Absolute negative mobility of inertial Brownian particles induced by noise

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***Abstract*—We study transport of a harmonically driven inertial particle moving in a symmetric periodic potential and subjected to both unbiased Gaussian thermal equilibrium noise and biased non-equilibrium Poissonian shot noise. The dependence of the average velocity on noise parameters exhibits a rich variety of anomalous transport characteristics: We identify an absolute negative mobility around zero biasing noise, the emergence of a negative differential mobility and the occurrence of a negative nonlinear mobility. As a feasible physical system we propound a setup consisting of a single resistively and capacitively shunted Josephson junction device.**

I. Introduction

Absolute negative mobility (ANM) is a counterintuitive phenomenon: particles move in a direction opposite to a static bias force. It seems to be in contradiction to the Newton equation of motion, the second law of thermodynamics and observations of motion at a macroscopic scale. However, under non-equilibrium conditions, there is no fundamental principle which excludes ANM. What are essential ingredients for the occurrence of ANM? The minimal model is formulated in terms of a one-dimensional Newton equation for a Brownian classical particle moving in a symmetric spatially periodic potential, driven by an unbiased harmonic force $A\cos(\Omega t)$ and biased by a static force F [1]. The ANM response in a symmetric periodic potential is so that an average particle velocity $\langle v(F)\rangle$ obeys the relation: $\langle v(F)\rangle = -\langle v(-F)\rangle$, which follows from the symmetry arguments. In particular, $\langle v(0)\rangle = 0$. So, for $F = 0$ there is no directed transport in the long-time regime (in a stationary state). The static force $F \neq 0$ breaks the symmetry and therefore induces a directed motion of particles. In the paper, we replace the static force F by a random force $\eta(t)$ of a time-independent non-zero mean value $\langle \eta(t)\rangle = \eta_0$. As an example of the random force $\eta(t)$, we consider non-equilibrium Poissonian shot noise, which is composed of a random sequence of δ-shaped pulses with random amplitudes. We assume that the particle is coupled to its environment (thermostat) of temperature T and thermal fluctuations $\xi(t)$ are included as well. We study the dependence of the long-time average velocity $\langle v\rangle = \eta_0$ on parameters of both random forces $\eta(t)$ and $\xi(t)$ and find a rich variety of anomalous transport regimes including the absolute negative mobility regime.

The layout of the present work is as follows. In Sec. II we present details of the model of a driven inertial Brownian particle exposed to biased white Poissonian shot noise. In next section we elaborate on various anomalous transport processes occurring in this system. Sec. IV provides conclusions.

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II. The model

We consider a Brownian particle of mass M moving in a one-dimensional spatially periodic potential $V(x) = V(x + L) = \Delta V \sin(2\pi x/L)$ and driven by both an unbiased time periodic force $F(t) = F(t + T) = A\cos(\Omega t)$ and a biased stochastic force $\eta(t)$. The dynamics of the particle is modelled by the Langevin equation [2]:

$$M\ddot{x} + \Gamma\dot{x} = -V'(x) + F(t) + \eta(t) + \sqrt{2\Gamma k_B T}\,\xi(t). \quad (1)$$

Here, the dot and prime denote differentiation with respect to time t and the Brownian particle's coordinate x, respectively. Thermal equilibrium fluctuations due to the coupling of the particle with the environment are modelled by Gaussian white noise $\xi(t)$ of zero mean $\langle\xi(t)\rangle = 0$ and the correlation function $\langle\xi(t)\xi(s)\rangle = \delta(t - s)$. The parameter Γ is the friction coefficient, k_B is the Boltzmann constant and T is temperature of the environment. The biased stochastic force $\eta(t)$ is chosen in the form of a random sequence of δ-shaped pulses with random amplitudes defined in terms of generalized white Poissonian shot noise [3]

$$\eta(t) = \sum_{i=1}^{n(t)} z_i \delta(t - t_i) \quad (2)$$

where t_i are random instants of δ-shaped pulses characterized by the Poissonian counting process $n(t)$ with the parameter λ, i.e. the number $n(t)$ of pulses in the time interval $[0, t]$ follows the Poisson distribution

$$\Pr\{n(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}. \quad (3)$$

The parameter λ determines the mean number of pulses per unit time. The amplitudes z_i of kicks are random variables independent of each other and of counting process $n(t)$. They are assumed to be exponentially distributed,

$$\rho(z) = \zeta^{-1}\theta(z)e^{-z/\zeta}, \quad \zeta > 0. \quad (4)$$

where $\theta(z)$ is the Heaviside step function. Therefore all amplitudes z_i are *positive* and all realizations of Poissonian noise are non-negative, i.e. the process $\eta(t) \geq 0$. It models non-equilibrium drift noise of a positive mean value and a covariance function in the form

