Two-mode Bose-Einstein condensate in a high-frequency driving field that directly couples the two modes

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A two-mode Bose-Einstein condensate coupled by a high-frequency modulation field is found to display rich features. An effective stationary Hamiltonian approach reveals the emergence of additional degenerate eigenstates as well as additional topological structures of the spectrum. Possible applications, such as the suppression of nonlinear Landau-Zener tunneling, are discussed. An interesting phenomenon, which we call "deterministic symmetry-breaking trapping" associated with separatrix crossing, is also found in an adiabatic process.

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I. INTRODUCTION

Significant research efforts have been devoted to the nonlinear dynamics of interacting cold atoms, e.g., a Bose-Einstein condensate (BEC) in an external driving field. One main motivation is to understand how we can actively control the nonlinear dynamics and how the self-interaction of cold atoms can be used to simulate fundamental models [1]. Examples include the control of the BEC self-trapping [2–4], effective turning off of the self-interaction [5], controlled Mott-insulator transitions associated with a BEC in an optical lattice [6], stabilization of bright BEC solitons by an oscillating magnetic field tuned close to the Feshbach resonance [7], as well as production of ultracold molecules using stimulated Raman adiabatic passage [8,9]. On a deeper level, BEC systems offer a useful tool to explore more aspects of many-body systems. In particular, the dynamics of a BEC in the large-particle-number limit is described by a mean-field nonlinear Schrödinger equation (Gross-Pitaeviskii equation). The resulting nonlinearity often challenges existing theories for linear systems. For example, the adiabatic following of a two-mode nonlinear system with an external field may necessarily break down [10].

Here we aim to examine how an adiabatic Landau-Zener (LZ) tunneling process of a BEC may be manipulated by an external driving field, thus extending an earlier study for linear systems [11]. As a second motivation on a more fundamental level, we shall expose some nonlinear dynamics phenomena that do not exist in the mean-field dynamics of a BEC under field-free conditions. Specifically, we consider a two-mode BEC under a high-frequency field that directly couples the two modes (hence called "off-diagonal" driving below). A biased static field is also considered for LZ processes.

It is well-known that a high-frequency "diagonal modulation" (e.g., high-frequency tilting of a double-well potential) can only rescale the natural two-mode coupling strength [6,12–14]. However, our findings below for a different type of high-frequency modulation are different. In particular, we show (i) that a high-frequency off-diagonal driving field can induce, in effect, additional nonlinear terms in the mean-field equations of motion, thus offering a control "knob" to tune different nonlinear terms and simulate some systems not considered before; (ii) that different topological structures of the eigenspectrum of the system can be generated and tuned by the driving field, also leading to additional degenerate eigenstates. We then suggest using a high-frequency driving field to realize the complete suppression of nonlinear LZ tunneling. In analyzing the adiabatic dynamics of a driven nonlinear system, we also find and qualitatively explain a phenomenon, called "deterministic symmetry-breaking trapping" associated with separatrix crossing.

II. TWO-MODE SYSTEM UNDER OFF-DIAGONAL MODULATION

The nonlinear two-mode system under off-diagonal modulation is described by

$$H(t)=\frac{1}{2}\begin{pmatrix}\gamma+c(|b|^2-|a|^2) & \Delta_0+A\sin(\omega t)\\ \Delta_0+A\sin(\omega t) & -\gamma-c(|b|^2-|a|^2)\end{pmatrix}, \qquad (1)$$

where γ denotes an external energy bias, $|a|^2$ and $|b|^2$ represent occupation probabilities for the two modes, c characterizes the nonlinear atom-atom interaction, and Δ_0 denotes the static coupling between the two modes. We put $\hbar=1$ throughout this work. There are a number of possibilities to experimentally realize this Hamiltonian. For example, one may consider a BEC in a double-well potential, with the height of the potential barrier periodically modulated, or a BEC in an optical lattice occupying two bands, with the well depth of the optical lattice periodically modulated. In principle, these procedures should be achievable, considering previous experiments on two-mode BEC's [16,18]. What might be even more feasible in realizing this two-mode system under off-diagonal modulation is to consider the internal states of a BEC, such as ^{87}Rb [17], where there exist two internal states separated by a relatively large hyperfine energy. Then, the energy bias γ can be effectively realized by

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the detuning of the coupling field from the resonance and the off-diagonal modulation may be achieved by modulating the intensity of the coupling field. Considering recent studies of two-mode nonlinear Schrödinger equations using nonlinear optical waveguides (for example, see Ref. [4]), it might also be possible to realize our system in nonlinear optics.

