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Locking equation with colored noise: Continued-fraction solution versus decoupling theory

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The locking equation in the presence of colored noise is studied. This system models, for example, the mean beat frequency $\langle\dot{\phi}\rangle$ of a ring-laser gyroscope in which weak noise with (dimensionless) noise correlation times $\tau\approx 10^{-2}$–10^{2} is used experimentally to overcome the locking. The non-Markovian, colored-noise dynamics is solved by use of a matrix-continued-fraction technique. The thusly calculated stationary probability and the mean beat frequency are compared to the decoupling theory introduced recently by Hänggi and co-workers [Phys. Rev. A **32**, 695 (1985); **33**, 4459 (1986)], as well as to the conventional small-noise-correlation-time approximation. The decoupling approximation, resulting in a Fokker-Planck equation with an effective diffusion which must be evaluated self-consistently, yields satisfactory agreement over the whole regime of physically relevant correlation times τ. The small-correlation-time approximation, however, breaks down for moderate-to-large τ. The mean beat frequency $\langle\dot{\phi}\rangle$ decreases at constant noise intensity with increasing noise color; i.e., increasing the noise correlation time τ increases the tendency to lock.

The effect of frequency locking is a ubiquitous phenomenon in nonlinear mechanics, optics, and electronics. This effect was apparently recognized long ago. Thus, for example, Van der Pol,[1] who started to develop the theory, remarks that "the synchronous timekeeping of two clocks hung on the same wall was already known to Huygens." Let us consider a nonlinear oscillator with a positive feedback, oscillating at some frequency ω_0, set by the type of nonlinearity of the amplifier. This free-running frequency can be shifted if we introduce an external signal of frequency ω_e. The detuning will be denoted by $a=\omega_0-\omega_e$. Then the time variation of the phase difference ϕ between the actual phase η and the synchronization phase η_e, i.e., $\phi=\eta-\eta_e$, can be modeled, as shown by Adler,[2] by the so-called "locking equation"

$$\dot{\phi}=a+b\sin\phi \ . \tag{1}$$

Here, the parameter b characterizes the effectiveness of the synchronization signal. Moreover, it is undesirable, if not impossible, to completely isolate the physical system from its surroundings. The influence of the environment is represented by an additive noise term $\varepsilon(t)$, whose strength of correlation D in the following will be assumed to be independent of the system variable $\phi(t)$. In particular, we assume for $\varepsilon(t)$ Gaussian statistics with vanishing mean and finite correlation time (colored noise)

$$\langle\varepsilon(t)\varepsilon(s)\rangle=(D/\tau)\exp(-|t-s|/\tau) \ . \tag{2}$$

For the idealized situation of a white-noise source (i.e., δ-correlated random forces; $\tau\to 0$) Eqs. (1) and (2) have been studied by Stratonovich,[3] and have since been applied in a multitude of physical systems.[4] For example, this model of a Brownian motion in tilted sinusoidal potential (sometimes with an additional second-order inertia term) has attracted attention in solid-state physics when describing superionic conductors,[5] overdamped soliton transport,[6] or the dynamics in Josephson junctions,[7] or also in chemical physics for the description of the rotation of dipoles in an external field.[8]

A particularly interesting application of (1) relates to ring-laser gyroscopes.[9] In this latter case, the stochastic variable $\phi(t)$ denotes the phase difference between two counterpropagating electromagnetic waves in a ring resonator. A rotation of the resonator around an axis perpendicular to the plane of the gyroscope causes between the two waves a Sagnac frequency shift a proportional to the rotation rate. Furthermore, the backscattering from the mirrors of the resonator couples the counterpropagating waves in a nonlinear fashion as described in (1), wherein b denotes the backscattering coefficient. The physics of the optical gyroscope is as follows: For small rotation rates $|a|\leq|b|$, the two waves lock,[9] yielding a vanishing mean beat frequency $\dot{\phi}$. The existence of this dead band constitutes a problem for the functioning of the gyroscope. Thus, in order to avoid locking one deliberately introduces external noise, resulting in a nonvanishing mean beat frequency. The case of white noise arising from spontaneous emission of the laser atoms has been studied in Refs. 10 and 11. Experimentally[12] the noise is introduced into the system by cementing one of the resonator mirrors on a piezoceramic element which is driven by a noise generator. Necessarily the correlation time τ of such imposed noise is nonzero. As a result of the noise, the area A of the ring, as well as its perimeter P, start to fluctuate. Because the Sagnac frequency a is proportional to A, and inversely proportional to P, one obtains for (1) an additive correlation noise source $\varepsilon(t)$ characterized by (2). Typical mirror displacements[12] are of the order of 0.1 μm and the bandwidth of the noise ranges from a few Hz to kHz. In units of the backscattering coefficient b, typically of a few hundred hertz, this translates into dimensionless correlation times $b\tau$ that range between 10^{-2} and 10^2, and noise correlation strengths D/b (corresponding to fluctuations of the mirror displacement) that range between 10^{-3} and unity. Therefore, (1) supplemented by the additive colored noise (2) defines a non-Markovian dynamics for $\phi(t)$. Thus the physics of the optical gyroscope is ruled by small noise intensities D and moderate-to-large correlation times τ. Formally, the single-event probability $P=P(t,\phi)$ of the non-Markovian process $\phi(t)$ obeys the exact equation[13]