$$\langle \eta(t) \rangle = \eta_0 = \lambda \langle z_i \rangle = \sqrt{D_P \lambda} \ > 0, \tag{5a}$$

$$\langle \eta(t)\eta(s) \rangle - \langle \eta(t) \rangle \langle \eta(s) \rangle = 2D_P \delta(t-s), \tag{5b}$$

where $D_P = \lambda \langle z_i^2 \rangle /2$ is the shot noise intensity and the statistical moments of amplitudes z_i are $\langle z_i^k \rangle = k! \zeta^k$ for $k = 1, 2, \ldots$. Moreover, we assume that Poissonian noise $\eta(t)$ is uncorrelated with Gaussian thermal noise $\xi(t)$, i.e. $\langle \eta(t)\xi(s) \rangle = \langle \eta(t) \rangle \langle \xi(s) \rangle = 0$.

Now, we transform eq. (1) to the dimensionless form. We propose to use the period L of the spatial potential $V(x)$ and time $\tau = L\sqrt{M/\Delta V}$ as length and time scales, respectively [4], [5]. Then the dimensionless form of eq. (1) reads

$$\ddot{\hat{x}} + \gamma \dot{\hat{x}} = -\hat{V}'(\hat{x}) + \hat{F}(\hat{t}) + \hat{\eta}(\hat{t}) + \hat{\xi}(\hat{t}), \tag{6}$$

where $\hat{x} = x/L$, $\hat{t} = t/\tau$ and $\gamma = \tau\Gamma/M$. The rescaled potential $\hat{V}(\hat{x}) = V(L\hat{x})/\Delta V = \sin(2\pi\hat{x})$ has the unit period. The dimensionless time periodic force $\hat{F}(\hat{t}) = a\cos(\omega\hat{t})$ possesses the amplitude $a = LA/\Delta V$ and the angular frequency $\omega = \tau\Omega$. The rescaled Poissonian shot noise $\hat{\eta}(\hat{t})$ of mean value $\langle \hat{\eta}(\hat{t}) \rangle = \hat{\lambda}\langle \hat{z}_i \rangle$ is δ-correlated: $\langle \hat{\eta}(\hat{t})\hat{\eta}(\hat{s}) \rangle - \langle \hat{\eta}(\hat{t}) \rangle \langle \hat{\eta}(\hat{s}) \rangle = 2\hat{D}_P\delta(\hat{t}-\hat{s})$, where $\hat{\lambda} = \tau\lambda$, $\hat{z}_i = z_i/\sqrt{M\Delta V}$ and $\hat{D}_P = \hat{\lambda}\langle \hat{z}_i^2 \rangle/2$. Likewise, the dimensionless thermal noise $\hat{\xi}(\hat{t})$ of zero mean has the correlation function $\langle \hat{\xi}(\hat{t})\hat{\xi}(\hat{s}) \rangle = 2\gamma\hat{D}_G\delta(\hat{t}-\hat{s})$, where $\hat{D}_G = k_BT/\Delta V$. From now on we will use only dimensionless variables and therefore omit hat above all quantities appearing in eq. (6).

Physical systems described by eq. (6) are widespread and well known. A significant example is semi-classical dynamics of a resistively and capacitively shunted Josephson junction which is driven by both a time periodic current and a noisy current [6]. In this case the phase difference between the macroscopic wave functions of the Cooper electrons in both sides of the junction translates to the Brownian particle coordinate and voltage across the junction translates to the particle velocity. Other specific systems include rotating dipoles in external fields [7], superionic conductors [8]–[10] or charge density waves [11], to name just a few.

III. Anomalous transport behaviour

The most important transport characteristic for the studied system is average velocity $\langle v \rangle$. We are particularly interested in the asymptotic long time regime, when it adopts the periodicity of the driving [12]–[14], namely

$$\langle v \rangle = \lim_{t\to\infty} \frac{\Omega}{2\pi} \int_t^{t+2\pi/\Omega} \prec v(s) \succ \ ds, \tag{7}$$

where $\prec v(s) \succ$ denotes the average over noise realizations and initial conditions.

