Fractional Fokker-Planck subdiffusion in alternating force fields

E. Heinsalu,[1,2] M. Patriarca,[1,2] I. Goychuk,[3] and P. Hänggi[3]
[1]*National Institute of Chemical Physics and Biophysics, Rävala 10, Tallinn 15042, Estonia*
[2]*IFISC, Instituto de Física Interdisciplinar y Sistemas Complejos (CSIC-UIB), E-07122 Palma de Mallorca, Spain*
[3]*Institut für Physik, Universität Augsburg, Universitätsstrasse 1, D-86135 Augsburg, Germany*
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The fractional Fokker-Planck equation for subdiffusion in time-dependent force fields is derived from the underlying continuous time random walk. Its limitations are discussed and it is then applied to the study of subdiffusion under the influence of a time-periodic rectangular force. As a main result, we show that such a force does not affect the universal scaling relation between the anomalous current and diffusion when applied to the biased dynamics: in the long-time limit, subdiffusion current and anomalous diffusion are immune to the driving. This is in sharp contrast with the unbiased case when the subdiffusion coefficient can be strongly enhanced, i.e., a zero-frequency response to a periodic driving is present.

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I. INTRODUCTION

The theoretical investigation of *anomalously* slow relaxation processes in time-dependent force fields constitutes a challenge of current research interest which is not free of ambiguity. It is known that there is no unique physical mechanism responsible for the occurrence of subdiffusion in condensed media [1]. One possible mechanism, which will be addressed in this work, corresponds to disordered glassy-like media consisting of trapping domains which the transporting particle can dwell for a random time with divergent mean value [2–4]. The successive residence times in traps are assumed to be mutually uncorrelated. Diffusion is nevertheless a non-Markovian (semi-Markovian) process exhibiting long (quasi-infinite) time correlations among the particle positions with a weak ergodicity breaking [5]. This physical model has been successfully applied, e.g., to explain experiments on transient photocurrents in amorphous semiconductor films [6,7]. Mathematically, it can be described by a continuous time random-walk (CTRW) model [3,4] which in the continuous space limit leads to the fractional Fokker-Planck equation (FFPE) [8–10]. This latter formulation is incomplete, in the sense that no (single event) non-Markovian master equation can fully characterize the underlying (multitime event) non-Markovian stochastic process [11,12]. This non-Markovian FFPE is very useful nevertheless; it can be derived from a closely associated complete description in terms of a (ordinary) Langevin equation in subordinated random operational time [13–15].

The generalization of the CTRW and FFPE to time-dependent forces is a nontrivial matter since the force changes in real physical and not in mathematical operational time [16,17]. Also, how a field varying in time affects the distribution of the residence times in the traps is not clear without specifying a concrete mechanism or some plausible model, especially when the mean residence time does not exist [18]. The FFPE describing the dynamics in time-dependent force fields $F(x,t)$ becomes ambiguous with a frequently (ab)used *ad hoc* version [19,20] which lacks a clear theoretical basis [16]. The correct version of the FFPE for time-dependent fields was first given in Refs. [16,21]: differently from the FFPE for a time-independent force, in the case of a time-dependent field the fractional derivative does not stand in front of the Fokker-Planck operator but only after it. As we explain with this work in more detail, such a FFPE can be justified beyond the linear-response approximation within a CTRW approach only for a special class of dichotomously fluctuating fields. The derivation of the FFPE for subdiffusion in such time-dependent fields is presented in Sec. II.

In Sec. III we apply the derived FFPE to study the influence of time-periodic rectangular fields on subdiffusive motion. Analytical solutions of the FFPE are confirmed by stochastic Monte Carlo simulations of the underlying CTRW. In particular, we show with this work that the universal scaling relation between the biased anomalous diffusion and subcurrent [3,4,9,10] is not affected by the periodic driving. Neither current nor diffusion is influenced asymptotically by the time-periodic field. This is in spite of the fact that the unbiased subdiffusion of the studied kind can be strongly enhanced in the time-periodic field [16,21].

II. DERIVATION OF THE FFPE FOR TIME-DEPENDENT FIELDS FROM THE UNDERLYING CTRW

Since the FFPE does not define the underlying stochastic non-Markovian process, its generalization to include the influence of a time-dependent field should start from the underlying CTRW [8,9]. Following the general picture of the CTRW, we introduce a one-dimensional lattice $\{x_i = i\Delta x\}$ with a lattice period Δx and $i=0,\pm 1,\pm 2,\ldots$. Let us first assume that there is no time-dependent field. After a random trapping time τ, a particle at site i hops with probability $q_i^{\pm}$ to one of the nearest-neighbor sites $i\pm 1$; $q_i^{+}+q_i^{-}=1$. The random time τ is extracted from a site-dependent residence time distribution (RTD) $\psi_i(\tau)$. The corresponding generalized master equation for populations $P_i(t)$ reads as [7,22,23]

$$\dot{P}_i(t) = \int_0^t \{K_{i-1}^{+}(t-t')P_{i-1}(t') + K_{i+1}^{-}(t-t')P_{i+1}(t') - [K_i^{+}(t-t') + K_i^{-}(t-t')]P_i(t')\}dt' . \tag{1}$$

