A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

January 2011

EPL, **93** (2011) 20002
doi: 10.1209/0295-5075/93/20002

www.epljournal.org

Markovian embedding of fractional superdiffusion

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received 11 November 2010; accepted in final form 4 January 2011
published online 2 February 2011

PACS 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion
PACS 87.16.Uv – Active transport processes
PACS 02.50.-r – Probability theory, stochastic processes, and statistics

Abstract – The Fractional Langevin Equation (FLE) describes a non-Markovian Generalized Brownian Motion with long time persistence (superdiffusion), or anti-persistence (subdiffusion) of both velocity-velocity correlations, and position increments. It presents a case of the Generalized Langevin Equation (GLE) with a singular power law memory kernel. We propose and numerically realize a numerically efficient and reliable Markovian embedding of this superdiffusive GLE, which accurately approximates the FLE over many, about $r = N \log_{10} b - 2$, time decades, where N denotes the number of exponentials used to approximate the power law kernel, and $b > 1$ is a scaling parameter for the hierarchy of relaxation constants leading to this power law. Besides its relation to the FLE, our approach presents an independent and very flexible route to model anomalous diffusion. Studying such a superdiffusion in tilted washboard potentials, we demonstrate the phenomenon of transient hyperdiffusion which emerges due to transient kinetic heating effects.

Introduction. – Anomalous diffusion and transport processes possess a rich variety of applications spanning many different research fields from plasma physics and nonlinear dynamical systems to condensed-matter physics, biophysics, epidemiology, and even quantitative finance [1–3]. There are several very different theoretical approaches to describe anomalous diffusion, from continuous time random walks (CTRW) [4] including Levy flights and Levy walks [1,2] to the Generalized Langevin Equation (GLE) [5–20]. CTRW-based anomalous diffusion involves such unusual concepts as fractal time and subordination to a random clock which does not possess a finite mean period. In the continuous space limit it is often associated with Fractional Fokker-Planck Equations (FFPEs) [1]. The corresponding Langevin equations are local in *random* time [21] and describe a doubly random process with infinite memory. These latter Langevin equations should not be confused [14] with the Fractional Langevin Equations (FLEs) [8,11–14,17,20]. Random time clocks without mean period entail the remarkable phenomenon of weak ergodicity breaking [22,23]. The position, or velocity increments in this approach are independent in all basic models. Ergodicity can also be broken in other approaches, *e.g.* in nonlinear Brownian motion [24]. Furthermore, GLEs including FLEs present a quite different approach incorporating long-time correlations or anti-correlations of the position increments, similar to the fractional Brownian motion (fBm) devised by Mandelbrot and van Ness [25] which can also be derived from a FLE [14]. This is a benchmark feature [16,20] of the latter approach. Another one is that this approach does not rely on the concept of random time with divergent mean clock period [16] and is almost always ergodic, except for ballistic GLE diffusion [10,16–19].

Generally such non-Markovian processes are not characterized completely by a master equation for conditional probabilities, or Fokker-Planck equations [26]. However, the corresponding Langevin equations specify the stochastic process completely, containing all the information on trajectories. Phenomenologically, the GLE

$$m\ddot{x} + \int_{-\infty}^{t} \eta(t-t')\dot{x}(t')\,\mathrm{d}t' + \frac{\partial V(x,t)}{\partial x} = \zeta(t), \qquad (1)$$

formally presents a Newtonian equation of motion for a particle with mass m and coordinate x (we consider a one-dimensional model), subjected apart from a regular force, $f(x,t) = -\partial V(x,t)/\partial x$, to a zero-mean stochastic force $\zeta(t)$ (which adds energy to the Brownian particle) and a non-local in time frictional, or dissipative force (which takes off energy from the Brownian particle). Both processes are balanced at thermal equilibrium, *i.e.* Brownian motion never ceases and obeys the fluctuation-dissipation theorem, FDT. In (1), the dissipative force is assumed to have the form of a linear velocity-dependent friction with its memory characterized by the integral kernel $\eta(t)$. The FDT is obeyed, when the noise is Gaussian [27] and the memory kernel and the noise autocorrelation function are related by the fluctuation-dissipation relation (FDR) [5] reading

$$\langle\zeta(t)\zeta(t')\rangle = k_B T\eta(|t-t'|) \quad (2)$$

with T being the environmental temperature.