Depending on the behaviour of the asymptotic long time average velocity $\langle v \rangle$ as a function of the mean bias η_0 in eq. (5a) one can classify transport of the particle as either normal or anomalous. For this purpose it is useful to generalize the standard definition of a mobility coefficient μ of the Brownian particle [1] in the linear response regime to the case of the biased stochastic force $\eta(t)$, namely

$$\langle v \rangle = \mu\eta_0, \quad \eta_0 = \langle \eta(t) \rangle. \tag{8}$$

Usually the average velocity should be an increasing function of the bias η_0, similarly as in the case of the deterministic static force F. In such regimes the mobility coefficient μ is positive. This situation corresponds to normal, Ohmic-like behaviour. However, more interesting are those regimes where mobility assumes negative values, exhibiting anomalous transport in the form of: (i) an absolute negative mobility [1] (near $\eta_0 = 0$), (ii) a negative differential mobility [15] or (iii) a negative nonlinear mobility [16]–[18] (away from $\eta_0 = 0$).

Before we turn to detailed analysis of the anomalous transport processes let us make some general comments about the dynamics corresponding to eq. (6). In the deterministic case ($D_G = D_P = 0$) it embodies a three-dimensional phase space, i.e. $\{x, y = \dot{x}, z = \omega t\}$ with three parameters $\{\gamma, a, \omega\}$. Depending on parameter values periodic, quasiperiodic and chaotic motion can be observed [19]. In some regimes ergodicity may be broken and the direction of transport depends strongly on the choice for the initial conditions. This leads to the conclusion that different attractors may coexist. At nonzero temperature, thermal fluctuations enable diffusive dynamics for which stochastic escape events among possibly coexisting attractors are probable. In particular, transitions between neighbouring locked states can lead to diffusive directed transport when the random force $\eta(t)$ is applied.

A. *Normal transport regime*

Eq. (6) cannot be studied by use of known analytical methods. In particular, the Fokker-Planck-Kolmogorov-Feller master equation [3] corresponding to the Langevin equation (6) cannot be solved analytically. For this reason we have carried out comprehensive numerical simulations of the Langevin dynamics. Full information about the employed numerical scheme can be found in [20]. We refer the reader to [2] for further details of our numerical simulations. Here, we want to point out that all calculations have been done by use of CUDA environment implemented on a modern desktop GPU. This gave us possibility to speed the numerical calculations up to few hundreds of times more than on typical present-day CPU [21].

We begin our analysis of the transport processes of inertial Brownian particles described by eq. (6) with brief comment on the influence of the white Poissonian shot noise parameters λ and D_P on the characteristics of the stochastic realizations. The reader can find detailed discussion on this topic in Ref. [2]. Here, we only mention two extreme regimes. The first limiting case is when both λ and D_P are large. Then the Brownian particle is very *frequently* kicked by *large* δ-pulses. On the contrary, when both λ and D_P are small, then the particle is very *rarely* kicked by *weak* δ-pulses.

Presence of the white Poissonian shot noise allows for symmetry breaking and consequently the emergence of transport phenomenon. We recall that due to the choice of the amplitude distribution of δ-pulses (cf. eq. (4)) only positive kicks are allowed. Therefore we expect the average velocity to be positive as well. Any deviation would be counterintuitive. In particular, we expect that ensemble and time averaged velocity $\langle v \rangle$ will be an increasing function of the mean value η_0. Such a normal transport regime is depicted in the inset of Fig. 1(a). The corresponding shot noise realizations correspond to rare