The Laplace transform of the kernel $K_i^{\pm}(t)$ is related to the

Laplace transform of the RTD $\psi_i(\tau)$ via $\tilde{K}_i^{\pm}(s) = q_i^{\pm} s\tilde{\psi}_i(s)/[1-\tilde{\psi}_i(s)]$. In the presence of a time-dependent field, the kernels become generally functions of both instants of time and not only of their difference, i.e., $K_i^{\pm}(t-t') \to K_i^{\pm}(t,t')$. One can relate $K_i^{\pm}(t,t')$ with the corresponding time-inhomogeneous RTDs $\psi_i^{\pm}(t+\tau,t) \equiv \psi_i^{\pm}(\tau|t)$, which are conditioned on the entrance time t. However, one always needs a concrete and physically meaningful model to proceed further [18]. A simple example is a Markovian process with time-dependent rates $g_i^{\pm}(t)$, where $\psi_i^{\pm}(\tau|t) = g_i^{\pm}(t+\tau)\exp\{-\int_t^{t+\tau}[g_i^{+}(t')+g_i^{-}(t')]dt'\}$ and $K_i^{\pm}(t,t') = 2g_i^{\pm}(t)\delta(t-t')$. This yields in Eq. (1) the standard master equation for a time-inhomogeneous Markovian process.

The modality how the nonexponential RTDs will be modified for a time-inhomogeneous process is generally neither obvious nor simple [18]. In the present case, one can assume that the trapping occurs due to the existence of direction(s) transverse to the x coordinate. According to the modeling in Refs. [15,21], an external field directed along the x direction would not affect the motion in the transverse direction(s). However, it is not correct to think that the RTD in the trap itself will not be influenced by the time-dependent field acting in the direction of x, as it will change the rates for moving toward the left or toward the right when escaping from the trap. Therefore, the RTD will generally be affected (for further details, see also Ref. [24]). Obviously, a situation for which this RTD in the trap surely will remain unaffected is when the sum of the two escape rates $g_i = g_i^{+}(t) + g_i^{-}(t)$ to escape from the trap—either toward the left or the right—is *time independent*. In that case, only the hopping probabilities $q_i^{\pm}(t) = g_i^{\pm}(t)/g_i$ acquire additional time dependence and not the RTD $\psi_i(\tau)$ itself [16]. This corresponds to the special class of dichotomously fluctuating force fields $F(x_i,t) = F(x_i)\xi(t)$, where $\xi(t) = \pm 1$. Beyond this class, at most the linear-response approximation can work [21].

Therefore, we restrict our treatment to the above class of fluctuating potentials. In this case, we can write $K_i^{\pm}(t,t') = q_i^{\pm}(t)K_i(t-t')$, where $\tilde{K}_i(s) = s\tilde{\psi}_i(s)/[1-\tilde{\psi}_i(s)]$. Furthermore, we use the Mittag-Leffler distribution for the residence times [8],

$$\psi_i(\tau) = -\frac{d}{d\tau}E_\alpha[-(\nu_i\tau)^\alpha]. \tag{2}$$

Here $E_\alpha(z) = \sum_{n=0}^{\infty} z^n/\Gamma(n\alpha+1)$ denotes the Mittag-Leffler function, $\alpha \in (0,1)$ is the index of subdiffusion, and $\nu_i = [g_i^{+}(t)+g_i^{-}(t)]^{1/\alpha}$ is the time scaling parameter; $g_i^{\pm}(t) = q_i^{\pm}(t)\nu_i^{\alpha}$. Then $\tilde{K}_i(s) = \nu_i^{\alpha} s^{1-\alpha}$ and we get

$$\begin{aligned}\dot{P}_i(t) &= g_{i-1}^{+}(t)\,{}_0\hat{D}_t^{1-\alpha}P_{i-1}(t) + g_{i+1}^{-}(t)\,{}_0\hat{D}_t^{1-\alpha}P_{i+1}(t)\\ &\quad - [g_i^{+}(t)+g_i^{-}(t)]\,{}_0\hat{D}_t^{1-\alpha}P_i(t),\end{aligned} \tag{3}$$

where the symbol ${}_0\hat{D}_t^{1-\alpha}$ stands for the integrodifferential operator of the Riemann-Liouville fractional derivative acting on a generic function of time $\chi(t)$ as

$${}_0\hat{D}_t^{1-\alpha}\chi(t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t dt'\frac{\chi(t')}{(t-t')^{1-\alpha}}. \tag{4}$$

$\Gamma(\alpha)$ is the gamma function. In a time-dependent potential $U(x,t)$, one can set

$$\begin{aligned}g_i^{\pm}(t) &= (\kappa_\alpha/\Delta x^2)\exp\{-\beta[U_{i\pm 1/2}(t) - U_i(t)]\}\\ &\approx (\kappa_\alpha/\Delta x^2)\exp[\pm\beta F(x_i,t)\Delta x/2],\end{aligned} \tag{5}$$

so that the Boltzmann relation $g_{i-1}^{+}(t)/g_i^{-}(t) = \exp\{\beta[U_{i-1}(t) - U_i(t)]\}$ is satisfied exactly and the time independence of $g_i^{+}(t)+g_i^{-}(t) = \nu_i^{\alpha} = \text{const}$ is also maintained for small Δx and a sufficiently smooth potential. We have used here the notation $U_i(t) \equiv U(i\Delta x,t)$ and $U_{i\pm 1/2}(t) \equiv U(i\Delta x \pm \Delta x/2,t)$; $\beta = k_B T$ is the inverse of temperature and κ_α is free fractional diffusion coefficient with dimension cm^2 s$^{-\alpha}$. By passing to the continuous space limit $\Delta x \to 0$ as in Ref. [9], one finally obtains,

$$\frac{\partial}{\partial t}P(x,t) = \left[-\frac{\partial}{\partial x}\frac{F(x,t)}{\eta_\alpha} + \kappa_\alpha\frac{\partial^2}{\partial x^2}\right]{}_0\hat{D}_t^{1-\alpha}P(x,t). \tag{6}$$

