Activation rates in dispersive optical bistability with amplitude and phase fluctuations: A case without detailed balance

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For the two-dimensional model of dispersive optical bistability put forward by Graham and Schenzle [Phys. Rev. A 23, 1302 (1981)] the activation rates of the metastable states at low noise are evaluated explicitly. The rates are calculated in terms of the mean first-passage time of a two-variable Fokker-Planck equation which does not obey detailed balance and which has a drift field not expressable as the gradient of the corresponding nonequilibrium potential. The forward rate, describing the transition from the state with low transmission to the state with high transmission, is exponentially decreased with increasing detuning δ^2, whereas the backward rate is exponentially enhanced with increasing δ^2. The prefactor of the rate exhibits a complicated dependence on the detuning parameter which for the absorptive case with zero detuning, $\delta=0$, reduces to the familiar Kramers result of a Fokker-Planck system with a drift field derivable from a potential.

I. INTRODUCTION

The phenomenon of optical bistability of a Fabry-Pérot étalon, driven by an external monochromatic field, is observed if the étalon is being filled with an optically nonlinear medium which possesses a strong and fast nonlinear response to the applied field. Two extreme cases can be distinguished: For the absorptive bistability,[1,2] the frequency of the driving field coincides with both an absorption frequency of the nonlinear medium and a resonance frequency of the Fabry-Pérot étalon. For dispersive bistability,[1–4] one detunes the Fabry-Pérot étalon and drives the nonlinear medium off resonance.

In the presence of fluctuations, there occur transitions between the two locally stable states of the transmitted field at random times. The sojourn time between successive transitions presents a basic quantity of this bistable system. The inverse of the mean sojourn time determines the decay rate of a metastable state. The mean sojourn time itself is given in terms of the mean first passage time for a realization of leaving the corresponding domain of attraction. Because the optical field has at the boundary of the domain of attraction an equal chance either to return to its previous point of stability or to switch into the new point of stability, the mean sojourn time equals twice the mean first passage time.

By neglecting phase fluctuations of the optical field, the rate of decay of a metastable state was investigated previously.[5,6] In this approximation the amplitude dynamics can adequately be described by a single-variable Fokker-Planck equation. In this case, the mean first passage time can be obtained in closed form by exact integrations.[7] Including the phase fluctuations, the stochastic dynamics can be described by a two-variable Fokker-Planck equation[1,8] which does not exhibit detailed balance, in general.

For the problem of evaluation of the transition rate, the work by Kramers[9] for Brownian motion in a potential represents a milestone. Therein, the rate was calculated from a current-carrying solution of the corresponding stationary two-variable Fokker-Planck equation. For a large class of multidimensional thermal equilibrium Fokker-Planck processes, the method of Kramers has been generalized by Landauer and Swanson[10] and by Langer.[11] A discussion of a special class of two-dimensional nonpotential Fokker-Planck systems was recently presented by Gardiner.[12] For one-variable systems, the path-integral approach has also been utilized.[13] A different approach is the direct calculation of the mean lifetime of the metastable state at weak noise in terms of the mean first passage time rather than of the rate. In this case, the problem of evaluating the activation rate can be formulated without relying on special assumptions about the nature of the metastable state or the nature of exit points from the domain of attraction.

Matkowsky and Schuss[14] investigated a related problem and obtained a lower bound for the mean first passage time. However, the lifetime of the metastable state at low noise has not been obtained in Ref. 14, as claimed by Matkowsky and Schuss later on.[15] The general problem of calculating the lifetime of metastable states in multivariable Fokker-Planck systems has been treated by Talkner and Ryter.[16] Unfortunately, the original work[16] contained a minor mistake which was corrected in Ref. 17 and has also been noted in Ref. 15.

The paper is organized as follows. For the sake of completeness we review in Sec. II the results of Ref. 17 as needed for the case of a two-dimensional Fokker-Planck system. In Sec. III, a two-variable model for the optical bistability is presented and the transition rates are calculated explicitly.

