

# Entanglement swapping in presence of dephasing

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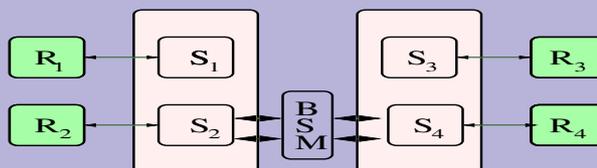
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Entanglement which originates from a pair of non-interacting qubits by means of entanglement swapping is discussed when dephasing mechanisms are present. Then, the exact reduced dynamics can be evaluated for two pairs of independent qubits which each are coupled to their own bosonic environment. Each of two pairs of qubits is initially maximally entangled. The conditions for the entanglement survival at asymptotic long times in terms of environment spectral properties are elaborated. It is shown that these conditions hold true also for the *inter-pair* entanglement stemming. Entanglement swapping can effectively be controlled by a proper choice of initial state of the *finite* environment.



Two non-interacting pairs of qubits,  $\{S_1, S_2\}$  and  $\{S_3, S_4\}$ , are in a separable state, wherein each pair of qubits is maximally entangled. The Bell state measurement (BSM) on the pair  $\{S_2, S_4\}$  creates the entanglement of the qubits  $S_1$  and  $S_3$ . The qubits are coupled to their own environments  $R_i$ ,  $i = 1, \dots, 4$ .

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**1 Introduction** The phenomenon of entanglement of quantum degrees of freedom occurring in open quantum systems has attracted considerable attention due to its prominent significance for both, fundamental physics and applications of quantum information processing [1]. The open system impacts the otherwise unitary time-evolution, giving rise to loss of coherence properties [2–5]. There have been proposed several procedures to sustain coherence and to minimize decoherence processes in such systems. E.g. the way how the system interacts with its outer environment decides about entanglement survival in the long time regime [6, 7]. In this work we consider one of the more challenging applications involving the concept of entanglement in quantum communication: the entanglement swapping [8]. This procedure, originally proposed in Ref. [9] has been developed both theoretically and experimentally in Refs. [10–13], allows to generate entanglement between qubits which neither come from the same source nor have they ever interacted before. In the following we focus on the (most) simple system composed of two pairs of non-interacting qubits which are coupled in a non-demolition

manner to their own, independent, environments which are either of finite or infinite (thermodynamic) size. Put differently, the two qubits undergo pure dephasing only with those surroundings. We demonstrate that this deteriorating coherence mechanism in such a system does not render entanglement swapping ineffective even at long asymptotic measurements times, provided that the environments possess a superohmic spectrum. We also propose a method of controlling the swapped entanglement by a proper choice of the initial preparation of the surroundings.

There exist several quantitative measures of entanglement degree of bipartite systems [14]. Two common such measures are the concurrence [15] and the negativity [16], which can uniquely be related to each other [17]. In this work, we employ as a measure for physical entanglement the negativity; i.e.,  $N(\rho) = \max(0, -\sum_i \lambda_i)$  [16], where  $\lambda_i$  denote the negative eigenvalues of the partially transposed density matrix  $\rho(t)$  of a pair of qubits [18]. For entangled mixed states, this negativity assumes positive values, whereas it identically vanishes for disentangled states.

**2 Entanglement swapping** Let us consider four qubits, denoted as  $\{S_1, S_2, S_3, S_4\}$ . Importantly, the four qubits do not interact with each other. At initial time of preparation  $t = 0$ , the two subsystems, formed by two pairs of qubits each, i.e.,  $A = \{S_1, S_2\}$  and  $B = \{S_3, S_4\}$  are prepared in the separable state

$$\rho(0) = \rho_A(0) \otimes \rho_B(0), \quad (1)$$

wherein each pair  $A$  and  $B$  of the qubits is maximally entangled,

$$\rho_k(0) = \frac{1}{2} (|01\rangle + |10\rangle) (\langle 01| + \langle 10|)_k, \quad (2)$$