Apart from this phenomenological justification, the GLE can be derived microscopically from a Hamiltonian model involving coupling of the diffusing particle to a thermal bath of harmonic oscillators obeying initially canonical distribution at temperature T [6,7,9]. Anomalous diffusion can be related to a power law memory kernel

$$\eta(t) = \frac{|\sin(\pi\alpha/2)|}{\pi/2}\Gamma(\alpha)\eta_\alpha \mathrm{Re}(it+1/\omega_c)^{-\alpha}, \quad (3)$$

which we write with a short-time cutoff $1/\omega_c$ corresponding to the largest frequency of the bath oscillators ω_c, $\alpha>0$, and $\Gamma(x)$ is the gamma-function. By eq. (2), for $0<\alpha<2$ such $\eta(t)$ corresponds in the singular limit $\omega_c\to\infty$ to a fractional Gaussian noise $\zeta(t)$ with the Hurst exponent $H=1-\alpha/2$ [25]. Then the solution of the GLE yields a fractional Brownian motion in the limit $m\to 0$ [14]. In the case of free diffusion ($f(x,t)=0$) and for $0<\alpha<2$ independently of ω_c the noise-averaged position variance $\sigma^2(t)=\langle\Delta x^2(t)\rangle=\langle[x(t)-\langle x(t)\rangle]^2\rangle$ grows with time *asymptotically* as $\sigma^2(t)\sim 2k_BTt^\alpha/[\eta_\alpha\Gamma(1+\alpha)]$ [9]. This corresponds to sub-diffusion in the case $0<\alpha<1$ (sub-linear growth, sub-Ohmic thermal bath), normal diffusion for $\alpha=1$ (linear growth, Ohmic bath), and superdiffusion for $1<\alpha<2$ (super-linear growth, super-Ohmic bath). In terms of the integral frictional strength, $\tilde{\eta}(0)=\int_0^\infty\eta(t')\mathrm{d}t'$, these behaviors are intuitively clear: subdiffusion corresponds to $\tilde{\eta}(0)\to\infty$, normal diffusion to $\tilde{\eta}(0)=$ *const* and superdiffusion to $\tilde{\eta}(0)\to 0$. For $\alpha>2$, the free diffusion is always ballistic, $\sigma^2(t)\sim t^2$, and nonergodic because the velocity autocorrelation function (VACF) does not decay to zero. Therefore, the emergence of hyperdiffusion $\sigma^2(t)\sim t^\lambda$, with $\lambda>2$ within the GLE model is rather surprising. All these results can be easily obtained from a general expression for the stationary VACF, found for an arbitrary memory kernel first by Kubo [5] and reading $\tilde{K}_v(s)=k_BT/[ms+\tilde{\eta}(s)]$ in the Laplace space, by taking into account that the position variance is the twice-integrated VACF. One has to remark at this point, that the Laplace-transformed $\eta(t)$ is $\tilde{\eta}(s)=\eta_\alpha s^{\alpha-1}$ in this model in the limit $\omega_c\to\infty$. However, the inverse Laplace transform for $\tilde{\eta}(s)=\eta_\alpha s^{\alpha-1}$ only exists for $\alpha\leqslant 1$, *i.e.* for normal and subdiffusive friction. For $1<\alpha<2$ and a finite ω_c, the kernel $\eta(t)$ in eq. (3) starts from a positive part and then becomes negative so that its total integral is zero, independently of ω_c. When the cutoff frequency tends to infinity, the memory kernel becomes singular, starting from a positive singularity and being negative otherwise. The frictional term can be recast in this limit with the help of a fractional Riemann-Liouville derivative, see below.