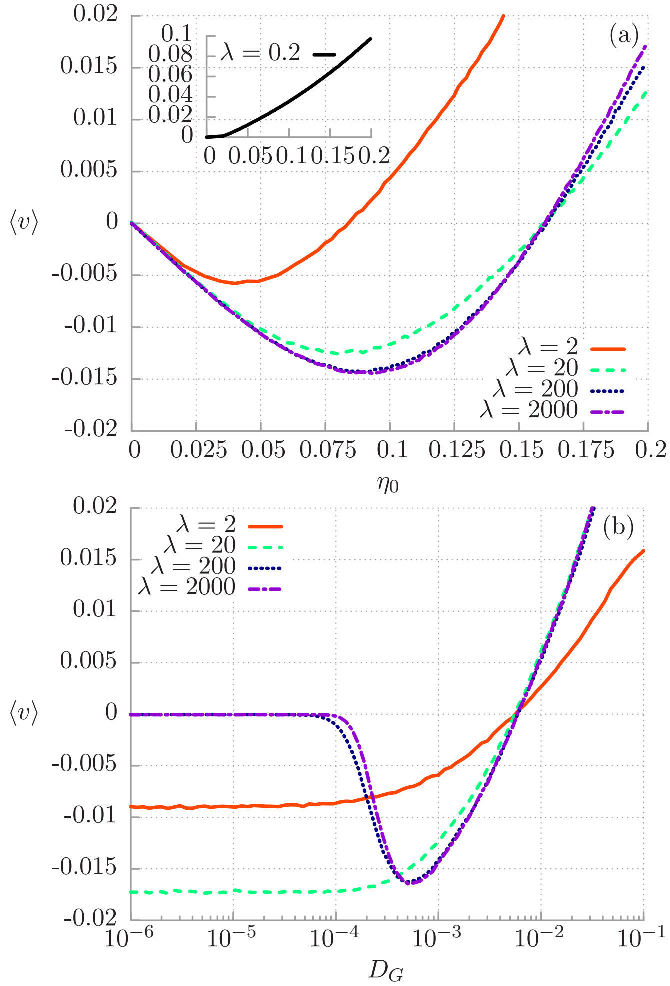

Fig. 1. Absolute negative mobility. Panel (a): the average velocity $\langle v \rangle$ as a function of the mean value of the white Poissonian shot noise η_0 for different frequencies λ and fixed thermal fluctuation intensity $D_G = 0.001$. In the inset, normal, Ohmic-like transport regime is observed for $\lambda = 0.2$. Panel (b): the influence of thermal fluctuations (temperature) is depicted for: $\lambda = 2$ (solid line, $D_P = 8 \cdot 10^{-4}$), $\lambda = 20$ (dashed line, $D_P = 3.8 \cdot 10^{-4}$), $\lambda = 200$ (dotted line, $D_P = 3.6 \cdot 10^{-5}$) and $\lambda = 2000$ (dashed-dotted line, $D_P = 3.8 \cdot 10^{-6}$). Note that for high λ the absolute negative mobility is solely induced by thermal fluctuations. In both panels other parameters read: friction coefficient $\gamma = 0.9$, harmonic driving amplitude $a = 4.2$ and angular frequency $\omega = 4.9$.

but large δ-pulses. Consequently, one can observe almost linear dependence of the average velocity $\langle v \rangle$ on the mean value η_0 which in the case of the static force corresponds to the Ohmic-like transport behaviour.

B. Absolute negative mobility

Our numerical search has shown that the Ohmic-like regime dominates in the parameter space. However, it is possible to find such parameter regimes for which the average velocity assumes negative values. It means that although only positive δ-pulses are realized, the particle moves on average in the negative direction, opposite to the drift Poissonian noise. Figure 1 presents this situation. In Fig. 1(a) we show the influence of the mean value η_0 of Poissonian noise on transport of the Brownian particle. The unique feature is the occurrence of intervals of η_0 where $\langle v \rangle$ is negative. Moreover, starting out from zero, $\langle v \rangle$ assumes negative response, takes its local minimum for definite η_0 and then heads towards positive values. We can see that this minimum is lowered when the frequency λ of δ-pulses is increased. What is also interesting, there exists a limiting minimal value of the average velocity which is reached when $\lambda \to \infty$.

The influence of thermal fluctuations on the particle transport is depicted in panel (b) of Fig. 1. Two separate mechanisms responsible for the absolute negative mobility effect are detected. For small values of λ, the ANM is caused by deterministic chaotic dynamics + Poissonian noise because even at zero temperature ($D_G = 0$) the average velocity is negative. In this case thermal fluctuations have destructive influence on the ANM - increase of temperature results in decrease of the negative value of velocity and for high temperatures $\langle v \rangle > 0$. On the contrary, for large λ the ANM is solely induced by thermal fluctuations: At low temperature $\langle v \rangle = 0$; for higher temperatures $\langle v \rangle < 0$. There exists an optimal temperature for which this phenomenon is most pronounced. Further increase of the temperature reduces the ANM and finally high temperature destroys it resulting in $\langle v \rangle > 0$.

C. Negative differential mobility

The average velocity $\langle v \rangle$ is often non-monotonic function of η_0 and for this reason a differential mobility μ_D defined by the relation

$$\mu_D = \mu_D(\eta_0) = \frac{d\langle v \rangle}{d\eta_0}. \tag{9}$$

can assume negative values. In Fig. 2(a) an example of such behaviour is depicted. For small to moderate λ the average velocity starts out from zero, assumes negative response in a form of the ANM and then is almost linearly increasing function of η_0. However, another situation occurs for larger values of λ. E.g. for $\lambda = 2000$, when the particle is very frequently kicked by small δ-pulse, after crossing zero the average velocity takes positive values, reaches its local maximum and then in the interval $\eta_0 \in (0.3, 0.37)$ is still positive but a decreasing function of η_0. Indeed, this is the regime of the negative differential mobility where $\mu_D < 0$.