$k = A, B.$

The notation  $|01\rangle$  means that for  $k = A$ , the qubit  $S_1$  is in the state  $|0\rangle$  and the qubit  $S_2$  is in the state  $|1\rangle$ . *Mutatis mutandis* for  $k = B$ : the qubit  $S_3$  is in the state  $|0\rangle$  and the qubit  $S_4$  is in the state  $|1\rangle$ . As there is no interaction acting between the pairs  $A$  and  $B$ , the separability remains preserved at any later time  $t > 0$ ,

$$\rho(t) = \rho_A(t) \otimes \rho_B(t). \quad (3)$$

Let at time  $t = \tau$  the entanglement swapping procedure being performed: it consists of projecting the system on the Bell state of the qubits  $\{S_2, S_4\}$ , i.e. measuring the system by means of the Bell state measurement [8]

$$\rho(\tau) \rightarrow \rho^{\text{swap}}(\tau) = \text{Tr}_{S_2, S_4} (|\Psi_{24}\rangle \langle \Psi_{24}| \rho(\tau)), \quad (4)$$

where

$$|\Psi_{24}\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)_{24} \quad (5)$$

is the symmetric Bell state of the qubits  $S_2$  and  $S_4$ . As a result, one can create an entanglement between the qubits  $S_1$  and  $S_3$  which in fact have never interacted with each other before. Because the density matrix  $\rho^{\text{swap}}(\tau)$  is a  $4 \times 4$  matrix, we can utilize the well-known, established methods to analyze the degree of entanglement, e.g. one can evaluate the negativity  $N(\rho^{\text{swap}})$ . To do so, we have to specify the dynamics (the Hamiltonian) of the system of four qubits.

**3 Hamiltonian and reduced dynamics** The system of four qubits consists of two independent and identical subsystems A and B and which are characterized by the total Hamiltonian

$$H = H_A + H_B, \quad (6)$$

$$H_A = H_1 + H_2, \quad H_B = H_3 + H_4.$$

Because the subsystems A and B are identical, we describe the subsystem A only.

The subsystem A consists of a pair of qubits, i.e.,  $A = \{S_1, S_2\}$ , and its Hamiltonian reads

$$H_1 = \omega_0 S_1^z + S_1^z \otimes \sum_{k=1}^{\infty} g_k (a_k^\dagger + a_k) + \sum_{k=1}^{\infty} \omega_k a_k^\dagger a_k \quad (7)$$

for the qubit  $S_1$  and

$$H_2 = \frac{1}{2} |0\rangle \langle 0| \otimes K_+ + \frac{1}{2} |1\rangle \langle 1| \otimes K_- \quad (8)$$

for the qubit  $S_2$ .

The qubit  $S_1$  is represented by the spin-1/2 operator  $S_1^z$ . It interacts with a heat bath  $R_1$  modeled by an infinite quasi-free reservoir, composed of bosonic harmonic oscillators of angular frequencies  $\omega_k$ , the operators  $a_k$  and  $a_k^\dagger$  are Bose annihilation and creation operators. The strength of the interaction between the qubit  $S_1$  and the  $k$ -th mode of the heat bath  $R_1$  is described by the coupling constant  $g_k$ .

The qubit  $S_2$  is coupled to its own environment  $R_2$  represented by a finite (or infinite) quantum system and the interaction is described in terms of the operators  $K_\pm$  which are elements of a Lie algebra  $\mathcal{G}$  generating the symmetry group  $G$  [19],

$$K_\pm = \sum_{k=1}^N h_\pm^k(t) X_k \pm \varepsilon_0, \quad [X_r, X_j] = \sum_l C_{rj}^l X_l, \quad (9)$$

where  $h_\pm^k(t)$  are scalar control functions and  $X_k$  are basis elements of the Lie algebra with the structural constants  $C_{rj}^l$ . For the qubit  $S_2$ , in its standard basis  $\{|0\rangle, |1\rangle\}$ , e.g.  $S_2^z = (|0\rangle \langle 0| - |1\rangle \langle 1|)/2$ . The energy levels of the qubit  $S_2$  are  $\varepsilon_0/2$  and  $-\varepsilon_0/2$ . The Hamiltonian for the subsystem  $B = \{S_3, S_4\}$  (consisting of two qubits  $S_3$  and  $S_4$ ) has the same form as for the subsystem A, provided that the qubit  $S_3$  is coupled to the heat bath  $R_3$  and the qubit  $S_4$  is coupled to the heat bath  $R_4$ .