For example, a spherical particle of radius R moving with velocity $v(t)=\dot{x}(t)$ in an incompressible liquid of kinematic viscosity μ and density ρ experiences a hydrodynamic force [28]

$$F_v(t) = -2\pi\rho R^3\left(\frac{1}{3}\ddot{x}+\frac{3\mu}{R^2}\dot{x} + \frac{3}{R}\sqrt{\frac{\mu}{\pi}}\int_{-\infty}^t\frac{\ddot{x}(\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau\right). \quad (4)$$

This classical result due to Boussinesq and Basset [11] generalizes the well-known by Stokes. The first term yields a mass renormalization of the Brownian particle $m\to m+\Delta m$ with $\Delta m=2\pi\rho R^3/3$, which is assumed to be implicitly done. It is present also in the absence of dissipation, *i.e.* for $\mu\to 0$. The second term corresponds to the Stokes friction, and the third term is due to a finite relaxation time $\tau_r=R^2/\mu$ of the disturbed velocity field of the liquid. It reflects hydrodynamic memory. An interesting mathematical interpretation of this term can be given within the formalism of fractional derivatives [1,11]. Namely, using the definition of the Riemann-Liouville fractional derivative, ${}_{t_0}\hat{D}_t^\gamma f(t):=\frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^t\mathrm{d}t' f(t')/(t-t')^\gamma$, $0<\gamma<1$, acting on some test function $f(t)$, it can be recast in the form [11]

$$F_{ad}(t) = -\eta_\alpha\, {}_{-\infty}\hat{D}_t^{\alpha-1}\,\dot{x}(t) \quad (5)$$

with $\eta_\alpha=\eta_1 R/\sqrt{\mu}$ and $\alpha=3/2$, where $\eta_1=6\pi R\mu\rho$ is the Stokes friction coefficient. The corresponding GLE was termed FLE [11]. Independently of this interpretation it is known [29] to yield the famous power law decay of the VACF, which was revealed in molecular-dynamic simulations by Alder and Wainwright [30].

The presence of a normal Stokes friction term makes the corresponding diffusion asymptotically normal. However, anomalous superdiffusive motion can emerge on a transient time scale $t<\tau_r=R^2/\mu$ for light particles [11]. If to neglect *ad hoc* the Stokes term and to set $\dot{x}(t)=0$ for $t<t_0=0$ (*i.e.* the particle starts to move at $t=0$) the corresponding superdiffusive FLE reads [8,11],

$$m\ddot{x}+\eta_\alpha\, {}_0\hat{D}_t^{\alpha-1}\,\dot{x}(t)+\frac{\partial V(x,t)}{\partial x}=\zeta(t) \quad (6)$$

with $\alpha=1.5$. This FLE serves as one of the basic models for superdiffusion with $1<\alpha<2$. It corresponds to a GLE with a singular memory kernel, which in mathematical sense is a generalized function. As discussed above, this kernel starts from a positive singularity at $t=0$ and then is negative, decaying to zero in accordance with a power law $t^{-\alpha}$, so that its total integral is zero.

The presence of a nonlinear time-dependent force $f(x,t)=-\partial V(x,t)/\partial x$ modifies this well-established picture considerably. Since general analytical results are then scarce and most likely generally nonexistent, the reliability of numerical simulations is a key issue. In particular, numerically tractable models can be obtained by approximating the given power law memory kernel by a finite sum of exponentials, see in [13,16,18]. By means of this approximation it is possible to represent the non-Markovian dynamics in the (x,v)-plane as a projection of a fully Markovian dynamics in a hyperspace of dimension $D=N+2$, where N is the number of exponentials in the approximation [13,16,18]. Then the central point is that one can propagate the corresponding Markovian dynamics locally in time by very reliable algorithms with a well-controlled numerical precision and by increasing N one can approximate the FLE dynamics ever better. Surprisingly one finds that the practical embedding dimension D need not be large to achieve an excellent approximation within statistical errors of stochastic simulations. Moreover, this approach can be used independently of the FLE with some advantages: i) The stochastic propagation is local in time and can readily be continued beyond end point. There is no need to generate a realization of long-correlated noise $\zeta(t)$ for the *whole* time span of simulation *fixed* in advance (without a possibility to continue) and solving the intregro-differential equation numerically for any such noise realization. This dramatically saves computer memory and enables extremely long simulations with appreciably small statistical errors. ii) The corresponding multi-dimensional Fokker-Planck equation is known explicitly for arbitrary $f(x,t)$ which can be used to develop an analytical theory for nonlinear dynamics.