Consider first the nondriven case, i.e., $A=0$. Then $H(t)$ reduces to the standard model of nonlinear LZ tunneling [10]. Therein the eigenspectrum diagram as a function of γ is known to display a loop structure at the tip of the lower (upper) level for $c>\Delta_0$ ($c<-\Delta_0$). Such a loop structure, absent in linear systems, directly leads to a nonzero LZ transition probability even when γ changes adiabatically. As shown below, different system properties emerge if the driving field is turned on. Without loss of generality we will restrict ourselves to the $c>0$ case, which requires an attractive interaction for bosons in a double-well potential or a repulsive interaction for bosons in two energy bands of an optical lattice.

In the general case of $A\neq 0$ with $\omega \gg \gamma, c, \Delta_0$, it is found that $|a|^2$ $(=1-|b|^2)$ also oscillates at the frequency ω. To expose possibly new physics hidden in the oscillations, another pair of wave function parameters (a', b') are found to be very useful, i.e.,

$$a' = \frac{a+b}{2}e^{-i(A/2\omega)\cos(\omega t)} + \frac{a-b}{2}e^{i(A/2\omega)\cos(\omega t)},$$

$$b' = \frac{a+b}{2}e^{-i(A/2\omega)\cos(\omega t)} - \frac{a-b}{2}e^{i(A/2\omega)\cos(\omega t)}. \qquad (2)$$

Their equations of motion are given by

$$i\frac{da'}{dt} = \frac{1}{2}[\gamma\cos(\theta) + c\cos^2(\theta)(|b'|^2 - |a'|^2) - ic\sin(\theta)\cos(\theta) \times (a'^*b' - a'b'^*)]a' + \frac{1}{2}[\Delta_0 + i\gamma\sin(\theta) + c\sin^2(\theta) \times (a'^*b' - a'b'^*) - ic\sin(\theta)\cos(\theta)(|a'|^2 - |b'|^2)]b'],$$

$$i\frac{db'}{dt} = \frac{1}{2}[\Delta_0 - i\gamma\sin(\theta) - c\sin^2(\theta)(a'^*b' - a'b'^*) + ic\sin(\theta)\cos(\theta)(|a'|^2 - |b'|^2)]a' + \frac{1}{2}[-\gamma\cos(\theta) - c\cos^2(\theta)(|b'|^2 - |a'|^2) + ic\sin(\theta)\cos(\theta)(a'^*b' - a'b'^*)]b', \qquad (3)$$

where $\theta \equiv \frac{A}{\omega}\cos(\omega t)$. Along with previous studies that focused on high-frequency driving fields [6,12–14], we consider now sufficiently large ω such that the oscillation in θ is much faster than the natural time scale of the system as characterized by Δ_0, γ, and c (numerically, we find that the regime of $\omega > 10\Delta_0$, $\omega > 10c$, and $\omega > 10\gamma_0$, where γ_0 is the initial value of $|\gamma|$ that is sufficiently large to ensure LZ dynamics, can be safely regarded as a high-frequency regime; experimentally, a high-frequency driving field should not interfere with the two-mode descriptions). Then Eq. (3) can be significantly reduced by considering the averages of (a', b') over $2\pi/\omega$. Speaking more rigorously, with the large-frequency condition, a zeroth-order approximation of a "$1/\omega$" expansion can be used to yield

$$i\frac{da'}{dt} = \frac{1}{2}[\gamma' + c_Z(|b'|^2 - |a'|^2)]a' + \frac{1}{2}[\Delta_0 + c_Y(a'^*b' - a'b'^*)]b',$$

$$i\frac{db'}{dt} = \frac{1}{2}[\Delta_0 - c_Y(a'^*b' - a'b'^*)]a' + \frac{1}{2}[-\gamma' - c_Z(|b'|^2 - |a'|^2)]b', \qquad (4)$$

where $\gamma' = \gamma\langle\cos(\theta)\rangle_T = \gamma J_0(A/\omega)$ [13], $c_Z = c\langle\cos^2(\theta)\rangle_T = c(\frac{1+J_0(2A/\omega)}{2})$, and $c_Y = c\langle\sin^2(\theta)\rangle_T = c(\frac{1-J_0(2A/\omega)}{2})$ (J_0 is the zeroth-order Bessel function of the first kind). Evidently, these newly defined parameters reflect the action of the high-frequency driving field. We stress that the validity of this kind of high-frequency approximation has been checked numerically and has been used in many situations.