The reduced dynamics of the qubits can be determined *exactly for arbitrary model parameters* [20–23] provided the initial state  $\rho(0)$  of the total system described by the Hamiltonian (6) can be factorized; namely,

$$\rho(0) = \rho_1 \otimes \rho_A(0) \otimes \rho_2 \otimes \rho_3 \otimes \rho_B(0) \otimes \rho_4. \quad (10)$$

We next assume that the state  $\rho_1$  of the heat bath  $R_1$  is an equilibrium Gibbs state of temperature  $T_1$  and the initial state of the environment  $R_2$  is any state of the form  $\rho_2 = |\Omega\rangle \langle \Omega|$ . The state  $\rho_3$  of the heat bath  $R_3$  is an equilibrium Gibbs state of temperature  $T_3$  and the initial state of the environment  $R_4$  is a similar state as that for  $R_2$ . For a time  $t > 0$ , the state  $\rho(t)$  of total system of four qubits assumes the form (cf. Eq. (5.19) in [20])

$$\rho(t) = \Lambda(t) \rho(0) = \Lambda_1(t) \otimes \Lambda_2(t) \otimes \Lambda_3(t) \otimes \Lambda_4(t) \rho(0), \quad (11)$$

where

$$\Lambda_n(t) \rho = C_n^{(1)}(t) \rho + 2C_n^{(2)}(t) [S_n^z, \rho] + 4C_n^{(3)}(t) S_n^z \rho S_n^z \quad (12)$$

for  $n = 1, 2, 3, 4$  and with  $\rho$  an arbitrary operator. The functions  $C_n^{(i)}(t); i = 1, 2, 3$ , read explicitly

$$\begin{aligned} C_n^{(1)}(t) &= \frac{1}{2}[1 + F_n(t) \cos \phi_n(t)], \\ C_n^{(2)}(t) &= \frac{1}{2}iF_n(t) \sin \phi_n(t), \\ C_n^{(3)}(t) &= \frac{1}{2}[1 - F_n(t) \cos \phi_n(t)]. \end{aligned} \quad (13)$$

For the qubits  $S_1$  and  $S_3$ , we then obtain the result [20]

$$\phi_1(t) = \phi_3(t) = \omega_0 t, \quad F_n(t) = \exp[-f_n(t)], \quad (14)$$

where the damping function

$$f_n(t) = \int_0^\infty d\omega \frac{J_n(\omega)}{\omega^2} \coth(\hbar\omega\beta_n/2)[1 - \cos \omega t] \quad (15)$$

for  $n = 1, 3$ . The parameters  $\beta_n = 1/k_B T_n$  and  $k_B$  is the Boltzmann constant. The spectral functions  $J_n(\omega)$  are assumed to take the form [23,24]

$$J_n(\omega) = \lambda_n \omega^{1+\mu_n} \exp(-\omega/\omega_n^c), \quad \mu > -1, \quad (16)$$

where the cut-off frequency  $\omega_n^c$  determines the largest energy scale of the heat bath  $R_n$ . The parameter  $\lambda_n$  is the coupling constant of the qubit  $S_n$  and  $R_n$ . The spectral exponent  $\mu_n$  characterizes low frequency properties of the heat baths  $R_n$ . According to the classification proposed in Ref. [24], the heat bath is called sub-Ohmic for  $\mu \in (-1, 0)$ , Ohmic for  $\mu = 0$  and super-Ohmic for  $\mu \in (0, \infty)$ . This classification shall be reflected in the dynamical properties of entanglement.

