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BISTABILITY DRIVEN BY COLORED NOISE: THEORY AND EXPERIMENT

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We consider a nonequilibrium, bistable system exposed to external, additive, Gaussian, colored noise. A new approximate theory, which results in a nonlinear Fokker-Planck type equation capable of accurately modeling the long time dynamics is constructed, and the mean sojourn time (MST) (in a potential well) is evaluated in the limits of weak noise and small correlation time. An electronic circuit which accurately mimics the bistable system has been constructed. Experimental measurements on this circuit are in good quantitative agreement with the predictions of the theory, which indicate an exponential increase of the MST with increasing noise correlation time. This observed exponential dependence is the clear signature that an Arrhenius law governs the process of escape from the well.

Recent years have witnessed an increasing interest in the stochastic behaviour of non-equilibrium systems coupled to a fluctuating external environment.[1] The influence of these fluctuations on nonlinear systems is not trivial.[2] For example, it has recently been realized that the usual Fokker-Planck descriptions are generally beset with difficulties for colored noise problems (or systems with state space dimension $d > 2$).[3]

We have studied the bistable system defined by

$$\dot{x} = ax - bx^3 + \xi(t) \qquad (1)$$

with the time correlated noise

$$\langle \xi(t)\ \xi(s) \rangle = \frac{D}{\tau} \exp - |t-s|/\tau \qquad (2)$$

both experimentally, as described below, and theoretically, using a new approximate procedure suggested by previous work.[4] The rate of change of the single event probability $p_t(x)$ of the bistable, colored dynamics obeys the exact equation[5]

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}(ax - bx^3)p_t + \frac{D}{\tau}\frac{\partial^2}{\partial x^2}\int_0^t ds \exp - (t-s)/\tau \cdot \langle \delta(x(t) - x)\frac{\delta x(t)}{\delta \xi(s)} \rangle \qquad (3)$$

where the functional derivative $(\delta x(t)/\delta\xi(s))$ obeys the integral equation

$$\frac{\delta x(t)}{\delta\xi(s)} = \Theta(t-s)\left\{1 + \int_0^t d\tau\ (a - 3bx^2(\tau))\ \frac{\delta x(\tau)}{\delta\xi(s)}\right\} . \qquad (4)$$

Upon repeated use of (4) we can rewrite the average in (3) in the form

$$\langle\delta(x(t) - x)\left[1 + \int_0^t d\tau_1 (a - 3bx^2(\tau_1))\right.$$
$$\left.\left[1 + \int_s^{\tau_1} d\tau_2 (a - 3bx^2(\tau_2))[1 + \ldots]\right]\ldots]\rangle. \qquad (5)$$

If we now neglect transients $(t\to\infty)$, and make use of a decoupling ansatz, thereby neglecting induced correlations among the successive functionals in (5), we end up with an effective, "nonlinear" Fokker-Planck dynamics

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}(ax - bx^3)p_t(x)$$
$$+ \frac{D}{[1 + \tau(3b\langle x^2\rangle - a)]}\frac{\partial^2}{\partial x^2} p_t(x), \qquad (6)$$

where $\langle x^2\rangle$ depends itself on the (unknown) solution $p_t(x)$. This causes no trouble, because the stationary 2-nd moment varies only weakly with D in this bistable. (Indeed, $\langle x^2\rangle \simeq a/b$, which insures that the effective diffusion constant in (6) is always > 0). This approximation will clearly work best for small τ and converges for $\tau \to 0$ to the proper white noise limit (the Smoluchowski equation).

We have used this theory to calculate the mean sojourn time (MST), a quantity which is sensitive to the wings of the stationary distribution.[5] An example (measured) time trajectory is shown in Fig. 1(b), where a clear time separation between the time spent in the well (sojourn time) and the time in transit between wells is evident. Thus transitions between wells can be considered as independent random events. The sojourn times are thus exponentially distributed

$$p(T) = (1/\langle T\rangle)\exp{-T/\langle T\rangle} \qquad (7)$$

The MST now depends only on the ratio of the stationary density evaluated at either of its maxima at $x_{1/2} = \pm (a/b)^{1/2}$ to the unstable point at $x = 0$,

$$\langle T\rangle = p_{st}(x_{1/2})/p_{st}(x = 0) \qquad (8)$$

Because (6) accurately models the long-time dynamics of x(t), one obtains for the leading behavior of the MST[4] (by the method of steepest descent)

$$\langle T\rangle = \frac{\pi}{a\sqrt{2}} \exp \frac{\Delta\phi(\tau)}{D} \qquad (9)$$

where

$$\Delta\phi = (a^2/4b)[1 + \tau(3b\langle s^2\rangle - a)] . \qquad (10)$$

thus the approximation scheme in (6) predicts an exponential increase of $\langle T\rangle$ with noise correlation time.

We found (9) and (10) to be in quite good agreement with measurements of <T> using an analog electronic circuit similar to one used previously to study noise induced phase transitions[7] and other bistable systems.[8] The circuit is shown in Fig. 1(a). It was designed to obey (to an accuracy of a few percent) the integral of equation (1) with a = b = 1, and with an integrating time constant which scales the time and noise intensity in the experiment as explained in Ref. 7. A measured time trajectory x(t) is shown in Fig. 1(b). Typically 5000 such trajectories were digitized by the computer from which the mean time between zero crossings (the MST) and the distribution of these times were assembled.

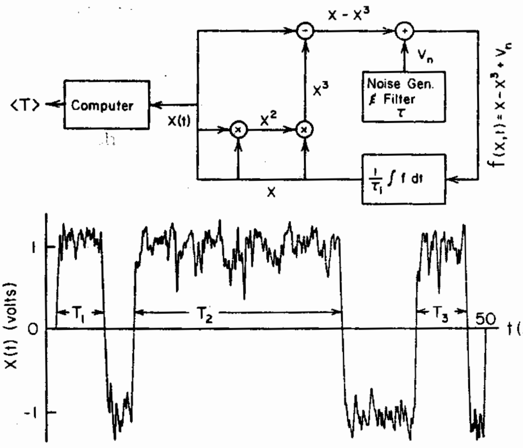

Figure 1. (a) the circuit, and (b) an example trajectory

Measurements of <T> were obtained for four different noise intensities, held constant as the noise correlation time was varied.

Equation (9) was then used to obtain the Arrhenius factor ($\Delta\phi$), and the results are shown in Fig. 2. The theory, as given by (9) and (10) is represented by the solid lines for the two values of $\langle x^2 \rangle$ shown. Arrhenius behavior (straight lines on this plot) is clearly indicated by the agreement evident over a surprisingly wide range of τ.

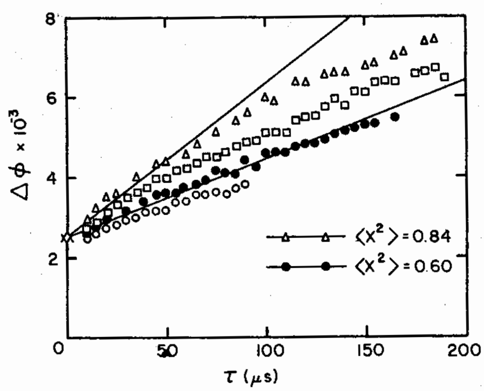

Figure 2. Data derived from MST measurements via Eq. (9). Triangles, squares, solid and open circles are dimensionless noise intensities: 0.21, 0.15, 0.11 and 0.083 respectively.

1. For a recent review, see Fluctuations and Sensitivity in Nonequilibrium Systems, W. Horsthemke and D. K. Kondepudi, eds. (Springer, Berlin, 1984).

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