Model. – We generalize the modeling of viscoelastic subdiffusion in ref. [16] to superdiffusive case, $1<\alpha<2$, and approximate the superdiffusive memory kernel as

$$\eta(t) = \frac{\eta_\alpha C_\alpha(b)}{|\Gamma(1-\alpha)|}\left[2\sum_{i=1}^{N}\left(\frac{\nu_0}{b^i}\right)^{\alpha-1}\delta(t) - \sum_{i=1}^{N}\left(\frac{\nu_0}{b^i}\right)^{\alpha}\exp\left(-\frac{\nu_0}{b^i}t\right)\right]. \qquad (7)$$

The first singular term mimics the positive singularity in the memory kernel of the FLE (6). The sum of exponentials obeys a fractal scaling with negative weights and approximates the power law decay [2] of this memory kernel; *i.e.*, $\eta(t)\propto -t^{-\alpha}$, for $t>0$, so that $\tilde{\eta}(0)=0$. Choosing $\nu_0=\omega_c$, the power law regime extends in this approximation from a short-time (high-frequency) cutoff, $\tau_l=\omega_c^{-1}$, to a large-time (small-frequency) cutoff, $\tau_h=\tau_l b^N$, where b is a scaling dilation parameter. Such a fit is known to exhibit logarithmic oscillations superimposed on the power law [2]. Their amplitude is, however, small and can be controlled by the choice of b. By adjusting b and N for a given α one can approximate $t^{-\alpha}$ over about $r=N\log_{10}b-2$ time decades between two time cutoffs, which are always physically present, beyond FLE modeling. Figure 1 illustrates the quality of the approximation of $t^{-1.5}$ with the parameters: $\nu_0=10^3$, $b=5$, $N=13$, and $C_\alpha(b)=1.78167$. One can detect a good agreement over

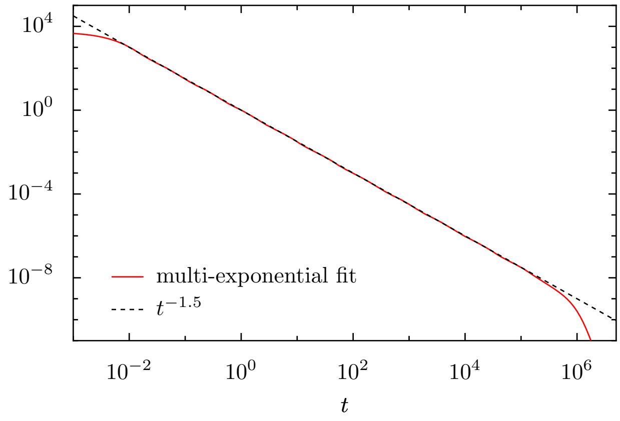

Fig. 1: (Color online) Approximation of the power law behavior of the friction kernel by a sum of exponentials $C_\alpha(b)\sum_{i=1}^{N}\left(\nu_0/b^i\right)^\alpha\exp\left(-\nu_0 t/b^i\right)$ (see eq. (7)) with the parameters: $N=13$, $\nu_0=10^3$, $b=5$, and $C_\alpha(b)=1.78167$.

about $r\approx 7$ decades in time. Clearly, with decreasing b and increasing N, one can further improve and control the quality of the approximation [16] which should be consistent with statistical errors of Monte Carlo simulations to avoid unnecessary numerical load.