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial\phi}[(a+b\sin\phi)P]+\frac{D}{\tau}\frac{\partial^2}{\partial\phi^2}\times\int_0^t ds\exp(-|t-s|/\tau)\left\langle\delta(\phi(t)-\phi)\frac{\delta\phi(t)}{\delta\varepsilon(s)}\right\rangle , \tag{3a}$$

wherein the functional derivative $\delta\phi(t)/\delta\varepsilon(s)$ is given by [$H(t)$ denotes the Heaviside step function]

$$\frac{\delta\phi(t)}{\delta\varepsilon(s)}=H(t-s)\exp\left[\int_s^t dr\left(\frac{\partial\dot\phi(r)}{\partial\phi}\right)\right] . \tag{3b}$$

Alternatively, if we represent the noise by the stochastic differential equation for an Ornstein-Uhlenbeck process, i.e., $\dot\varepsilon=-\varepsilon/\tau+\eta(t)$ with

$$\langle\eta(t)\eta(s)\rangle=(2D/\tau^2)\delta(t-s) ,$$

we can recast[4,14] the non-Markovian dynamics (3) as the two-dimensional Fokker-Planck process $P=P(t,\phi,\varepsilon)$,

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial\phi}[(a+b\sin\phi+\varepsilon)P]+\tau^{-1}\frac{\partial}{\partial\varepsilon}(\varepsilon P)+\frac{D}{\tau^2}\frac{\partial^2 P}{\partial\varepsilon^2} . \tag{4}$$

However, neither (3) nor (4) can be solved analytically in a closed form; in particular, detailed balance does not hold for the Markovian dynamics in (4). Only the matrix-continued-fraction (MCF) method[4] provides a formally exact solution of Eq. (4).[14] However, the solutions can only be evaluated on a computer. Suitable approximate treatments of colored noise are therefore of great interest. The usual approximation scheme, widely used in the previous literature,[15] uses an expansion around the Markovian limit (zero correlation time), with $b\tau$ being a small parameter. In our case, this yields[14] a Fokker-Planck equation for $P=P(t,\phi)$ similar to multiplicative-noise process

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial\phi}[(a+b\sin\phi)P]+D\frac{\partial^2}{\partial\phi^2}[(1+b\tau\cos\phi)P] . \tag{5}$$

A different approximation scheme, which is tailored to weak noise, but otherwise does not restrict the magnitude of the noise correlation time, decouples[16-18] the correlation in (3a) from the average of the functional derivative (3b). With (3b), we then obtain the decoupling approximation

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial\phi}[(a+b\sin\phi)P]+\frac{D}{(1-b\tau\langle\cos\phi\rangle)}\frac{\partial^2 P}{\partial\phi^2} . \tag{6}$$

Comparing this equation to the corresponding Fokker-Planck equation for white noise [i.e., Eq. (6) for $\tau=0$], we note that the influence of colored noise is contained in a renormalized diffusion coefficient involving the average $\langle\cos\phi\rangle$ which must be determined self-consistently via the corresponding stationary probability $P_{SS}(\phi)$. Making use of the white-noise results[3,4] we find

$$P_{SS}(\phi)=[4\pi^2 I_{id_1}(d_2)I_{-id_1}(d_2)\exp(-\pi d_1)]^{-1}\exp[d_1\phi+d_2\cos\phi]\int_\phi^{\phi+2\pi}\exp[-(d_1\psi+d_2\cos\psi)]d\psi , \tag{7a}$$

where I_{ix} denotes the Bessel function of imaginary order and argument.[19] In contrast to Refs. 3 and 4, however, the coefficients $d_{1,2}$ now depend on the noise correlation time τ as well as on the average $\langle\cos\phi\rangle$, i.e.,

$$d_1=\frac{a}{D}(1-b\tau\langle\cos\phi\rangle),\quad d_2=-\frac{b}{D}(1-b\tau\langle\cos\phi\rangle) . \tag{7b}$$