II. ACTIVATION RATE OF A METASTABLE STATE AT LOW NOISE

Let us consider a dynamical two-variable system whose deterministic motion is given by the two coupled nonlinear first-order differential equations

$$\dot{x}_i = K_i(\vec{x}), \quad i=1,2 . \tag{2.1}$$

Suppose further that the system possesses a stable attracting point $\hat{x}$ within a bounded domain Ω of attraction. This is the situation tacitly assumed in Refs. 16 and 17. Later it will be shown how the assumption of a bounded domain of attraction can be relaxed. If the system under consideration is perturbed by white Gaussian noise, the sojourn time within Ω is generally finite—even in presence of arbitrarily weak noise.

The perturbed system is described by a probability $p_t(\vec{x})$ whose time evolution is governed by the Fokker-Planck operator L,

$$L=-\frac{\partial}{\partial x_1}K_1(\vec{x})-\frac{\partial}{\partial x_2}K_2(\vec{x})+\tfrac{1}{2}Q\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right], \tag{2.2}$$

with the adjoint operator $L^\dagger$,

$$L^\dagger=K_1(\vec{x})\frac{\partial}{\partial x_1}+K_2(\vec{x})\frac{\partial}{\partial x_2}+\tfrac{1}{2}Q\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right]. \tag{2.3}$$

Hereby we have assumed that the noise is additive and isotropic, i.e., the noise is characterized by a single diffusion constant Q.

The mean time $t(\vec{x})$ at which a trajectory starting at a point $\vec{x}$ in Ω reaches the boundary $\partial\Omega$ of Ω for the first time is given by[7,12–17]

$$L^\dagger t(\vec{x})=-1 \quad \text{for } \vec{x} \text{ in } \Omega \tag{2.4}$$

and

$$t(\vec{x})=0 \quad \text{for } \vec{x} \text{ on } \partial\Omega\ .$$

If one integrates (2.4) over Ω with an integrable solution $p(\vec{x})$ of the stationary Fokker-Planck equation

$$Lp(\vec{x})=0\ , \tag{2.5}$$

one obtains by use of the Gaussian theorem

$$\frac{Q}{2}\int_{\partial\Omega}p(\vec{x})\vec{\nabla}t(\vec{x})\cdot d\vec{S}=-\int_{\Omega}p(\vec{x})d^2x\ , \tag{2.6}$$

where $d\vec{S}$ denotes the oriented surface element on $\partial\Omega$.

In the limit of small noise ($Q\to 0$), a trajectory starting within Ω will typically first approach the attractor and stay around its neighborhood for a long time as compared with the time constants of the deterministic motion, until an occassional fluctuation drives the system towards the boundary. Hence, the mean absorption time $t(\vec{x})$ assumes the same large value T everywhere in Ω except for a thin layer $\Delta\Omega$ along the boundary $\partial\Omega$, where the noise strength is still sufficient to cause a direct exit. Accordingly, one may define a function $f(\vec{x})$ which is unity in the inner part of Ω:

$$t(\vec{x})=Tf(\vec{x})\ , \tag{2.7}$$

where

$$f(\vec{x})=0 \quad \text{for } \vec{x} \text{ on } \partial\Omega\ ,$$

$$f(\vec{x})\approx 1 \quad \text{for } \vec{x} \text{ on the inner boundary of } \Delta\Omega\ .$$

Since T is exponentially large in Q^{-1} and since clearly $\Delta\Omega$ shrinks to $\partial\Omega$ for $Q\to 0$, the inhomogeneity in the equation for $f(\vec{x})$ following from (2.4) becomes negligible on the boundary layer $\Delta\Omega$, i.e.,

$$L^\dagger f(\vec{x})=0\ , \tag{2.8}$$

with the boundary conditions stated below (2.7). With (2.6), and (2.7), the quantity T may be expressed in terms of $p(\vec{x})$ and of the gradient of f on $\partial\Omega$, i.e.,

$$T=\frac{-2}{Q}\frac{\int_\Omega p(\vec{x})d^2x}{\int_{\partial\Omega}p(\vec{x})\vec{\nabla}f\cdot d\vec{S}}\ . \tag{2.9}$$