Next, we introduce N auxiliary variables u_i and the corresponding multi-dimensional Markovian dynamics in the hyperspace of dimension $D=N+2$,

$$\begin{aligned} \dot{x}(t) &= v(t) \\ m\dot{v}(t) &= -V'(x,t)-\sum_{i=1}^{N}u_i(t)-\eta_0 v(t)+\sqrt{2k_BT\eta_0}\xi_0(t) \\ \dot{u}_i(t) &= -\eta_i v(t)-\nu_i u_i(t)+\sqrt{2k_BT\eta_i\nu_i}\xi_i(t), \end{aligned} \qquad (8)$$

where $\nu_i=\nu_0/b^i$, $\eta_i=C_\alpha(b)\eta_\alpha\nu_i^\alpha/|\Gamma(1-\alpha)|$, for $i=1,\ldots,N$, and $\eta_0=\sum_{i=1}^{N}\eta_i/\nu_i$. Furthermore, $\xi_0(t)$ and the $\xi_i(t)$ are $N+1$ delta-correlated white Gaussian noise sources of zero-mean and unit intensity. N of them are totally uncorrelated, $\langle\xi_i(t)\xi_j(t')\rangle=\delta_{ij}\delta(t-t')$, for $i,j=1,\ldots,N$, and for $i=0, j=0$. However, the noise $\xi_0(t)$ is chosen as a weighted, normalized sum of the other *independent* noises,

$$\xi_0(t)=\sum_{i=1}^{N}\sqrt{\frac{\eta_i}{\nu_i\eta_0}}\xi_i(t)\,. \qquad (9)$$

In the limiting case $N=1$, our present model yields the minimal 3-dimensional embedding of ballistic GLE superdiffusion developed in refs. [19,31] and presents thus a generalization of this earlier model to the sub-ballistic case. Given the lower integral limit $t_0=0$ in the GLE (instead of minus infinity), the exact reduction requires that the $u_i(0)$ are independently Gaussian distributed with zero-mean and variance $\langle u_i(0)u_j(0)\rangle=k_BT\eta_i\delta_{ij}$ to ensure the stationarity of the noise $\zeta(t)$ and FDR (2) for all times.

Furthermore, on the time scale $t>\tau_h$, the diffusion becomes ballistic in our modeling. However, τ_h becomes *exponentially* larger with increasing N, which ensures, that the practical embedding dimension can be reasonably small ($D = 15$ in simulations below). When speaking about asymptotic behavior below we yet assume that $t < \tau_h$, but choose τ_h so that it cannot be reached numerically. This procedure is similar to the subdiffusive case [16].

Numerical simulations. – Below, we investigate superdiffusion with $\alpha = 1.5$ in a tilted washboard potential of the form $V(x) = -V_0 \cos(2\pi x/x_0) - Fx$. For the sake of convenience we transform the equations above into dimensionless units by scaling time t in units of $\tau_0 = (\eta_\alpha/m)^{1/(\alpha-2)}$, distance in x_0, energy (which applies to V_0, k_BT and Fx_0) in $\Delta E = m(x_0/\tau_0)^2$ and u_i in mx_0/τ_0^2. As a consequence of the time scaling, ν_0 must be scaled in $1/\tau_0$, which implies, that η_0/m is scaled in $1/\tau_0$ and η_i/m in $1/\tau_0^2$. We integrated the dimensionless equations with a standard stochastic Euler algorithm using a combination of Mersenne-Twister and Box-Muller algorithms to generate the Gaussian random numbers. In each simulation an ensemble of 10^4 particles was propagated with a time step $\Delta t = 10^{-4}$ to achieve (weak) convergence of the ensemble averaged results. The end point of simulations is $t_{\max} = 10^4$. We used the friction kernel parameters of the approximation shown in fig. 1 and distributed the particle velocities initially thermally with $\langle v(0)\rangle = 0$ and $\langle v^2(0)\rangle = k_BT/m = v_T^2$. All particles were set initially to the position $x(0) = 0$.

Free superdiffusion. We first test our method by comparison of the numerical results for free superdiffusion, *i.e.*, $V_0 = 0$ and $F = 0$, with the available analytical solution of FLE [12],

$$\langle \Delta x^2(t)\rangle = 2v_T^2 t^2 E_{2-\alpha,3}[-(t/\tau_0)^{2-\alpha}]. \qquad (10)$$

Here, $E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} t^n/\Gamma(\alpha n + \beta)$ is the generalized Mittag-Leffler function. Furthermore, for $\eta(t)$ in eq. (7) the Laplace-transformed variance $\widetilde{\sigma^2}(s)$ is a rational function which can also be inverted to the time domain. The agreement in fig. 2 among the analytical FLE result, the analytical result for the Markovian embedding and the simulation results is indeed very good. Initially, the diffusion is always ballistic, $\langle \Delta x^2(t)\rangle \sim t^2$, turning over into the asymptotic behavior $\langle \Delta x^2(t)\rangle \sim t^{1.5}$.