In the latter equation, $\eta_\alpha = (\beta\kappa_\alpha)^{-1}$ is the fractional friction coefficient. Our derivation also clarifies that the fractional derivative does not operate in front of the right-hand side of Eq. (6); i.e., in the case of a time-dependent force, this fractional Riemann-Liouville derivative does *not* act in front of the Fokker-Planck operator [16,21]. As already discussed in Ref. [16], we are not aware of such a physical time-dependent-driven CTRW which would correspond to such an *ad hoc* procedure.

In the following, we use Eq. (6) to study analytically the subdiffusion in time-periodic rectangular fields. Our study is complemented by stochastic simulations of the underlying CTRW using the algorithm detailed in Ref. [10].

III. DRIVEN SUBDIFFUSION

We consider a dichotomous modulation of a biased subdiffusion where the absolute value of the bias is fixed but its direction flips periodically in time, i.e.,

$$F(t) = F_0\xi(t), \tag{7}$$

with

$$\xi(t) = \begin{cases} +1 & \text{for } n\tau_0 < t < (n+r)\tau_0\\ -1 & \text{for } (n+r)\tau_0 < t < (n+1)\tau_0.\end{cases} \tag{8}$$

Here τ_0 is the period of the time-dependent force and $n = 0,1,2\ldots$ The quantity $r \in (0,1)$ determines the value of the average force,

$$\bar{F} = \langle F(t)\rangle_{\tau_0} = F_0(2r-1). \tag{9}$$

For $r=0.5$, the average bias is zero and we recover the model investigated in Ref. [16]. Notice that the force $F(t)$ can be decomposed in the following way: $F(t) = \bar{F} + \tilde{F}(t)$. The asymmetric driving,

$$\tilde{F}(t) = \begin{cases} 2F_0(1-r) & \text{for } n\tau_0 < t < (n+r)\tau_0 \\ -2F_0 r & \text{for } (n+r)\tau_0 < t < (n+1)\tau_0, \end{cases} \tag{10}$$

has a zero-mean value, $\langle \tilde{F}(t)\rangle_{\tau_0}=0$, and the driving root-mean-squared (rms) amplitude is $\sigma=\langle \tilde{F}^2(t)\rangle_{\tau_0}^{1/2}=2F_0\sqrt{r(1-r)}$. For a fixed average bias $\bar{F}$, this yields

$$\sigma = 2\bar{F}\frac{\sqrt{r(1-r)}}{2r-1}, \tag{11}$$

and therefore one can vary the ratio $\sigma/\bar{F}$ between 0 for $r=1$ and ∞ for $r=0.5+\epsilon$, $\epsilon\to 0$. This offers the way to study the influence of an asymmetric zero-mean driving $\tilde{F}(t)$ with period τ_0 and rms amplitude σ on the subdiffusion under constant bias $\bar{F}$.

Let us begin by finding the recurrence relation for the moments $\langle x^n(t)\rangle$. Assuming in Eq. (6) the force of the form (7) with Eq. (8), multiplying both sides of Eq. (6) by x^n, and integrating over the x coordinate, one obtains,

$$\frac{d\langle x^n(t)\rangle}{dt} = n v_\alpha \xi(t)\,{}_0\hat{D}_t^{1-\alpha}\langle x^{n-1}(t)\rangle + n(n-1)\kappa_\alpha\,{}_0\hat{D}_t^{1-\alpha}\langle x^{n-2}(t)\rangle, \tag{12}$$

with subvelocity $v_\alpha=F_0/\eta_\alpha (n>1)$. For $n=1$, the last term on the right-hand side of Eq. (12) is absent,

$$\frac{d\langle x(t)\rangle}{dt} = \frac{v_\alpha}{\Gamma(\alpha)}\xi(t)t^{\alpha-1}. \tag{13}$$

Equations (12) and (13) will be used to calculate the average particle position and the mean-square displacement.

A. Average particle position

Upon integrating Eq. (13) in time with $\xi(t)$ given by Eq. (8), the solution for the average particle position reads as

$$\langle x(t)\rangle = \begin{cases} x_N + \dfrac{v_\alpha t^\alpha}{\Gamma(\alpha+1)} & N\tau_0 \le t < (N+r)\tau_0 \\ x_N' - \dfrac{v_\alpha t^\alpha}{\Gamma(\alpha+1)} & (N+r)\tau_0 \le t < (N+1)\tau_0, \end{cases} \tag{14}$$

with

$$x_N = \langle x(0)\rangle - \frac{v_\alpha (N\tau_0)^\alpha}{\Gamma(\alpha+1)} + \frac{v_\alpha \tau_0^\alpha}{\Gamma(\alpha+1)} \times \sum_{n=0}^{N-1}[2(n+r)^\alpha - n^\alpha - (n+1)^\alpha], \tag{15}$$

$$x_N' = x_N + \frac{2v_\alpha \tau_0^\alpha}{\Gamma(\alpha+1)}(N+r)^\alpha, \tag{16}$$

N counts the number of time periods passed.