We note that for the following application the stationary solution $p(\vec{x})$ is known to be of the form

$$p(\vec{x})=Z^{-1}\exp[-\Phi(\vec{x})/Q]\ , \tag{2.10}$$

where Z denotes the state-independent normalizing factor. One can show that $\Phi(\vec{x})$ is a Lyapunov function of the deterministic system (2.1). Hence, in the limit of weak noise, the Ω integral in (2.9) is dominated by the value of the absolute minimum of $\Phi(\vec{x})$ at the attracting point $\hat{x}$. The leading order contribution to the integral comes from the Gaussian approximation of (2.10) at $\hat{x}$. For the discussion of the $\partial\Omega$ integral in (2.9) we must distinguish the case with a constant $\Phi(\vec{x})$ on $\partial\Omega$ from the case with varying $\Phi(\vec{x})$ on $\partial\Omega$. The first case is not of relevance for the following application and we refer to Ref. 17 for details. In view of the following application we restrict ourselves to a function $\Phi(\vec{x})$ which on $\partial\Omega$ possesses one minimum at a point $\hat{y}$. Because of the Lyapunov character of $\Phi(\vec{x})$, the unrestricted function $\Phi(\vec{x})$ actually has a saddle point at $\hat{y}$ which corresponds to a hyperbolic point of the deterministic system (2.1). The leading order contribution to the $\partial\Omega$ integral again comes from the Gaussian approximation of $p(\vec{x})$ at $\hat{y}$ and the quantity $\vec{\nabla} f(\vec{x})$ can be replaced by its value at $\hat{y}$. The remaining unknown quantity is $\vec{\nabla}f$. In order to solve (2.8) we use for $f(\vec{x})$ near $\partial\Omega$ the ansatz

$$f(\vec{x})=\left[\frac{2}{\pi Q}\right]^{1/2}\int_0^{\rho(\vec{x})}dz\,\exp\left[-\frac{z^2}{2Q}\right]. \tag{2.11}$$

In view of (2.8), the function $\rho(\vec{x})$ introduced in (2.11) satisfies

$$K_1(\vec{x})\frac{\partial\rho(\vec{x})}{\partial x_1}+K_2(\vec{x})\frac{\partial\rho(\vec{x})}{\partial x_2}-\tfrac{1}{2}(\vec{\nabla}\rho)^2\rho(\vec{x})=0\ . \tag{2.12}$$

Because the normal component of the drift field $\vec{K}(\vec{x})$ vanishes on $\partial\Omega$, (2.12) admits a solution $\rho(\vec{x})$ which equals zero on $\partial\Omega$. Now, we may choose a coordinate system in the boundary layer with one axis r along $-d\vec{S}$ on $\partial\Omega$ and another axis α in $\partial\Omega$. Typically, the r component of the drift field does vanish linearly with r, i.e.,

$$K_r=gr,\quad g>0\ . \tag{2.13}$$

Thus, the function $\rho(\vec{x})$ near $\partial\Omega$ is given by

$$\rho = ar \;, \tag{2.14}$$

where a is a function of α which obeys from (2.12) the equation

$$ga + K_\alpha \frac{da}{d\alpha} - \tfrac{1}{2}(\vec{\nabla} r)^2 a^3 = 0 \;. \tag{2.15}$$

Note that in Ref. 16 a possible α dependence of a, yielding the second term in (2.15), has been disregarded. However, K_α vanishes at the saddle point $\hat{y}$ and we recover for the gradient of f at $\hat{y}$ the old result[16]

$$\frac{\partial f}{\partial r}(\hat{y}) = 2\left[\frac{g}{\pi Q(\vec{\nabla} r)^2}\right]^{1/2}, \tag{2.16}$$
$$\frac{\partial f}{\partial \alpha}(\hat{y}) = 0 \;.$$

Combining the result in (2.16) with (2.9) yields the central result

$$T = \frac{-\int_\Omega p(\vec{x}) d^2x}{[gQ(\vec{\nabla} r)^2/\pi]^{1/2} \int_{\partial\Omega} p(r=0,\alpha) dS_r} \;. \tag{2.17}$$

From (2.7) and (2.16) and the fact that $f(\vec{x})$ is unity at the maximum of the sharply peaked stationary probability $p(\vec{x})$, one finds that T coincides with the mean first passage time averaged over an ensemble $\bar{p}(\vec{x})$

$$\bar{p}(\vec{x}) = p(\vec{x}) \Big/ \int_\Omega p(\vec{x}) d^2x \tag{2.18}$$

of starting points.