Superdiffusion in a tilted periodic potential. Next, we study superdiffusion in a tilted periodic potential, for which no analytical solution is available. Similar to the free case, the superdiffusion starts again with ballistic diffusion and asymptotically again yields, $\langle \Delta x^2(t)\rangle \sim t^{1.5}$, cf. fig. 3 for $F = 10$. However, on an intermediate time scale a *hyperdiffusive regime* is developed, with $\langle \Delta x^2(t)\rangle \sim t^\lambda$, where $\lambda > 2$. Such an puzzling intermediate regime (note that free superdiffusion cannot be faster than ballistic within the GLE description) with a highly enhanced power law dependence of the position variance was also found recently for ballistic superdiffusion, see [19], and seems to be a generic feature of driven superdiffusion in non-confining potentials. Within the GLE description with a power law memory kernel and a quite different numerical approach (not based on Markovian embedding), such a hyperdiffusion was revealed first in ref. [15]. However, it was perceived there as an asymptotic regime probably because of insufficiently long time propagation. It is clear from fig. 3 that for a sufficiently small bias $F = 0.25$ we also cannot arrive at the asymptotic regime within two weeks of simulations. For larger bias strengths this is, however, possible. The mean velocity of diffusing particles also grows with time and when the corresponding mean kinetic energy becomes so large that the periodic potential ceases to play a role, the diffusion attains asymptotically the regime of free superdiffusion. However, on an intermediate time scale such a confined superdiffusion can be much faster than the free one. This surprising phenomenon somewhat resembles giant acceleration of normal diffusion in washboard potentials [32], but has a distinctly different origin [19].

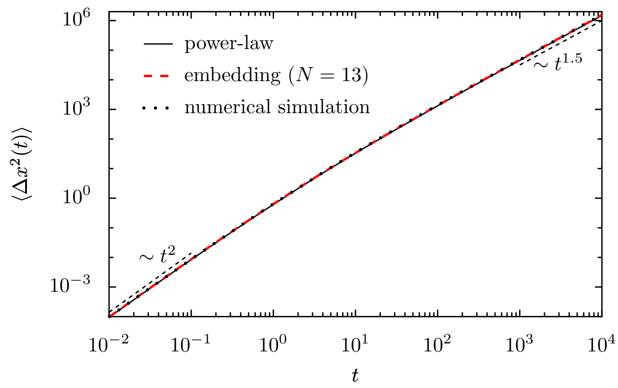

Fig. 2: (Color online) Comparison of the analytical FLE solution for the mean-squared displacement in eq. (10) with the analytical results for the Markovian embedding and the corresponding numerical results, $\alpha = 1.5$.

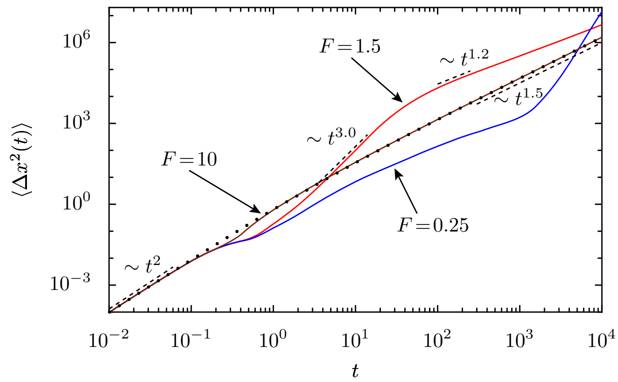

Fig. 3: (Color online) Position mean square displacement of particles in a tilted washboard potential for several different values of the bias strength F and the fixed potential amplitude $V_0 = 1$ and the bath temperature $T = 1$. The dotted line corresponds to the free case.