Equation (4) no longer explicitly contains a time-dependent field. We can then define an effective static Hamiltonian H_{eff} that generates Eq. (4). That is,

$$H_{\text{eff}} = \frac{1}{2}\begin{pmatrix} \gamma + c_Z(|b|^2 - |a|^2) & \Delta_0 + c_Y(a^*b - ab^*) \\ \Delta_0 - c_Y(a^*b - ab^*) & -\gamma - c_Z(|b|^2 - |a|^2) \end{pmatrix}, \qquad (5)$$

where, for simplicity, we have replaced a' by a, b' by b, and so on. If we compare H_{eff} with the original Hamiltonian in Eq. (1) for $A=0$, we see that the nonlinear parameter c_Z can be regarded as a rescaled parameter c, and the nonlinear term containing c_Y is new. In addition, the ratio of c_Z and c_Y is given by $[1+J_0(2A/\omega)]/[1-J_0(2A/\omega)]$, easily adjustable by choosing different ω and A.

The effective Hamiltonian in Eq. (5) can be recognized as the one describing a single spin in a biaxial crystal field, with the c_Y (c_Z) term describing the anisotropy in the Y (Z) direction. Indeed, H_{eff} can also be written as $H_{\text{spin}} = \gamma S_Z + \Delta_0 S_X - c_Z S_Z^2 - c_Y S_Y^2$, where $S_Z = \frac{|a|^2 - |b|^2}{2}$, $S_X = \frac{a^*b + b^*a}{2}$, and $S_Y = \frac{a^*b - b^*a}{2i}$. The corresponding second-quantization Hamiltonian exactly describing the quantum system with N bosons on the two modes is given by

$$\hat{H}_Q = \gamma\frac{(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})}{2} + \Delta_0\frac{(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger)}{2} - \frac{c_Z}{N}\left(\frac{\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}}{2}\right)^2 + \frac{c_Y}{N}\left(\frac{\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a}}{2}\right)^2. \qquad (6)$$

III. DETAILED RESULTS

We now present in Fig. 1 the eigenspectrum of H_{eff} as a function of γ. Evidently, the typical level structures (such as the loop structure) for a nonlinear LZ tunneling model [10] are also possessed by our system. On top of that, additional

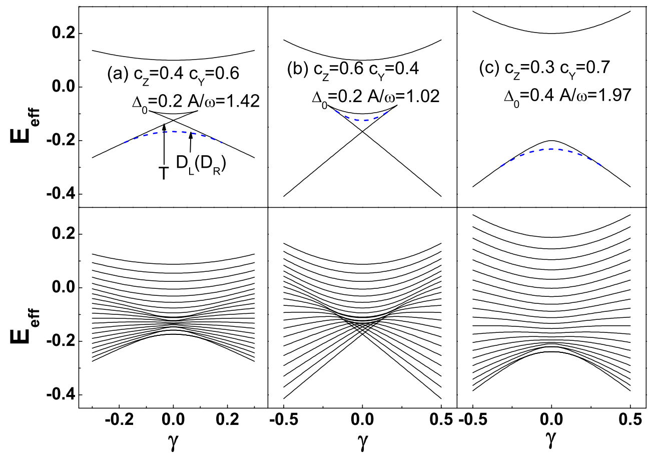

FIG. 1. (Color online) Upper panels: Level structures of the stationary effective Hamiltonian H_{eff} [see Eq. (5)], as a function of γ. The dashed lines denote the new mean-field levels that are absent in a nondriven two-mode BEC. The symbols T, D_R, and D_L indicate how the involved levels are connected with the phase space structures shown in Fig. 2. Bottom panels: Parallel results in a fully quantum treatment for $N=20$.