For the following application the respective domains of attraction are not bounded. Because the mean first passage time will then, in general, not be constant almost everywhere in each single domain of attraction, the above considerations seemingly do not apply. However, if one is only interested in the mean first passage time T averaged over the ensemble $\bar{p}(x)$, the remote parts of Ω are of negligible weight and Ω can be replaced by a bounded domain Ω' contained in Ω. Of course, Ω' must have a shape such that there is deterministically no outgoing flow. Consequently, Ω' must contain the attracting point $\hat{x}$. Then $t(\vec{x})$ assumes again a constant value T almost everywhere in Ω', which is given by (2.9) if Ω and $\partial\Omega$ are replaced by Ω' and $\partial\Omega'$, respectively. For small noise, the integrals are again dominated by the maxima of $p(\vec{x})$ which for the Ω' integral and the $\partial\Omega'$ integral are located at $\hat{x}$ and $\hat{y}$, respectively.

III. FLUCTUATION THEORY OF DISPERSIVE OPTICAL BISTABILITY

A. Modeling and stationary behavior

Following Graham and Schenzle,[8] the stochastic dynamics of the complex-valued transmitted electric field, $E = x_1 + ix_2$, is modeled by the Fokker-Planck equation

$$\dot{p}_t(x_1,x_2) = \frac{\partial}{\partial x_1}\left[x_1 - \delta x_2 - E_0 + \Gamma^2 \frac{x_1 - \delta x_2}{1+x_1^2+x_2^2}\right] p_t(x_1,x_2) + \frac{\partial}{\partial x_2}\left[x_2 + \delta x_1 + \Gamma^2 \frac{x_2+\delta x_1}{1+x_1^2+x_2^2}\right] p_t(x_1,x_2)$$
$$+ \tfrac{1}{2} Q \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right] p_t(x_1,x_2) \;. \tag{3.1}$$

$\Gamma^2 > 0$ denotes a coupling constant, E_0 is the real-valued incident deterministic electric field, and Q measures the combined strength of the quantum noise, thermal noise, and the noise of the external laser field.[8] We restrict our considerations to the case for which the atomic detuning δ and the cavity detuning γ are of equal strength, i.e., we set $\delta = \gamma$. For the Fokker-Planck dynamics in (3.1), the drift is not derivable from a potential (detailed balance is not satisfied); but the stationary probability

$$\bar{p}(x_1,x_2) = Z^{-1} \exp[-\Phi(x_1,x_2)/Q] \;, \tag{3.2}$$

with Z being the normalization, is known nevertheless:[8]

$$\Phi(x_1,x_2) = \left[x_1 - \frac{E_0}{1+\delta^2}\right]^2 + \left[x_2 + \frac{\delta E_0}{1+\delta^2}\right]^2$$
$$+ \Gamma^2 \ln(1+x_1^2+x_2^2) \;. \tag{3.3}$$

The case $\delta = 0$ corresponds to absorptive optical bistability obeying detailed balance with a potential drift field.

Next we introduce a description in terms of an amplitude a and phase φ, i.e.,

$$x_1 = a\cos\varphi, \quad x_2 = a\sin\varphi \;. \tag{3.4}$$

The deterministic equations of motions, obtained from (3.1) in the limit $Q \to 0$, are then given by

$$\dot{a} = -a + E_0 \cos\rho - a\Gamma^2/(1+a^2) \;, \tag{3.5a}$$

$$\dot{\varphi} = -\delta - (E_0/a)\sin\varphi - \delta\Gamma^2/(1+a^2) \;, \tag{3.5b}$$

and the steady states $(\hat{a}, \hat{\varphi})$ are determined by (see Fig. 1)

$$\mathscr{E}_0 \equiv \frac{E_0}{(1+\delta^2)^{1/2}} = \hat{a}\left[1 + \frac{\Gamma^2}{1+\hat{a}^2}\right], \tag{3.6a}$$

$$\tan\hat{\varphi} = -\delta \;, \tag{3.6b}$$

where we have introduced the scaled electric field $\mathscr{E}_0$. Bistability with three positive roots for the amplitude $\hat{a}$ occurs for $\Gamma^2 > 8$ and $\mathscr{E}_0$ being inside the window