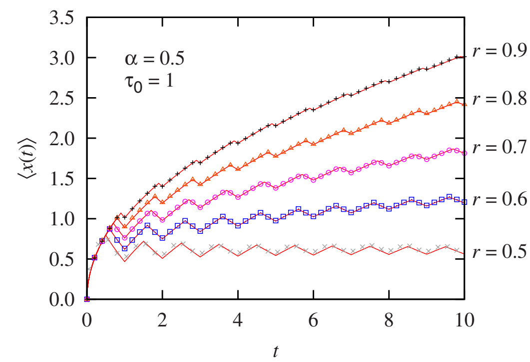

FIG. 1. (Color online) Average particle position $\langle x(t)\rangle$ for various values of the parameter r (average force $\bar{F}$). Symbols represent the results from the numerical simulations of the CTRW obtained by averaging over 10^5 trajectories (for $r=0.5$ over 10^6 trajectories). Continuous lines represent the analytical solution (14) of the FFPE (6). The time period of the rectangular force (7) is $\tau_0=1$ and the fractional exponent $\alpha=0.5$; in numerical simulations, $F_0/(\eta_\alpha\sqrt{\kappa_\alpha})=1$ is used. The value $r=0.9$ corresponds to $\bar{F}=0.8F_0$ and $\sigma=0.6F_0$; $r=0.8$ corresponds to $\bar{F}=0.6F_0$ and $\sigma=0.8F_0$; $r=0.7$ corresponds to $\bar{F}=0.4F_0$ and $\sigma\approx 0.92F_0$; $r=0.6$ corresponds to $\bar{F}=0.2F_0$ and $\sigma\approx 0.98F_0$; and $r=0.5$ to $\bar{F}=0$ and $\sigma=F_0$.

When the average bias is zero, i.e., $r=0.5$, in the long-time limit the mean particle position approaches the constant value

$$\langle x(\infty)\rangle = v_\alpha \tau_0^\alpha b(\alpha)/\Gamma(\alpha+1), \tag{17}$$

with $b(\alpha)=\sum_{n=0}^{\infty}[2(n+1/2)^\alpha - n^\alpha - (n+1)^\alpha]$. The function $b(\alpha)$ changes monotonously from $b(0)=1$ to $b(1)=0$. It describes the initial field phase effect which the system remembers forever when $\alpha\in(0,1)$ (see also Ref. [16]). This is one of the main differences between the anomalous motion in the absence of a force and in the presence of a time-dependent field with zero average value.

In Fig. 1, the analytical solution (14) for the mean particle position $\langle x(t)\rangle$ obtained from the FFPE (6) is compared with the numerical solution of the CTRW for different values of r, i.e., for different values of the average bias $\bar{F}$. In Fig. 2 the solution (14) is presented in the long-time limit for various values of r and α. Figures 1 and 2 demonstrate that in the presence of an average bias, the mean particle position grows as t^α. For all values of r, the asymptotic value of subvelocity corresponds to the averaged bias $\bar{F}=F_0(2r-1)$, indicating that the periodic unbiased field does not affect the subdiffusion current for different values of $\bar{F}$ and the field rms amplitude σ.

Furthermore, the results depicted in Figs. 1 and 2 clearly show the phenomenon of the "death of linear response" of the fractional kinetics to time-dependent fields in the limit $t\to\infty$: the amplitude of the oscillations decays to zero as $1/t^{1-\alpha}$ [Eq. (13)] (see also Refs. [21,25]). The amplitude of the oscillations is larger for larger values of α and of τ_0; however, $\Gamma(1+\alpha)\langle x(t)\rangle/t^\alpha$ reaches asymptotically the same value for any α and τ_0.

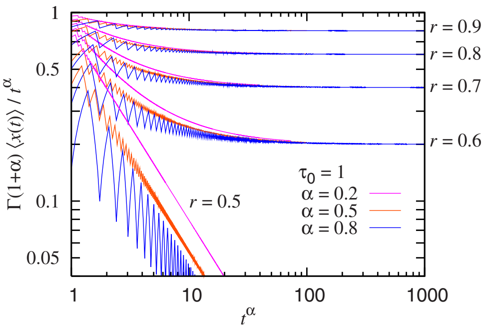

FIG. 2. (Color online) The analytical solution (14) for the average particle position $\langle x(t)\rangle$ obtained from the FFPE (6) is presented for various values of the parameter r (average bias $\bar{F}$) and anomalous exponent α. The time period of the force is $\tau_0=1$; however, in the long-time limit the same asymptotic value is obtained for any value of τ_0. The relation between r and $\bar{F}$ and σ is the same as in Fig. 1.

B. Mean-square displacement

Let us now study the mean-square displacement defined as

$$\langle \delta x^2(t)\rangle = \langle x^2(t)\rangle - \langle x(t)\rangle^2. \tag{18}$$

For $n=2$ one obtains from Eq. (12),

$$\frac{d\langle x^2(t)\rangle}{dt} = 2v_\alpha \xi(t)\, {}_0\hat{D}_t^{1-\alpha}\langle x(t)\rangle + \frac{2\kappa_\alpha}{\Gamma(\alpha)}t^{\alpha-1}. \tag{19}$$