For the qubits  $S_2$  and  $S_4$ , we find with  $\rho_2 = |\Omega_2\rangle\langle\Omega_2|$  and  $\rho_4 = |\Omega_4\rangle\langle\Omega_4|$  the result for phase [25]

$$\begin{aligned} \phi_n(t) &= \varepsilon_0 t \\ &+ \arg \left[ \langle \Omega_n | T_n^\dagger(g_-^{(n)}(t)) T_n(g_+^{(n)}(t)) | \Omega_n \rangle \right] \end{aligned} \quad (17)$$

and the amplitude

$$F_n(t) = |\langle \Omega_n | T_n^\dagger(g_-^{(n)}(t)) T_n(g_+^{(n)}(t)) | \Omega_n \rangle| \quad (18)$$

for  $n = 2, 4$ , where  $T^{(n)}(\cdot)$  is a representation of the group  $G^{(n)}$  acting in the space of the controlling system  $R_n$  and functions  $g_\pm^{(n)}(t) \in G^{(n)}$  depend on the specific form given in eq. (9). A relevant example will be given below.

**4 Entanglement swapping in environments producing pure dephasing** The swapped entanglement, quantified by the negativity, for qubits with reduced dynamics (11)-(17) reads

$$N = N(\rho^{\text{swapped}})(\tau) = \frac{1}{2} F_1(\tau) F_2(\tau) F_3(\tau) F_4(\tau). \quad (19)$$

In the absence of coupling to the environments, when  $F_n = 1$  for  $n = 1, 2, 3, 4$ , there occurs maximal entanglement swapping with  $N = 1/2$ . The effect of the coupling to the environments producing pure dephasing is two-fold. The infinite baths cause dissipation of information. This effect

can either result in a full deterioration of the entanglement swapping, or only partially, i.e. such that the swapping is still effective (i.e. the qubits 1 and 3 become entangled) at arbitrary time instants of measurement.

First let us consider the case when the subsystems  $A$  and  $B$  are coupled to the infinite baths  $R_1$  and  $R_3$  only (i.e.  $F_2(\tau) = F_4(\tau) = 1$ ). For convenience, the baths have exactly the same characteristics, i.e.  $\lambda_1 = \lambda_3 = \lambda, \mu_1 = \mu_3 = \mu, \omega_1^c = \omega_3^c = \omega_c$ . We assume first that the baths operate at vanishing temperature, i.e.,  $T_1 = T_3 = 0$ . In order to make the work self-contained we quote here the formulas derived in [20]. For the Ohmic bath ( $\mu = 0$ ), one obtains

$$F_1(\tau) = F_3(\tau) = (1 + \omega_c^2 \tau^2)^{-\lambda/2}. \quad (20)$$

For the sub-Ohmic and super-Ohmic baths one finds instead

$$\begin{aligned} F_1(\tau) = F_3(\tau) &= \exp\{-\lambda\Gamma(\mu)\omega_c^\mu \\ &\times [1 - (1 + \omega_c^2 \tau^2)^{-\mu/2} \cos(\mu \arctan(\omega_c \tau))]\}, \end{aligned} \quad (21)$$

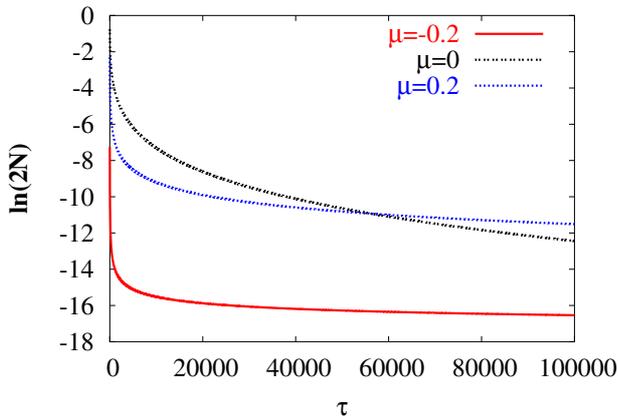
where  $\Gamma(z)$  is the Euler gamma function. One can deduce that for both Ohmic ( $\mu = 0$ ) and sub-Ohmic ( $\mu \in (-1, 0)$ ) reservoirs,  $F_1(\infty) = F_3(\infty) = 0$ . In consequence, the long-time negativity is zero,  $N = 0$ , and there occurs no entanglement swapping in the asymptotic long-time limit. For the super-Ohmic bath ( $\mu > 0$ ), however,

$$F_1(\infty) = F_3(\infty) = \exp(-\lambda\Gamma(\mu)\omega_c^\mu) \neq 0. \quad (22)$$