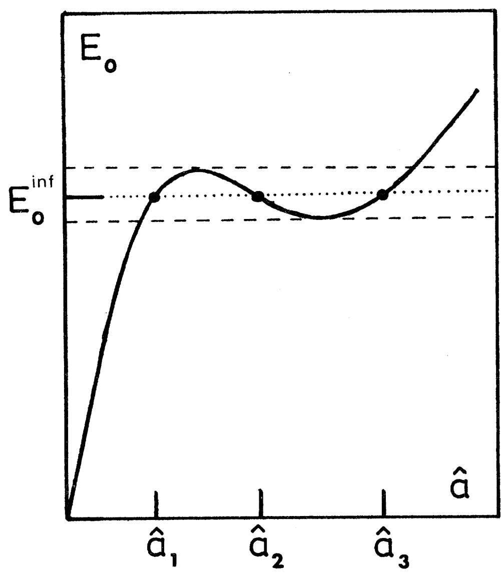

FIG. 1. Plot of the deterministic steady applitudes $\hat{a}$ vs incident coherent electric field E_0, Eq. (3.6a). The bistable window is bounded by two dashed lines, and the dotted line gives the steady amplitudes at the value $E_0^{\rm inf}$ through the inflection point of (3.6a).

$\mathscr{E}_0^{(1)} < \mathscr{E}_0 < \mathscr{E}_0^{(2)}$, where[18]

$$\mathscr{E}_0^{(1),(2)} = \left[\frac{\Gamma^2}{2} - 1 \pm \frac{\Gamma}{2}(\Gamma^2 - 8)^{1/2} \right]^{1/2} \times \left[1 + \frac{2\Gamma}{\Gamma \pm (\Gamma^2-8)^{1/2}} \right], \tag{3.7}$$

where superscripts (1) and (2) refer to the plus and minus signs, respectively. With $\Gamma^2 > 8$, (3.6a) always has a single inflection point, located at $(a^{\rm inf}, \mathscr{E}_0^{\rm inf})$,

$$\begin{aligned} a^{\rm inf} &= \hat{a}_2 = \sqrt{3} \, , \\ \mathscr{E}_0^{\rm inf} &= \sqrt{3}(1+\Gamma^2/4) \, , \end{aligned} \tag{3.8}$$

and with the locally stable states for the transmitted optical amplitudes at

$$\begin{aligned} \hat{a}_1 &= \tfrac{1}{8}\{\sqrt{3}\Gamma^2 - [3\Gamma^4 - 4^3(1+\Gamma^2/4)]^{1/2}\} \\ &\to \frac{\sqrt{3}}{3} \ \text{ as } \Gamma^2 \to \infty \, , \\ \hat{a}_3 &= \tfrac{1}{8}\{\sqrt{3}\Gamma^2 + [3\Gamma^4 - 4^3(1+\Gamma^2/4)]^{1/2}\} \\ &\to \frac{\sqrt{3}}{4}\Gamma^2 \ \text{ as } \Gamma^2 \to \infty \, , \end{aligned} \tag{3.9}$$

Note that for $\mathscr{E}_0 = \mathscr{E}_0^{\rm inf}$ all three steady amplitudes do not depend on the detuning parameter δ.

B. Activation Rate

Compared to those situations where the drift term in the Fokker-Planck equation is derivable from a potential, the nonpotential character of the drift field together with the nonthermal dynamics of the nonequilibrium discontinuous (first-order-type) phase transition in (3.1) complicates considerably the evaluation of the asymptotic mean first passage time (2.17). The quantity that essentially determines the sojourn time at low noise is the stationary probability $p(\vec{x})$, which, fortunately, has been determined exactly [see (3.3)]. Because the local directions of the separatrix around the saddle point $(\hat{a}_2, \hat{\varphi})$ for a nonpotential drift field do not coincide with the principal directions of the nonequilibrium potential (3.3), the details of the prefactor must be evaluated with care. A linearization of the deterministic flow around the saddle point $(\hat{a}_2, \hat{\varphi})$ yields with $a = \hat{a}_2 + x$, $\varphi = \hat{\varphi} + \beta$,

$$\dot{\vec{z}} = B\vec{z} \, , \tag{3.10}$$

where $\vec{z} = (x, \beta)$ and the matrix

$$B = \begin{bmatrix} b & \delta\mathscr{E}_0 \\ -\delta c & -d \end{bmatrix} \tag{3.11}$$