In order to find the analytical solution for the mean-square displacement, we use the Laplace-transform method and the Fourier series expansion for $\xi(t)=\xi(t+\tau_0)$ given with Eq. (8),

$$\xi(t) = \sum_{n=-\infty}^{\infty} f_n \exp(in\omega_0 t), \tag{20}$$

with

$$f_n = \frac{1}{\tau_0}\int_0^{\tau_0} \xi(t)\exp(-in\omega_0 t)dt = [1-\exp(-inr2\pi)]/(in\pi) \tag{21}$$

and $\omega_0=2\pi/\tau_0$. Applying them to Eq. (19) and assuming $\langle x(0)\rangle=0$ and $\langle x^2(0)\rangle=0$, we obtain that in the long-time limit (see Appendix A),

$$\langle x^2(t)\rangle = \frac{2v_\alpha^2(2r-1)^2}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{2\kappa_\alpha}{\Gamma(\alpha+1)}t^\alpha + \frac{2v_\alpha^2(2r-1)S_1}{\omega_0^\alpha\Gamma(\alpha+1)}t^\alpha + \frac{8v_\alpha^2\cos(\alpha\pi/2)}{\pi^2\omega_0^\alpha\Gamma(\alpha+1)}\left[\zeta(2+\alpha) - \sum_{n=1}^{\infty}\frac{\cos(nr2\pi)}{n^{2+\alpha}}\right]t^\alpha, \tag{22}$$

here $\zeta(x)$ is the Riemann's zeta function, S_1 is a function of α, and r as given by Eq. (A7) in Appendix A.

For $r=0.5$ (average zero bias) the first and third terms in Eq. (22) are equal to zero. Furthermore, in the long-time limit the average particle position $\langle x(\infty)\rangle$ is a finite constant. The asymptotic behavior of the mean-square displacement is thus proportional to t^α as in the force free case, however, characterized by an effective fractional diffusion coefficient $\kappa_\alpha^{(\mathrm{eff})}$ instead of the free fractional diffusion coefficient κ_α, i.e., $\langle \delta x^2(t)\rangle = 2\kappa_\alpha^{(\mathrm{eff})}t^\alpha/\Gamma(1+\alpha)$ for $t\to\infty$. The effective diffusion coefficient is [16],

$$\kappa_\alpha^{(\mathrm{eff})} = \kappa_\alpha + \frac{8F_0^2}{\pi^2\eta_\alpha^2\omega_0^\alpha}\zeta(\alpha+2)\left(1-\frac{1}{2^{\alpha+2}}\right)\cos(\alpha\pi/2). \tag{23}$$

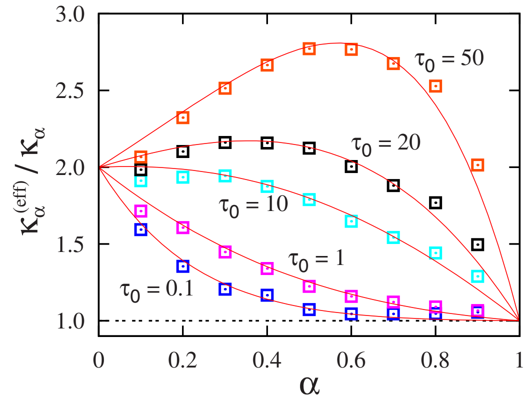

FIG. 3. (Color online) Scaled effective fractional diffusion coefficient $\kappa_\alpha^{(\mathrm{eff})}$ versus fractional exponent α for different driving periods τ_0. The average bias is zero ($r=0.5$). The analytical prediction (23) (continuous lines) is compared with the results (symbols) obtained from the numerical simulation of the CTRW by averaging over 10^5 trajectories. For $\tau_0>\tau_0^*\approx 8.818$ [16], the effective fractional diffusion coefficient $\kappa_\alpha^{(\mathrm{eff})}(\alpha)$ exhibits a maximum.

The driving-induced part of the effective subdiffusion coefficient is directly proportional to the square of driving amplitude F_0 and inversely proportional to ω_0^α. For slowly oscillating force fields, this leads to a profound acceleration of subdiffusion compared with the force free case: an optimal value of the fractional exponent α can exist, at which the driving-induced part of the effective fractional diffusion coefficient possesses a maximum (see Fig. 3).

When $r\neq 0.5$ (finite average force), we obtain in the long-time limit for $\langle x(t)\rangle^2$ (see Appendix B),

$$\langle x(t)\rangle^2 = \frac{v_\alpha^2(2r-1)^2}{\Gamma^2(\alpha+1)}t^{2\alpha} + \frac{2v_\alpha^2(2r-1)S_1}{\omega_0^\alpha\Gamma(\alpha+1)}t^\alpha \tag{24}$$

[S_1 is given by Eq. (A7) in Appendix A]. Clearly, the leading term in Eq. (24) corresponds to the subvelocity in constant field $\bar{F}$ (averaged bias), i.e., the influence of periodic unbiased driving $\tilde{F}(t)$ dies out asymptotically, as illustrated in Fig. 2.