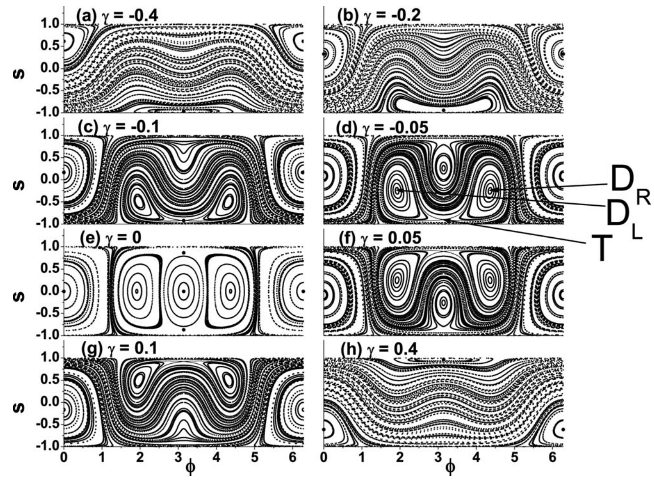

FIG. 2. Phase space structures of H_c defined in Eq. (7), for $\Delta_0=0.2$, $c_Z=0.4$, and $c_Y=0.6$ (or equivalently, $c=1$, $A/\omega=1.42$). T denotes an unstable fixed point with $\phi=\pi$. D_R and D_L denote two stable fixed points one on the right and one on the left.

mean-field eigenstates (dashed lines) that are absent in a nondriven case also emerge through level bifurcations. The new eigenstates are directly caused by the c_Y term induced by the driving field. In particular, if a loop structure exists and if $c_Y<c_Z$, then the additional level lies inside the loop, as shown in Fig. 1(b); and if $c_Y>c_Z$ and $c_Y>\Delta_0$, then level bifurcation takes place on the lowest branch and the additional level can be below the loop structure, as shown in Fig. 1(a). Figure 1(c) shows that the additional level may also exist in the absence of a loop structure. In the bottom panels of Fig. 1, we also show fully quantum mechanical levels calculated from Eq. (6) for $N=20$. The results confirm that the additional eigenstates we obtain on the mean-field level do have physical implications for fully quantum levels, even in the cases with a not very large N.

Let us now examine Eq. (4) from a phase space perspective, by mapping the mean-field trajectory of Eq. (4) to that of a well-defined classical Hamiltonian system. The associated phase space can be defined in terms of s and ϕ, where $\phi=\phi_b-\phi_a$ [14,15], $s=|b|^2-|a|^2$, with $a=|a|e^{i\phi_a}$ and $b=|b|e^{i\phi_b}$. Using this pair of canonical variables, the classical Hamiltonian involved is

$$H_c=\frac{1}{2}\left(-\gamma s-\frac{c_Z}{2}s^2+\Delta_0\sqrt{1-s^2}\cos(\phi)-\frac{c_Y}{2}(1-s^2)\sin^2(\phi)\right). \tag{7}$$

The nonlinear eigenstates of H_{eff} now become fixed points in the phase space of H_c. Figure 2 displays phase space portraits of H_c for the parameters used in Fig. 1(a), for several values of γ covering the regime of level bifurcation. In particular, the lower parts of Figs. 2(b)–2(d) near $\phi=\pi$ clearly show the splitting of one fixed point into three fixed points, thus associating the level bifurcation in Fig. 1 with the splitting of a fixed point. Because both the elliptic (stable) fixed points (marked by D_R and D_L in Fig. 2) yield the dashed line in Fig. 1(a), the additional level shown in Fig. 1 in fact denotes twofold-degenerate eigenstates. By contrast, the hyperbolic (unstable) fixed point marked by T in Fig. 2(d) yields the level right above the degenerate eigenstates [also marked by T in Fig. 1(a)]. In Figs. 2(f) and 2(g), the above-mentioned three fixed points start to merge back to one fixed point, in parallel with the level merging seen in Fig. 1(a) as γ increases further. On examining the phase space globally, it is also clear that the number of the fixed points and hence the number of the nonlinear eigenstates of H_{eff} can vary from two to six, a clear sign that the nonlinear dynamics of a driven BEC can be very rich.