with

$$b = \Gamma^2 \frac{\hat{a}_2^2 - 1}{(\hat{a}_2^2+1)^2} - 1 > 0 \, , \tag{3.12a}$$

$$c = \mathscr{E}_0/\hat{a}_2^2 - 2\Gamma^2\hat{a}_2/(1+\hat{a}_2^2)^2 < 0 \, , \tag{3.12b}$$

$$d = \mathscr{E}_0/\hat{a}_2 > 0 \, . \tag{3.12c}$$

The two principal frequencies $\lambda_\pm$ of the relaxation matrix B are readily evaluated to be

$$\lambda_\pm = \tfrac{1}{2}(b-d) \pm \tfrac{1}{2}[(b+d)^2 - 4\delta^2 c \mathscr{E}_0]^{1/2} \tag{3.13}$$

with corresponding eigenvectors $\vec{v}_\pm$ (see Fig. 2)

$$\vec{v}_+(-\delta\mathscr{E}_0, d+\lambda_-) \, , \tag{3.14a}$$

$$\vec{v}_-(-\delta\mathscr{E}_0, d+\lambda_+) \, . \tag{3.14b}$$

The eigenvector $\vec{v}^{\perp}$ of the transposed matrix B^T to the eigenvalue λ_+ is perpendicular to the stable direction, i.e., $\vec{v}^{\perp}\cdot\vec{v}_- = 0$ where

$$B^T \vec{v}^{\perp} = \lambda_+ \vec{v}^{\perp} \tag{3.15a}$$

with

$$\vec{v}^{\perp} = (\lambda_+ + d, \delta\mathscr{E}_0) \, . \tag{3.15b}$$

In terms of this vector, the equation of the separatrix around $(\hat{a}_2, \hat{\varphi})$ reads locally (see Fig. 2)

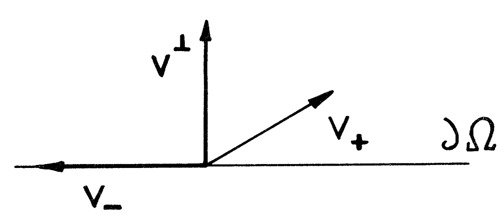

FIG. 2. Form of the local separatrix around the saddle point $(\hat{a}_2, \hat{\varphi})$ with corresponding eigenvectors introduced in the text.

$$\vec{z}\cdot\vec{v}^{\perp}=0=x(\lambda_{+}+d)+\delta\mathscr{E}_{0}\beta\ . \tag{3.16}$$

Next we introduce an appropriate coordinate system in the boundary layer with one axis (r) (see Sec. II), along the $v^{\perp}$ direction ($r=0$ on the separatrix and $r>0$ in the domain of attraction).

In particular, we use a coordinate transformation S: $(x,\beta)\rightarrow(r,\alpha)$ such that

$$\begin{bmatrix} r \\ \alpha \end{bmatrix}=S\begin{bmatrix} x \\ \beta \end{bmatrix}=\begin{bmatrix} \lambda_{+}+d & d\mathscr{E}_{0} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ \beta \end{bmatrix}\ . \tag{3.17}$$

The components K_r, K_β of the drift field in the new system of coordinates are then given by

$$\begin{bmatrix} K_r \\ K_\beta \end{bmatrix}=SBS^{-1}\begin{bmatrix} r \\ \alpha \end{bmatrix} =\begin{bmatrix} \lambda_{+} & 0 \\ -\dfrac{\delta c}{d+\lambda_{+}} & \dfrac{\delta^{2}c\mathscr{E}_{0}}{d+\lambda_{+}}-d \end{bmatrix}\begin{bmatrix} r \\ \alpha \end{bmatrix}\ . \tag{3.18}$$

This equation exhibits explicitly the vanishing of K_r linearly in r,

$$K_r\equiv gr=\lambda_{+}r\ , \tag{3.19}$$

and for $(\vec{\nabla}r)^2$ we find

$$(\vec{\nabla}r)^{2}=(\lambda_{+}+d)^{2}+(\delta\mathscr{E}_{0}/\hat{a}_{2})^{2}\ . \tag{3.20}$$