The results (22) and (24) indicate that in the presence of a rectangular time-periodic force with a finite average value, the general behavior of the mean-square displacement is similar to the case of a constant force, i.e., the mean-square displacement $\langle \delta x^2(t)\rangle$ consists of terms proportional to t^α and $t^{2\alpha}$. In fact, for the leading term proportional to $t^{2\alpha}$ in the mean-square displacement, one obtains the coefficient

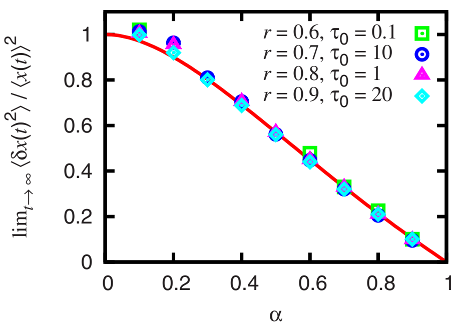

FIG. 4. (Color online) The asymptotic scaling relation (25). Symbols correspond to the numerical results obtained from the CTRW for different values of r ($r \neq 0.5$) and τ_0. Solid curve corresponds to the analytical result (25). Different values of the parameter r correspond to different values of the bias $\bar{F}$ and the periodic field rms σ, as described in Fig. 1. The field parameters do not influence the results within the statistical errors.

$$\frac{\bar{F}^2}{\eta_\alpha^2}\left[\frac{2}{\Gamma(2\alpha+1)} - \frac{1}{\Gamma^2(\alpha+1)}\right].$$

This coefficient is the same as in the case of the subdiffusive motion under the influence of a constant force if the value of the constant force would be $\bar{F}$.

Furthermore, similarly to the case of a constant bias, the asymptotic scaling relation holds ($r \neq 0.5$, $\bar{F} \neq 0$) between the mean-square displacement and average particle position. In the limit $t \to \infty$, the mean-square displacement grows as $\langle \delta x^2(t)\rangle \propto t^{2\alpha}$ and

$$\lim_{t\to\infty} \frac{\langle \delta x^2(t)\rangle}{\langle x(t)\rangle^2} = \frac{2\Gamma^2(\alpha+1)}{\Gamma(2\alpha+1)} - 1, \quad \bar{F} \neq 0. \tag{25}$$

It is illustrated by Fig. 4, where the analytical curve [Eq. (25)] is compared with the numerical results. The universality of relation (25) under the unbiased driving $\tilde{F}(t)$ means that the biased diffusion is not affected by the driving. This is in sharp contrast with the unbiased diffusion in Fig. 3.

IV. CONCLUSION

With this work, we presented the derivation of the FFPE (6) for a special class of space- and time-dependent force fields from the underlying CTRW picture. Our derivation shows along with the corresponding discussion that it is difficult to justify this equation for time-dependent forces different from $F(x,t) = F(x)\xi(t)$ with $\xi(t) = \pm 1$ beyond the linear-response approximation. Using the FFPE (6), we demonstrated that the universal scaling relation (25) for a biased subdiffusion is not affected by the additional action of a time-periodic zero-mean rectangular driving; neither is the asymptotic anomalous current nor the anomalous biased diffusion. We conjecture on physical grounds that this result is general and it is valid also for other forms of driving with zero-mean value. This driving immunity is due to the fact that the CTRW subdiffusion occurs in a random operational time which has no finite mean value, whereas any physical field changes in the real physical time. The CTRW-based subdiffusion fails to respond asymptotically to such time-dependent fields, while on its intrinsic random operational time scale any real alternating field is acting infinitely fast and it makes effectively no influence in a long run [16,17], unless the rate of its change is precisely zero. This is the main reason for the observed anomalies. The remarkable enhancement of the unbiased subdiffusion within the CTRW framework by time-periodic rectangular fields is rather the exception than the rule.

Finally, besides the case with amorphous semiconductors mentioned in Sec. I, the results presented in this work may as well be of use when investigating how various chemical, physical, and biological systems where an anomalous hydration phenomenon occurs respond to an external driving: indeed, hydrating water is known to behave in certain situations subdiffusively [26].

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APPENDIX A

Using the property $\mathcal{L}\{df(t)/dt\} = s\mathcal{L}\{f(t)\} - f(0)$ and assuming the initial conditions $\langle x(t)\rangle = 0$ and $\langle x^2(t)\rangle = 0$, we obtain from Eq. (19),

$$\mathcal{L}\{\langle x^2(t)\rangle\} = \frac{2\kappa_\alpha}{s^{\alpha+1}} + \frac{2v_\alpha}{s}\mathcal{L}\{\xi(t)\,{}_0\hat{D}_t^{1-\alpha}\langle x(t)\rangle\}. \tag{A1}$$

Considering that

$$\mathcal{L}\{{}_0\hat{D}_t^{1-\alpha}\langle x(t)\rangle\} = s^{1-\alpha}\mathcal{L}\{\langle x(t)\rangle\} = s^{-\alpha}\mathcal{L}\{d\langle x(t)\rangle/dt\},$$

one obtains,

$$\mathcal{L}\{\xi(t)\,{}_0\hat{D}_t^{1-\alpha}\langle x(t)\rangle\} = \sum_{n=-\infty}^{\infty} \frac{f_n}{(s - in\omega_0)^\alpha}\,\mathcal{L}_{s-in\omega_0}\left\{\frac{d\langle x(t)\rangle}{dt}\right\}, \tag{A2}$$

where the symbol $\mathcal{L}_{s-in\omega_0}$ denotes the corresponding Laplace transform at the shifted argument $s - in\omega_0$. Using Eq. (13) with Eq. (20), it follows:

$$\mathcal{L}_{s-in\omega_0}\left\{\frac{d\langle x(t)\rangle}{dt}\right\} = v_\alpha \sum_{m=-\infty}^{\infty} \frac{f_m}{[s - i\omega_0(n+m)]^\alpha}. \tag{A3}$$

Inserting Eqs. (A2) and (A3) into Eq. (A1), we obtain,

$$\mathcal{L}\{\langle x^2(t)\rangle\} = \frac{2\kappa_\alpha}{s^{\alpha+1}} + \frac{2v_\alpha^2}{s}\sum_{n=-\infty}^{\infty}\frac{f_n}{(s-in\omega_0)^\alpha} \times \sum_{m=-\infty}^{\infty}\frac{f_m}{[s-i\omega_0(n+m)]^\alpha}. \quad (A4)$$