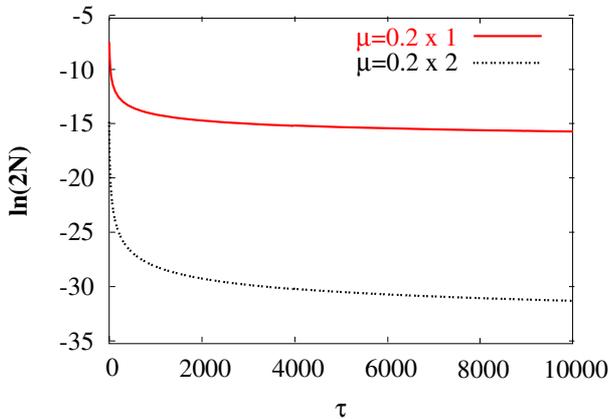
It follows that in this case the long-time negativity remains positive,  $N > 0$  and the information does not deteriorate completely, i.e., one can obtain a non-vanishing swapped entanglement also at  $\tau \rightarrow \infty$ , as depicted in Fig. 1.

When temperatures of the bosonic baths are non-zero,  $T_1 > 0, T_3 > 0$ , then the entanglement swapping can also be performed effectively still at asymptotic long times; but the environments need to be super-Ohmic with  $\mu > 1$  [25].

For the system considered so far, the negativity (and the swapped entanglement) is a monotonic decreasing function of measurement time  $t$ . The results are depicted with in Fig. 2. The saturation of the swapped entanglement at asymptotic large time is, for superohmic baths, readily observed. Any additional bath influences cause a decrease of the swapped entanglement which, remarkably, remains still finite provided the bath is superohmic. If one wants to manipulate the entanglement swapping in a desired way (e. g. via modulating or maintaining the entanglement in a desired interval), a control method of how to achieve this has to be devised. Below, we propose one of the possible scenarios how to decrease and increase in time  $t$  the entanglement swapping by controlling the dynamics by an external finite quantum system; i.e. by a proper choice of the controlling Hamiltonians  $H_2$  and  $H_4$ . As an example, we consider a control scheme with a single bosonic mode. This situation may typically occur for circuit-quantum electrodynamics (circuit-QED); see in Refs. [13,26]. Put differently, let in Eq. (9) for both  $S_2$  and  $S_4$ , the operators  $K_\pm$



**Figure 1** Entanglement swapped at the dimensionless time  $\tau = \omega_0 t$  in the system interacting with one bosonic bath ( $T = 0$ ) with different  $\mu$ , fixed  $\lambda = 1$  and  $\omega_c/\omega_0 = 10^3$ .



**Figure 2** Entanglement swapped performed at the dimensionless time  $\tau = \omega_0 t$  for the system interacting with one (solid line, red online) or two (dotted line or black online) super-Ohmic bosonic baths ( $T = 0$ ) with  $\mu = 0.2$ ,  $\lambda = 1$  and  $\omega_c/\omega_0 = 10^3$ .

be of the form

$$K_{\pm} = a^{\dagger} a \pm \gamma_{\pm} (a + a^{\dagger}) \pm 1. \quad (23)$$

In the following we limit ourselves to an isotropic coupling  $\gamma_{\pm} = \gamma$ . The corresponding functions  $F_2(t)$  and  $F_4(t)$  then read with

$$F_2(t) = F_4(t) = |\langle \Omega | D(\alpha(t)) | \Omega \rangle|, \quad (24)$$

