very large τ-values). From this view point the numerical method and the decoupling scheme by Hänggi (Section 9.7.3) – which works best for $D \ll 1$ – complement each other. In the asymptotic regime of very small noise intensity $D < 0.05$, the ratio $R(\tau)$ of maximal over minimal stationary probability is from the decoupling scheme



(c)



(d)

Figure 9.2. The stationary probability for the bistable colored noise dynamics in (9.7.9). (a) $D = 0.1$, $\tau = 0.2$; (b) $D = 0.1$, $\tau = 0.4$; (c) $D = 0.1$, $\tau = 0.99$; (d) $D = 0.1$, $\tau = 1$. In all (a)–(d): ——— numerical continued fraction evaluation of $p_{st}(x;\tau)$ (Jung and Hänggi, 1988; - - - - unified approximation in (9.7.7c); -·-·- decoupling approximation in (9.7.8);.... small correlation time approximation in (9.7.4); there we depict $p_{st}(x;\tau)$ only for $\tau = 0.2$ where it has bistable character, and at $\tau = 0.4$ where $p_{st}(x;\tau)$ is already no longer bistable; however it can be normalized for $\tau < 0.5$. Note that the unified approximation (9.7.7c) in (a) and (b) practically coincides with the numerical solution (solid line).

(9.7.8) with $\langle x^2 \rangle \approx 1$ given by

$$R(\tau) = \exp\left(\frac{1}{4D}(1+2\tau)\right). \tag{9.7.10}$$

This characteristic asymptotic dependence, i.e. $R(\tau) \propto \exp(\text{const}\cdot\tau/D)$, has independently been verified within a Kramers rate approach for weak noise (Marchesoni, 1987), and also via explicit numerical calculations in periodic, multistable potentials (Leiber, Marchesoni and Risken, 1987).

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Note added in proof

Recent precise calculations on the problem of colored noise driven bistability (Jung and Hänggi, *Phys. Rev. Lett.* **61**, 11 (1988)), see section (9.7.4), show that the escape rate $\Gamma(\tau)$ undergoes two different crossover behaviors as τ increases from $\tau = 0$ to $\tau = \infty$. At very small $\tau \ll 1$, $\tau/D \ll 1$ and weak noise $D \ll 1$, one has $\Gamma(\tau) = (2\pi^2)^{-1/2}(1 - 1.5\tau)\exp - (1/4D)$. This result is followed by a crossover to a behavior of the form $\Gamma(\tau) \propto \exp - (\alpha\tau/D)$, $\alpha \cong 0.1$ at small-to-moderate noise correlation time τ, being in qualitative agreement with the decoupling theory (see (9.7.10)). At even larger noise color $\tau \gg 1$, there occurs yet another very slow crossover to a limiting law $\Gamma(\tau) = (54\pi D(\tau + \frac{1}{2}))^{-1/2}\exp - \{(1/4D)[\frac{4}{9} + \frac{8}{27}\tau]\}$, as $\tau \to \infty$ (Hänggi, Jung and Marchesoni, 'Escape Driven by Strongly Correlated Noise', preprint, 1988).

Appendix. On the statistical treatment of dynamical systems*

L. PONTRYAGIN, A. ANDRONOV, and A. VITT

1 Formulation of the problem

Suppose we have a dynamical system determined by n differential equations of first order†

$$\frac{dx_i}{dt} = X^{(i)}(x_1, x_2, \ldots, x_n); \quad i = 1, 2, \ldots, n. \tag{I}$$

For given initial conditions, these equations uniquely determine the behavior of the point that 'represents' our system in the phase space.‡ We assume that our system, which satisfies equation (I), is subject to 'impulses' or 'perturbations', which act in accordance with laws of chance (different probability hypotheses are possible here).

These 'random' impulses are considered for two reasons connected with the two problems that we pose in this paper.

First problem

There is no doubt that the processes in real dynamical systems are not completely represented by differential equations of the form (I); these equations determine the motion of the system only in the basic features, i.e. only approximately, and do not take into account random impulses and perturbations. Under favorable conditions, an experiment can reveal certain consequences of the existence of such random impulses. This leads to the following problem: *To establish the general behavior of a system in the presence of random impulses and, in particular, to give a theoretical construction that makes it possible to elucidate from experimental data the nature of the*

* Translated by Julian B. Barbour from Pontryagin, L., Andronov, A. and Vitt, A. (1933). *Zh. Eksp. Teor. Fiz.* **3**, pp. 165–80.

† We restrict the treatment to autonomous systems, i.e. systems for which the differential equations do not depend explicitly on time. An analogous treatment can also be given for nonautonomous systems.

‡ Translator's note: 'Phase space' has been translated literally; however, except in the final paragraph of the paper, it appears to mean configuration space.