In Eq. (A4) let us separate the terms $m=0$ and $n=0$,

$$\mathcal{L}\{\langle x^2(t)\rangle\} = \frac{2\kappa_\alpha}{s^{\alpha+1}} + \frac{2v_\alpha^2 f_0^2}{s^{2\alpha+1}} + \frac{2v_\alpha^2 f_0}{s^{\alpha+1}}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{f_m}{(s-im\omega_0)^\alpha} + \frac{2v_\alpha^2}{s}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{f_n}{(s-in\omega_0)^\alpha}\sum_{m=-\infty}^{\infty}\frac{f_m}{[s-i\omega_0(n+m)]^\alpha}. \quad (A5)$$

In the long-time limit, i.e., in the limit $s\to 0$, in the double sum only terms with $m=-n$ contribute, thus giving,

$$\mathcal{L}\{\langle x^2(t)\rangle\} \sim \frac{2\kappa_\alpha}{s^{\alpha+1}} + \frac{2v_\alpha^2 f_0^2}{s^{2\alpha+1}} + \frac{2v_\alpha^2 f_0}{\omega_0^\alpha s^{\alpha+1}}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{f_m}{(-im)^\alpha} + \frac{2v_\alpha^2}{\omega_0^\alpha s^{\alpha+1}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{|f_n|^2}{(-in)^\alpha}. \quad (A6)$$

Let us compute the sums. Considering that

$$\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{f_m}{(-im)^\alpha} = \sum_{m=1}^{\infty}\left[\frac{f_m}{(-im)^\alpha} + \frac{f_{-m}}{(im)^\alpha}\right],$$

and replacing here f_m from Eq. (21), one obtains,

$$\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{f_m}{(-im)^\alpha} = \frac{2}{\pi}\left\{\zeta(1+\alpha)\sin(\alpha\pi/2) - \sum_{m=1}^{\infty}\frac{\sin[(\alpha-4mr)\pi/2]}{m^{1+\alpha}}\right\} \equiv S_1, \quad (A7)$$

here $\zeta(x)$ is the Riemann's zeta function. Analogously,

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{|f_n|^2}{(-in)^\alpha} = \frac{4}{\pi^2}\cos(\alpha\pi/2)\left[\zeta(2+\alpha) - \sum_{n=1}^{\infty}\frac{\cos(nr2\pi)}{n^{2+\alpha}}\right]. \quad (A8)$$

Replacing these sums into Eq. (A6) and considering that $f_0=2r-1$ [see Eq. (21) together with Eq. (8)], we get,

$$\mathcal{L}\{\langle x^2(t)\rangle\} \sim \frac{2v_\alpha^2(2r-1)^2}{s^{2\alpha+1}} + \frac{2\kappa_\alpha}{s^{\alpha+1}} + \frac{2v_\alpha^2(2r-1)S_1}{\omega_0^\alpha s^{\alpha+1}} + \frac{8v_\alpha^2}{\pi^2\omega_0^\alpha s^{\alpha+1}}\cos(\alpha\pi/2) \times\left[\zeta(2+\alpha) - \sum_{n=1}^{\infty}\frac{\cos(nr2\pi)}{n^{2+\alpha}}\right]. \quad (A9)$$

Taking here the inverse Laplace transform, one obtains the expression for $\langle x^2(t)\rangle$ in the long-time limit [Eq. (22)].

APPENDIX B

Using Eq. (13), the quantity $\langle x(t)\rangle^2$ can be written in the following way:

$$\langle x(t)\rangle^2 = \frac{2v_\alpha^2}{\Gamma^2(\alpha)}\int_0^t dt'\int_0^{t'}\xi(t')t'^{\alpha-1}\xi(t'')t''^{\alpha-1}dt''. \quad (B1)$$

Exploiting the property $\mathcal{L}\{\int_0^t f(t')dt'\}=s^{-1}\mathcal{L}\{f(t')\}$ and denoting $t'=t$ and $t''=t'$, we can write,

$$\mathcal{L}\{\langle x(t)\rangle^2\} = \frac{2v_\alpha^2}{\Gamma^2(\alpha)s}\mathcal{L}\left\{\xi(t)t^{\alpha-1}\int_0^t\xi(t')t'^{\alpha-1}dt'\right\} = \frac{2v_\alpha^2}{\Gamma^2(\alpha)s}\sum_{n=-\infty}^{\infty}f_n\sum_{m=-\infty}^{\infty}f_m \times\mathcal{L}_{s-in\omega_0}\left\{t^{\alpha-1}\int_0^t t'^{\alpha-1}\exp(im\omega_0 t')dt'\right\}. \quad (B2)$$

For $\alpha=1$, the latter equation gives,

$$\mathcal{L}\{\langle x(t)\rangle^2\} = \frac{2v_\alpha^2}{s}\sum_{n=-\infty}^{\infty}\frac{f_n}{s-in\omega_0}\sum_{m=-\infty}^{\infty}\frac{f_m}{s-i\omega_0(n+m)}. \quad (B3)$$

Comparing this result with Eq. (A4) we see that $\mathcal{L}\{\langle\delta x^2(t)\rangle\}=2\kappa_\alpha/s^2$, and thus $\langle\delta x^2(t)\rangle=2\kappa_\alpha t$, as it should be for the normal Brownian motion.