Therefore, Eq. (7a) is an implicit expression. Similarly, replacing the coefficients d_1 and d_2 in Ref. 3 by Eq. (7b) the mean beat frequency $\langle\dot\phi\rangle$ within this decoupling approximation (6) reads

$$\langle\dot\phi\rangle\equiv\langle\omega\rangle-\omega_e=a\frac{\sinh(\pi d_1)}{\pi d_1|I_{id_1}(d_2)|^2} . \tag{8}$$

In the remainder of the present article we compare the two different approximation schemes (5) and (6) with the MCF approach[4,20] applied to Eq. (4). We first expand the probability $P(t,\phi,\varepsilon)$ into a series

$$P(t,\phi,\varepsilon)=\frac{1}{\sqrt{2\pi}}\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\mathcal{S}_n^m(t)\exp(in\phi)\mathcal{H}_0(\varepsilon)\mathcal{H}_m(\varepsilon) , \tag{9}$$

where $\mathcal{H}_m(\varepsilon)$ are the (orthonormalized) Hermite functions.[4] Substituting (9) into (4) yields for the coefficients $\mathcal{S}_n^m(t)$ a three-term vector recurrence relation which can be solved[4,14] by the MCF method. The stationary probability $P_{SS}=P_{SS}(\phi)$ and the physically important mean beat frequency $\langle\dot\phi\rangle=a+b\langle\sin\phi\rangle$ are then given[14] in terms of the stationary values $\{\hat{\mathcal{S}}_n^0\}=\lim_{t\to\infty}\{\mathcal{S}_n^0(t)\}$ as

$$P_{SS}(\phi)=\int_{-\infty}^{\infty}\lim_{t\to\infty}P(t,\phi,\varepsilon)d\varepsilon=\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\hat{\mathcal{S}}_n^0\exp(in\phi) , \tag{10a}$$

and

$$\langle\dot\phi\rangle=a-\sqrt{2\pi}b\,\mathrm{Im}\hat{\mathcal{S}}_1^0 . \tag{10b}$$

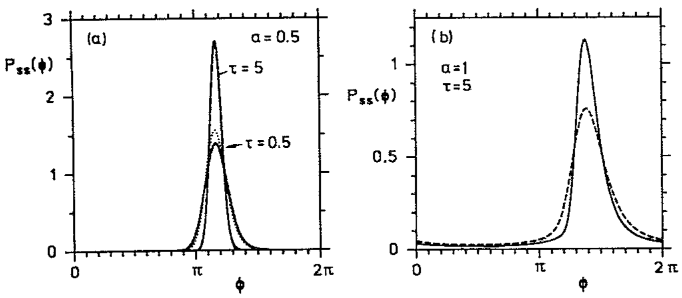

FIG. 1. Exact stationary probability $P_{SS}=P_{SS}(\phi)$ [Eq. (10a)] (solid line) compared with the decoupling approximation, (6) and (7) (dashed line), for $b=1$ and weak noise $D=0.1$. (a) Stationary probability in the locking regime ($a=0.5$) for two noise-correlation times $\tau=0.5$ and $\tau=5$. The small-noise-correlation-time approximation is shown for $\tau=0.5$ by a dotted line. (b) Stationary probability at the threshold to the unlocked regime ($a=1$) for $\tau=5$.