It should also be noted that the above-mentioned twofold degeneracy occurs in a high-dimensional parameter space. In particular, for fixed nonlinear parameter c and fixed field parameters A and ω, the degeneracy can still occur in a two-parameter space of Δ_0 and γ. This is in contrast to the well-studied nondriven model of a two-mode BEC where degeneracy occurs only along a line for fixed c, A, and ω.

So, how does the additional eigenstate shown in Fig. 1(a) [the cases in Figs. 1(b) and 1(c) are physically less appealing] affect the adiabatic dynamics? To answer this question we numerically solve Eq. (4) for the parameters used in Fig. 1(a), with the initial state put on the lowest level. As γ increases very slowly, the system's state is found to follow the nondegenerate lowest level up to the bifurcation point. When γ increases beyond the "phase transition" point where the new twofold-degenerate level emerges, the twofold-degenerate level becomes the lowest and the system is found to move along the new level. As γ increases further, the twofold degenerate level finally disappears and the system reaches the nondegenerate lowest level again, thus completing the LZ process. During the entire process, the system remains at the lowest level available and no transitions to any upper levels are found. Hence, the new twofold-degenerate level induced by the driving field offers a means to circumvent the loop structure and hence totally suppress the nonlinear LZ transition that is doomed to happen if the twofold-degenerate state is not there. To connect this observation of totally suppressed LZ tunneling in the (a',b') representation [see Eq. (2)] with the direct observable (a,b) in experiments, note that (i) initially if $A=0$ then $a=a'$ and $b=b'$, and (ii) if A is switched on slowly enough as compared with ω but fast enough as compared with the change rate of (a',b') (characterized by γ, Δ_0, and c), then the initial values of (a,b) are passed to (a',b').

One more aspect of the above nonlinear LZ process remains to be examined. Because the dashed line in Fig. 1(a) denotes a twofold-degenerate eigenstate, we should study which state the system will reside in when it slowly pass the level bifurcation point with an increasing γ. Since the phase space structure shown in Fig. 2 always possesses a mirror symmetry with respect to $\phi=\pi$, one may intuitively expect that during the LZ process the system is trapped by either of the two stable points D_R or D_L in a random fashion, with equal probability. However, we find that this picture is incorrect here. Instead, the system is found to be deterministically trapped by D_R [see Fig. 2(d)]. Physically, this deterministic trapping means that the relative phase between a' and b' is not random during the LZ process, i.e., it is robust to small fluctuations in the initial state. A careful analysis enables us to explain this intriguing observation qualitatively. As γ increases, the fixed point with $\phi=\pi$ [see the lower parts of Figs. 2(c) and 2(d)] moves upward in the phase space. When this fixed point becomes unstable [denoted T in Fig. 2(d)], the adiabatic following must break down and hence the actual trajectory will find itself slightly below the up-moving fixed point. As such, the trajectory starts to slowly move counterclockwise around a separatrix, or from left to right, as illustrated in Fig. 3. At the same time, because γ is increasing, the separatrix deforms and swells, and as a result the trajectory necessarily crosses the separatrix on the right and hence gets trapped by D_R. Indeed, if we reverse the adiabatic process, i.e., pass the bifurcation point with a decreasing γ, then one can predict that the system will be trapped by D_L, a prediction confirmed numerically. This counterintuitive deterministic symmetry-breaking trapping is complementary to a well-known separatrix-crossing-induced phenomenon, i.e., quasirandom trapping in classical mechanics (which has been systematically applied to a few BEC systems [9]). Encouraged by the finding here and in efforts to confirm its generality, we also studied the adiabatic following dynamics of a modified rotating pendulum system whose fixed point moves with an external parameter. Analogous results are also found in this pendulum case. Hence, it should be of some interest to carry out an experimental BEC study of the symmetry-breaking separatrix crossing observed here. This kind of experiment might also offer additional insights into the validity of the mean-field description of a BEC.