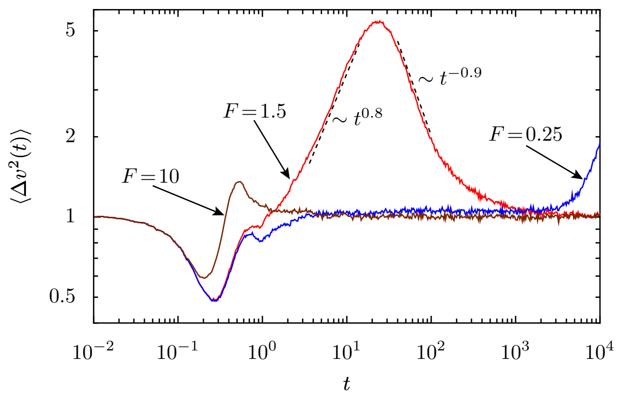

Fig. 4: (Color online) Mean square fluctuation of the particles velocity for the diffusion of fig. 3.

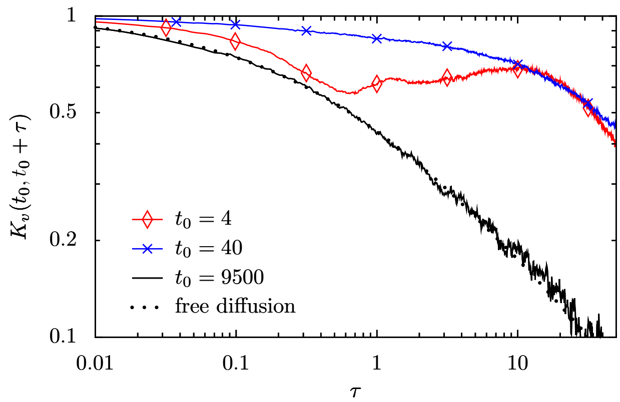

Fig. 5: (Color online) Normalized velocity autocorrelation function for the free superdiffusion (dotted line, analytical result) and for the biased diffusion of fig. 3 for $F=1.5$. One can see, that the latter decays much more slowly in the transient regime, for two values of t_0, which correspond to the start of the ascent of the velocity variance in fig. 4, $t_0=4$, and the descent, $t_0=40$. In the asymptotic regime, $t_0=9500$, the VACF is again the same as in the free case.

The intermediate behavior of the position mean-squared displacement (MSD) is a consequence of a transient heating of the particles from the bath temperature T to the kinetic temperature $T_{\rm kin}$, which we define [19] via the width of the velocity distribution $k_B T_{\rm kin}(t)/2 \doteqdot m\langle \Delta v^2(t)\rangle/2$, $\Delta v(t) = v(t) - \langle v(t)\rangle$. Such a kinetic temperature should not be confused with the thermodynamic temperature. Nevertheless, it presents a very helpful concept characterizing the kinetic energy of the disordered motion, when the kinetic energy of the directed motion is subtracted from the mean kinetic energy, in accordance with the physical meaning of temperature.