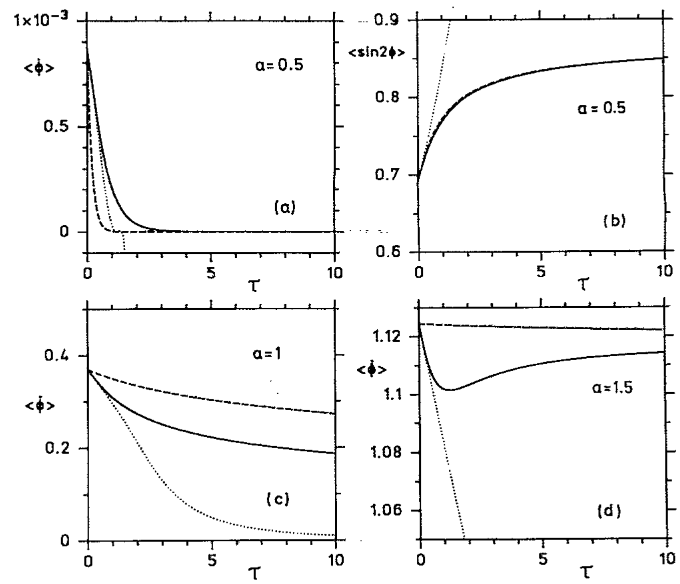

FIG. 2. Mean beat frequency $\langle\dot\phi\rangle=a+b\langle\sin\phi\rangle$ and mean value $\langle\sin 2\phi\rangle$ for $b=1$ and weak noise $D=0.1$ as a function of the noise correlation time τ. Solid lines represent exact results evaluated via the MCF method, dashed lines are based on the decoupling scheme (6) and (7), and doted curves correspond to the small-noise-correlation-time approximation (5). (a) Mean beat frequency, $\langle\dot\phi\rangle$ in the locked regime $a=0.5$. (b) Mean value $\langle\sin 2\phi\rangle$ for $a=0.5$. (c) Mean beat frequency $\langle\dot\phi\rangle$ at threshold $a=1$, and (d) in the unlocked regime ($a=1.5$).

For the explicit form of the MCF's determining the coefficients $\mathcal{S}_n^0$ we refer to Ref. 14.

In the remainder of the present article we have expressed all quantities in units of b and set $b=1$. Figure 1(a) depicts the normalized stationary probability $P_{SS}=P_{SS}(\phi)$ in the locked regime, i.e., for $|a|=0.5$, at weak noise $D=0.1$ for two different noise correlation times $\tau=0.5$ and $\tau=5$. Within this locked regime the exact probability (solid line) and the decoupling approximation (6) (dashed curve) are practically indistinguishable. The small-correlation-time approximation (5) which exists only for sufficiently small τ, however, exhibits an overshoot for the maximal probability. In Fig. 1(b) we show $P_{SS}=P_{SS}(\phi)$ for $\tau=5$ at the threshold to the (deterministically) unlocked regime, i.e., $a=1$. The stationary probability now exhibits a typical asymmetry. Again, the decoupling approximation shows the same asymmetry. The width, however, is slightly larger, which explains the undershoot for the maximal value. In Fig. 2 we display colored-noise results for the noise-modified mean beat frequency $\langle\dot\phi\rangle$ and the moment $\langle\sin(2\phi)\rangle$. The decoupling approximation yields qualitatively reliable results over the whole regime of small-to-moderate-to-large noise correlation times. For a small noise correlation time, the conventional approximation yields good results, starting with the exact slope. The approximation starts to fail, as must be expected from the nature of its construction, for moderate-to-large τ values. The better agreement for $\langle\dot\phi\rangle$ at small $\tau<0.5$ is accidental because the decoupling approximation for the stationary probability P_{SS} actually exceeds in quality that of the small-correlation-time approximation [see Fig. 1(a)]. Indeed, other transport quantities such as $\langle\sin(2\phi)\rangle$ or $\langle\cos\phi\rangle$, etc., no longer are produced with such good accuracy [see Fig. 2(b)]. In Fig. 2(b) the Hänggi ansatz closely fits the exact result over the whole τ regime. Thus, there occurs within the conventional scheme (5) a fortuitous cancellation of errors for the evaluation of $\langle\dot\phi\rangle$ at small noise correlation times $\tau\leqq 0.5$, which does not show up for other transport coefficients.

In conclusion, we have studied the role of noise color in the locking equation. For ring-laser gyroscopes colored noise of a relatively long correlation time plays an important role. As depicted in Fig. 2, noise color at a fixed noise intensity D tends to reduce the mean beat frequency $\langle\dot\phi\rangle$; i.e., the tendency "to lock" increases unavoidably with increasing noise correlation time τ. This effect can be readily understood by referring to the decoupling approximation (6). Note that for $\langle\cos\phi\rangle<0$ an increase in τ results in a decrease for the effective noise strength. The role of noise color on other quantities of interest in the gyro, such as, e.g., the variance of the mean beat frequency can, of course, be studied analogously.

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