For $0<\alpha<1$, it is more convenient to proceed as follows. Let us calculate the Laplace transform of $\langle x(t)\rangle$. Considering that $\mathcal{L}\{\langle x(t)\rangle\}=s^{-1}\mathcal{L}\{d\langle x(t)\rangle/dt\}$, one obtains [see Eq. (A3)],

$$\mathcal{L}\{\langle x(t)\rangle\} = v_\alpha\sum_{n=-\infty}^{\infty}\frac{f_n}{s(s-in\omega_0)^\alpha} = \frac{v_\alpha f_0}{s^{1+\alpha}} + v_\alpha\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{f_n}{s(s-in\omega_0)^\alpha}. \quad (B4)$$

In the limit $t\to\infty$, i.e., $s\to 0$, the latter equation becomes,

$$\mathcal{L}\{\langle x(t)\rangle\} = \frac{v_\alpha f_0}{s^{1+\alpha}} + \frac{v_\alpha}{s}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{f_n}{(-in\omega_0)^\alpha}. \quad (B5)$$

Taking into account Eq. (A7) we can write,

$$\mathcal{L}\{\langle x(t)\rangle\} = \frac{v_\alpha f_0}{s^{1+\alpha}} + \frac{v_\alpha S_1}{\omega_0^\alpha s}. \tag{B6}$$

Taking here the inverse Laplace transform we have for $t \to \infty$,

$$\langle x(t)\rangle = \frac{v_\alpha(2r-1)}{\Gamma(\alpha+1)} t^\alpha + \frac{v_\alpha S_1}{\omega_0^\alpha}. \tag{B7}$$

From here, one obtains the result (24) for $\langle x(t)\rangle^2$.

[1] J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[2] H. Scher, M. F. Shlesinger, and J. T. Bendler, Phys. Today **44** (1), 26 (1991); M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, Nature (London) **363**, 31 (1993); J. Klafter, M. F. Shlesinger, and G. Zumofen, Phys. Today **49**(2), 33 (1996).
[3] H. Scher and E. W. Montroll, Phys. Rev. B **12**, 2455 (1975).
[4] M. F. Shlesinger, J. Stat. Phys. **10**, 421 (1974).
[5] G. Bel and E. Barkai, Phys. Rev. Lett. **94**, 240602 (2005).
[6] G. Pfister and H. Scher, Adv. Phys. **27**, 747 (1978).
[7] B. D. Hughes, *Random Walks and Random Environments: Random Walks* (Clarendon, Oxford, 1995), Vol. 1.
[8] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000); R. Metzler, E. Barkai, and J. Klafter, Phys. Rev. Lett. **82**, 3563 (1999); E. Barkai, Phys. Rev. E **63**, 046118 (2001).
[9] I. Goychuk, E. Heinsalu, M. Patriarca, G. Schmid, and P. Hänggi, Phys. Rev. E **73**, 020101(R) (2006).
[10] E. Heinsalu, M. Patriarca, I. Goychuk, G. Schmid, and P. Hänggi, Phys. Rev. E **73**, 046133 (2006).
[11] H. Grabert, P. Hänggi, and P. Talkner, J. Stat. Phys. **22**, 537 (1980).
[12] P. Hänggi and H. Thomas, Phys. Rep. **88**, 207 (1982).
[13] H. C. Fogedby, Phys. Rev. E **50**, 1657 (1994).
[14] A. A. Stanislavsky, Phys. Rev. E **67**, 021111 (2003).
[15] M. Magdziarz, A. Weron, and K. Weron, Phys. Rev. E **75**, 016708 (2007).
[16] E. Heinsalu, M. Patriarca, I. Goychuk, and P. Hänggi, Phys. Rev. Lett. **99**, 120602 (2007).
[17] I. Goychuk, Phys. Rev. E **76**, 040102(R) (2007).
[18] I. Goychuk and P. Hänggi, Phys. Rev. E **69**, 021104 (2004).
[19] I. M. Sokolov, A. Blumen, and J. Klafter, Europhys. Lett. **56**, 175 (2001); I. M. Sokolov, A. Blumen, and J. Klafter, Physica A **302**, 268 (2001).
[20] See, e.g., J.-L. Déjardin and J. Jadzyn, J. Chem. Phys. **123**, 174502 (2005); M. Y. Yim and K. L. Liu, Physica A **369**, 329 (2006).
[21] I. M. Sokolov and J. Klafter, Phys. Rev. Lett. **97**, 140602 (2006).
[22] V. M. Kenkre, E. W. Montroll, and M. F. Shlesinger, J. Stat. Phys. **9**, 45 (1973); A. I. Burshtein, A. A. Zharikov, and S. I. Temkin, Theor. Math. Phys. **66**, 166 (1986); I. Goychuk, Phys. Rev. E **70**, 016109 (2004).
[23] G. H. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, Amsterdam, 1994).
[24] I. Goychuk, P. Hänggi, J. L. Vega, and S. Miret-Artes, Phys. Rev. E **71**, 061906 (2005).
[25] F. Barbi, M. Bologna, and P. Grigolini, Phys. Rev. Lett. **95**, 220601 (2005).
[26] A. R. Bizzarri and S. Cannistraro, J. Phys. Chem. B **106**, 6617 (2002); A. R. Bizzarri, A. Paciaroni, and S. Cannistraro, Phys. Rev. E **62**, 3991 (2000).