Figure 4 shows the time evolution of the velocity variance for the diffusion in fig. 3. One can immediately see that the transient hyperdiffusion is related to the transient kinetic heating. To explore this relation in more detail we notice that the position MSD equals the twice-integrated VACF, *i.e.*, $\langle \Delta x^2(t)\rangle = 2\int_0^t {\rm d}t' \int_0^{t'} {\rm d}t'' \langle \Delta v(t')\Delta v(t'')\rangle$. Introducing a normalized VACF $K_v(t'',t') = \langle \Delta v(t')\Delta v(t'')\rangle / \langle \Delta v^2(t'')\rangle$, one can rewrite the integrand as $\langle \Delta v(t')\Delta v(t'')\rangle = \langle \Delta v^2(t'')\rangle K_v(t'',t')$ and single out the evolution of the velocity variance. If $K_v(t', t'+\tau)$ remained constant, or it tended asymptotically to a constant with increasing τ (indicating the breaking of ergodicity), then a power law increase of the kinetic temperature with time, $T_{\rm kin}(t) \sim t^{\beta}$, would yield to a hyperdiffusive law for the position MSD $\langle \Delta x^2(t)\rangle \sim T_{\rm kin}(t) t^2 \sim t^{2+\beta}$. This is what occurs for ballistic superdiffusion in tilted washboard potentials which is nonergodic [19]. There, the kinetic temperature increases until it saturates at a large final value [19]. In contrast, the studied generalized Brownian motion is ergodic. Its free motion VACF does decay to zero. It also decays in the tilted washboard potentials, see in fig. 5 for the case $F=1.5$ in figs. 3, 4. However, this decay is enormously retarded in the transient regime, cf. fig. 5. This retardation explains qualitatively the quasi-nonergodic origin of hyperdiffusion also in the present case. Indeed, if $K_v(t', t'+\tau)$ were a constant, then the hyperdiffusion power law exponent would be $\lambda = 2+\beta \approx 2.8$ for the velocity variance growth depicted in fig. 4. This is not much different from the observed value $\lambda \approx 3$ in fig. 3. However, very different from the case of ballistic memory kernel [19], the kinetic temperature, associated with the velocity variance, starts to decline towards the bath temperature, see in fig. 4. This is because the motion is ergodic and after the periodic potential ceases to play a role the velocity ACF has to decay to zero much faster (like in the free case). Since the dissipation is much stronger than in the ballistic case, the particles are not strictly accelerated by the bias F. Their mean velocity grows rather sublinearly, $\langle v(t)\rangle \sim F t^{\alpha-1}$ and one can show that for any $\alpha < 2$ the mean kinetic energy of particles becomes negligible in the course of time as compared with the work done by the biasing force F on its increase. This is very different to the ballistic case, where asymptotically a finite portion of work is used for the heating (see online supplement of ref. [19]), and leads eventually to the decline of the kinetic energy of disordered motion, cooling back to the bath temperature T. If our explanation of the transient hyperdiffusion in terms of a delayed decay of VACF is correct, it should be able to explain another feature in fig. 3. Namely, that the hyperdiffusive regime first turns over into a transiently decelerated superdiffusion with $\lambda \approx 1.2 < \alpha$ (see for $F=1.5$), before λ grows back to α. Indeed, after reaching the maximum in fig. 4 the velocity variance starts to decay in accordance with a power law $\beta \approx -0.9$. Then, our reasoning with a strongly delayed VACF decay yields $\lambda = 2+\beta \approx 1.1$, which is close to the observed $\lambda \approx 1.2$ in fig. 3. This confirms that our line of reasoning is consistent.

Summary and conclusions. – In this work we proposed a general and simple method for Markovian embedding of a superdiffusive non-Markovian GLE dynamics and showed that it can be used to approximate FLE dynamics over many time decades, serving also as an independent approach to model superdiffusion. We studied numerically such a superdiffusion with $\alpha = 1.5$ in a tilted washboard potential. Concordant with our prior findings for the ballistic diffusion case [19], a hyperdiffusive transient regime was found. We gave a simple physical explanation of this transient regime as a kinetic heating effect in terms of the growing velocity variance and a strongly delayed decay of the velocity autocorrelation function. This is similar to the case of *nonergodic* ballistic superdiffusion in washboard potentials. The transient can be very long and superdiffusion can become enormously accelerated, compared to the free superdiffusion, during this transient regime. However, very different from the ballistic case, the transient heating is followed by a subsequent cooling back to the temperature of the thermal bath, after the kinetic energy of the particles, which grows in time, exceeds much the potential energy. This cooling effect is due to the fact, that the motion remains ergodic in the studied case and the friction is sufficiently strong to take off the extra part of the kinetic energy, which was built up during the transient heating regime. During this transient cooling regime the power law exponent of the diffusion becomes *less* than the one of the free superdiffusion and gradually grows to the latter in the course of time. Then the VACF coincides asymptotically with the one of the free motion. The influence of the periodic potential becomes forgotten, very differently from the nonergodic ballistic case.

In conclusion, we expect that our general methodology will be used in a number of future applications of anomalous GLE diffusion. Moreover, the surprising transient heating/cooling effects are expected to attract a further attention not only of theorists but also of the experimental community.

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This work was supported by the German Excellence Initiative via the Nanosystems Initiative Munich (NIM).

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