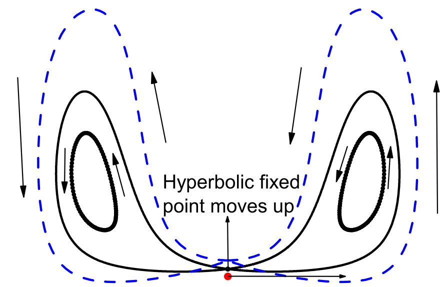

FIG. 3. (Color online) A classical trajectory falling below an up-moving unstable fixed point evolving counterclockwise around the associated separatrix (solid line). The separatrix for a slightly larger γ is represented by the dashed line. The shown trajectory is moving from left to right and will cross the swelling separatrix on the right and get trapped by the stable fixed point on the right.

IV. CONCLUSION

To conclude, we have theoretically examined the dynamics of a two-mode BEC driven by a high-frequency driving field that directly couples the two modes. Based on our results here we expect rich phenomena in general for a multimode BEC under a high-frequency driving field. Though our results are purely theoretical, it is our hope that the results here will stimulate future experiments on the dynamics of a BEC in high-frequency driving fields and on the control of nonlinear LZ tunneling dynamics.

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[1] O. Morsch and M. Oberthaler, Rev. Mod. Phys. **78**, 179 (2006).
[2] M. Albiez, R. Gati, J. Folling, S. Hunsmann, M. Cristiani, and M. M. Oberthaler, Phys. Rev. Lett. **95**, 010402 (2005).
[3] M. Holthaus, Phys. Rev. A **64**, 011601(R) (2001); C. Weiss and T. Jinasundera, *ibid.* **72**, 053626 (2005).
[4] X. Luo, Q. Xie, and B. Wu, Phys. Rev. A **76**, 051802(R) (2007).
[5] D. Jaksch, J. I. Cirac, and P. Zoller, Phys. Rev. A **65**, 033625 (2002).
[6] A. Eckardt, C. Weiss, and M. Holthaus, Phys. Rev. Lett. **95**, 260404 (2005); C. E. Creffield and T. S. Monteiro, *ibid.* **96**, 210403 (2006).
[7] H. Saito and M. Ueda, Phys. Rev. Lett. **90**, 040403 (2003); F. K. Abdullaev, J. G. Caputo, R. A. Kraenkel, and B. A. Malomed, Phys. Rev. A **67**, 013605 (2003).
[8] H. Pu, P. Maenner, W. Zhang, and H. Y. Ling, Phys. Rev. Lett. **98**, 050406 (2007); P. D. Drummond, K. V. Kheruntsyan, D. J.

Heinzen, and R. H. Wynar, Phys. Rev. A **65**, 063619 (2002).
[9] A. P. Itin and S. Watanabe, Phys. Rev. E **76**, 026218 (2007).
[10] B. Wu and Q. Niu, Phys. Rev. A **61**, 023402 (2000); J. Liu, B. Wu, and Q. Niu, Phys. Rev. Lett. **90**, 170404 (2003).
[11] M. Wubs *et al.*, New J. Phys. **7**, 218 (2005).
[12] F. Grossmann, T. Dittrich, P. Jung, and P. Hanggi, Phys. Rev. Lett. **67**, 516 (1991).
[13] Y. Kayanuma and Y. Mizumoto, Phys. Rev. A **62**, 061401(R) (2000).
[14] G. F. Wang, L. B. Fu, and J. Liu, Phys. Rev. A **73**, 013619 (2006).
[15] J. Liu, L. Fu, B. Ou, S. Chen, D. Choi, B. Wu, and Q. Niu, Phys. Rev. A **66**, 023404 (2002).
[16] Y. Shin, M. Saba, T. A. Pasquini, W. Ketterle, D. E. Pritchard, and A. E. Leanhardt, Phys. Rev. Lett. **92**, 050405 (2004); M. Albiez, R. Gati, J. Folling, S. Hunsmann, M. Cristiani, and M. K. Oberthaler, *ibid.* **95**, 010402 (2005).
[17] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, M. J. Holland, J. E. Williams, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **83**, 3358 (1999).
[18] M. Jona-Lasinio, O. Morsch, M. Cristiani, N. Malossi, J. H. Muller, E. Courtade, M. Anderlini, and E. Arimondo, Phys. Rev. Lett. **91**, 230406 (2003).