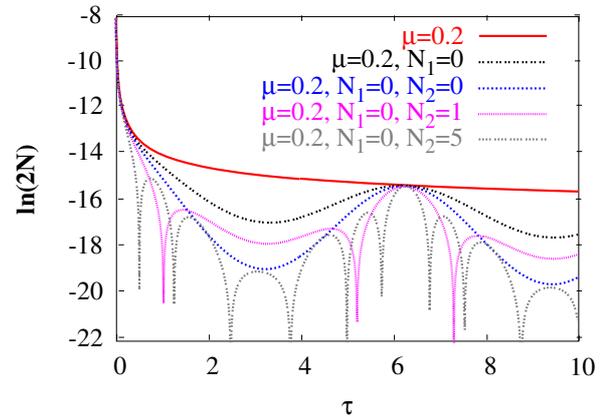
where

$$\alpha(t) = \gamma [1 - \exp(it)] \quad (25)$$

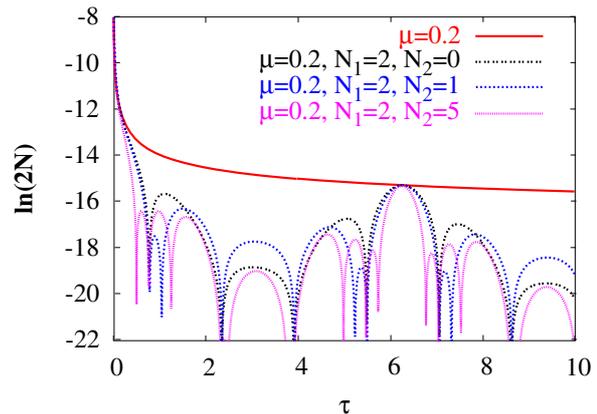
and  $D(z)$  is the displacement operator [19,27]

$$D(z) = \exp(za^{\dagger} - z^*a). \quad (26)$$

This operator generates the set of standard coherent states. The function  $F_2(t) = F_4(t)$  is the Weyl function studied



**Figure 3** Entanglement swapping at  $\tau = \omega_0 t$  in the system interacting with the single bosonic bath ( $T = 0$ ) with  $\mu = 0.2$  and none (solid line, red online), one (the case with  $N_1$  only) and two (the case with  $N_1$  and  $N_2$ ) finite controlling systems with  $\varepsilon_0 = \omega_0$  prepared in a number eigenstate either  $|N_1\rangle$  (one controlling system) or  $|N_1, N_2\rangle$  (two controlling finite systems), respectively.  $\lambda = 1$  and  $\omega_c/\omega_0 = 10^3$ .



**Figure 4** Entanglement swapped at  $\tau = \omega_0 t$  in the system interacting with the single bosonic bath ( $T = 0$ ) with  $\mu = 0.2$  and two finite controlling systems with  $\varepsilon_0 = \omega_0$  prepared in a number eigenstate  $|N_1, N_2\rangle$ .  $\lambda = 1$  and  $\omega_c/\omega_0 = 10^3$ .

intensively in the context of interference phenomena [28] and for mesoscopic devices controlled by non-classical external fields [29]. The effect of the control when applied in a particularly simple case with  $|\Omega\rangle$  being a number (Fock)-eigenstate, is shown in Figs. 3 and 4 for one and two controlling systems. The non-monotonic entanglement of the qubits swapped at the time  $\tau$  results from the non-Markovian properties of the evolving qubits. The only *non-local* operation performed on the system is the Bell state measurement. The time evolution of uncoupled qubits is clearly local and hence, in agreement with the common wisdom, cannot increase bipartite entanglement, which is never larger than the initial one. As the time homogeneity of

the non-Markovian system is broken the local operations related to the time evolution transform the reduced density matrices on the time interval  $t = 0 \rightarrow t = \tau$ . The resulting swapped entanglement can be effectively designed: one can predict the qubit–qubit correlations by a proper choice of the initial state of the controlling boson. Such a method can be of great importance when the measurement time  $\tau$  cannot easily and precisely be adjusted. There is an obvious advantage of using several controlling bosons instead of a single harmonic oscillator coupled to a single qubit: the rich structure of control scenarios can be maintained with the help of only lowest excitations with small  $N$ .

**5 Conclusions** In summary, we studied the problem of origination of qubit–qubit entanglement via entanglement swapping when the qubits are interacting with a surrounding environment producing dephasing onto the qubit dynamics. It has been shown that under certain conditions this procedure still remains effective even for asymptotically long times  $t$ : in particular, the decoherence of entanglement information is asymptotically not entirely destroyed for superohmic environments at non-vanishing, finite temperatures. The effective control by a suitable initial preparation of a controlling finite quantum system (e.g. via a coupling to a single mode bosonic oscillator) can be performed, leading to oscillatory-like behavior of the entanglement obtained by a swapping measurement.

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