The components of the oriented surface element $d\vec{S}$ occurring in (2.17) follow from (3.16) and (3.17):

$$dS_r=\frac{1}{\lambda_{+}+d}d\alpha\ ,\qquad dS_\alpha=0\ . \tag{3.21}$$

At low noise, the integrals in (2.17) can be calculated with the method of steepest descent to yield for the activation rate k_i the main result

$$k_i=\frac{1}{2T}=\frac{\hat{a}_{2}[\Phi''_{aa}(\hat{a}_{i},\hat{\varphi})\Phi''_{\varphi\varphi}(\hat{a}_{i},\hat{\varphi})]^{1/2}\{\lambda_{+}[(\lambda_{+}+d)^{2}+(\delta\mathscr{E}_{0}/\hat{a}_{2})^{2}]\}^{1/2}}{2\sqrt{2}\pi\hat{a}_{i}\{\,|\Phi''_{aa}(\hat{a}_{2},\hat{\varphi})|\,[\delta\mathscr{E}_{0}/(\lambda_{+}+d)]^{2}+\Phi''_{\varphi\varphi}(\hat{a}_{2},\hat{\varphi})\}^{1/2}(\lambda_{+}+d)}\exp(-\Delta\Phi/Q)\ , \tag{3.22}$$

where $i=1$ denotes the rate for the forward transition $\hat{a}_1\rightarrow\hat{a}_3$ and $i=3$ the backward rate k_3: $\hat{a}_3\rightarrow\hat{a}_1$. The Arrhenius factor $\Delta\Phi$ equals

$$\begin{aligned}\Delta\Phi&=\Phi(\hat{a}_{2},\hat{\varphi})-\Phi(\hat{a}_{i},\hat{\varphi})\\ &=\hat{a}_{2}^{2}-\hat{a}_{i}^{2}-2\mathscr{E}_{0}(\hat{a}_{2}-\hat{a}_{i})\\ &\quad+\Gamma^{2}\ln[(1+\hat{a}_{2}^{2})/(1+\hat{a}_{i}^{2})]\ ,\end{aligned} \tag{3.23}$$

and the curvatures of the potential are

$$\Phi''_{aa}=2[1+\Gamma^{2}(1-a^{2})/(1+a^{2})^{2}]\ , \tag{3.24}$$

$$\Phi''_{\varphi\varphi}=2\mathscr{E}_{0}a\ . \tag{3.25}$$

Expanding the Arrhenius factor $\Delta\Phi$ around $\delta=0$, one obtains

$$\Delta\Phi=\Delta\Phi^{0}+(\hat{a}_{2}^{0}-\hat{a}_{i}^{0})E_{0}\delta^{2} \tag{3.26}$$

with

$$\begin{aligned}\Delta\Phi^{0}&=\Gamma^{2}\ln\{[1+(\hat{a}_{2}^{0})^{2}]/[1+(\hat{a}_{i}^{0})^{2}]\}\\ &\quad+(\hat{a}_{2}^{0})^{2}-(\hat{a}_{i}^{0})^{2}-2E_{0}(\hat{a}_{2}^{0}-\hat{a}_{i}^{0})\ ,\end{aligned} \tag{3.27}$$

where the notation $\hat{a}^0$ denotes the steady-state amplitude at zero detuning. Therefore, the forward rate k_1 becomes exponentially decreased with increasing detuning δ^2, whereas the backward rate k_3, i.e., $\hat{a}_2^0$-$\hat{a}_3^0<0$, is exponentially increased with increasing detuning. The prefactor in (3.22) has also a compounded dependence on δ. For absorptive optical bistability with $\delta\equiv0$, the Fokker-Planck drift is derivable from the potential Φ^0, which simplifies considerably the result in (3.22):

$$k_i^{\delta=0}=\frac{1}{4\pi}\left[\frac{\hat{a}_{2}^{0}}{\hat{a}_{i}^{0}}\right]^{1/2}[\Phi^{0''}_{aa}(\hat{a}_{i}^{0})\,|\Phi^{0''}_{aa}(\hat{a}_{2}^{0})|\,]^{1/2}\exp\left[-\frac{\Delta\Phi^{0}}{Q}\right]\ . \tag{3.28}$$

This latter result follows also readily by use of the standard methods.[9–11]

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