D. Negative nonlinear mobility

Further increases of η_0 (for the case $\lambda = 2000$ in Fig. 2(a)) leads to another interesting phenomenon which we call negative nonlinear mobility. In contrast to the linear response regime, in the nonlinear response regime the average velocity can tend to zero even if the biasing man value η_0 assumes nonzero value. In particular, there can be two value of η_0 such that in between the average velocity is negative. Then, the nonlinear mobility coefficient μ_N defined as

$$\mu_N = \mu_N(\eta_0) = \frac{\langle v \rangle}{\eta_0} \tag{10}$$

is negative in this interval which means that the average velocity assumes the opposite sign to η_0. Indeed, this situation is presented in Fig. 2a in the case of large $\lambda = 2000$ for $\eta_0 \in (0.37, 0.51)$.

The influence of thermal noise on transport in the above considered regime $\lambda = 2000$ is depicted in Fig. 2(b). We can

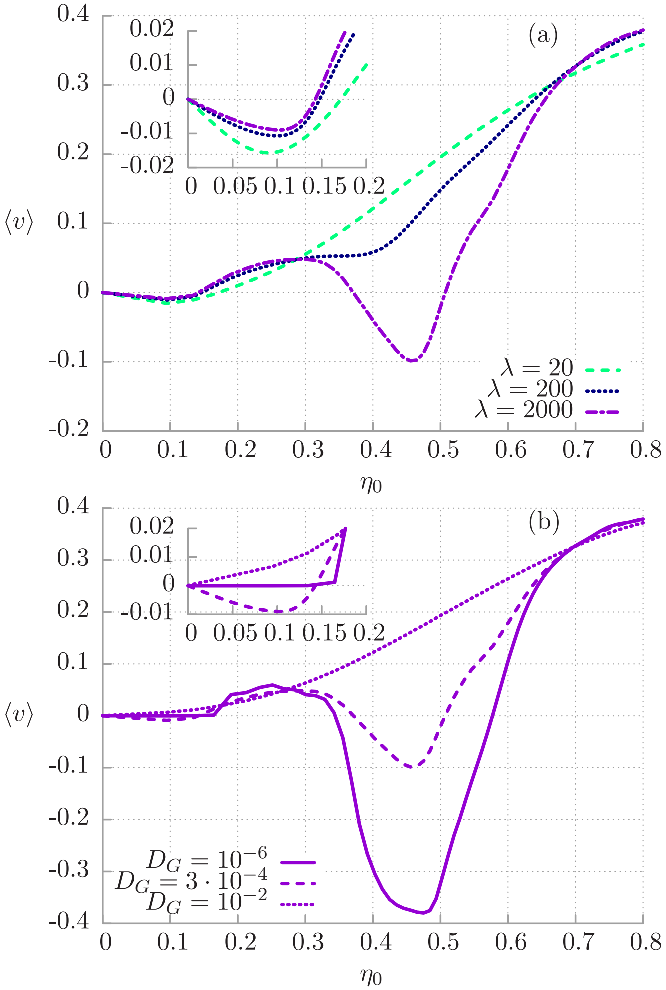

Fig. 2. Negative differential and nonlinear mobility. Panel (a): the average velocity $\langle v \rangle$ as a function of the mean of the white Poissonian shot noise η_0 for different frequencies λ and fixed thermal noise intensity $D_G = 0.0003$. The reentrant effect of the negative response is presented for $\lambda = 2000$. Panel (b): impact of the thermal noise (temperature) on the anomalous transport regime is shown for $\lambda = 2000$. Other parameters are the same as in Fig.1.

see that thermal fluctuations have destructive impact on the negative mobility of the Brownian particle. As temperature increases, the negative mobility is smaller and smaller and finally disappears.

IV. Conclusions

With this work, we took a closer look at the richness of anomalous transport processes occurring in an harmonically driven inertial Brownian particle which is in addition exposed to biased Poissonian white shot noise. It turns out that underlying chaotic dynamics combined with the influence of thermal fluctuations is able to exhibit a whole variety of transport characteristics. In particular, the average velocity $\langle v \rangle$ as a function of the mean value of Poissonian shot noise $\langle \eta(t) \rangle$ can manifest each of the three anomalous transport features, namely, absolute negative, negative differential and nonlinear mobility. Furthermore, reentrant phenomenon of negative mobility occurs as a function of the mean value $\eta_0 = \langle \eta(t) \rangle$ of the Poissonian shot noise.

Our results can readily be experimentally tested via appropriately designing the experimental working parameters for the single resistively and capacitively shunted Josephson junction device operating in its semi-classical regime.

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