Activation rates in bistable systems in the presence of correlated noise

Peter Hänggi and Peter Riseborough
Polytechnic Institute of New York, Department of Physics,
333 Jay Street, Brooklyn, New York, 11201
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In the case of telegraphic noise $F_\nu(t)$, the activation rates for bistable flows of the type $\dot{x}=f(x)+g(x)F_\nu(t)$ can be calculated exactly. For a noise $F_\nu(t)$ of constant intensity, the activation rates are enhanced exponentially with decreasing correlation time $\tau=1/\nu$.

Systems with a number of competing states of local stability play a key role in the analysis of various physical phenomena.[1,2] In a growing number of cases, such situations occur in open systems, i.e., systems which require a continual flux of energy or matter. Our focus will be on the one-dimensional dynamics of an order parameter x exhibiting bistability, i.e.,

$$\dot{x}=f(x,\alpha) \ , \tag{1}$$

where α denotes an external control parameter. The flow $f(x,\alpha)$ is assumed to possess three real roots $\{\bar{x}_1,\bar{x}_u,\bar{x}_2\}$. We define $\bar{x}_1<\bar{x}_2$, where $\bar{x}_1$ and $\bar{x}_2$ denote locally stable steady states and $\bar{x}_u$ is an intermediate locally unstable steady state. Typical examples would be a bistable Esaki diode[1,3] with α being the external constant supply current, an optical bistability with α being the externally injected coherent field[4] or the phase dynamics of externally synchronized oscillators.[5] In the presence of fluctuations, the phenomenon on bistability generates a number of interesting questions. A particularly important one is the rate of decay of the metastable state. Fluctuations can be of an intrinsic nature or can be imposed externally by dealing with a noisy control parameter $\alpha\rightarrow\alpha+F(t)$. If the system under consideration is already macroscopic, the influence of intrinsic noise plays a negligible role.[6] In what follows, we model the noise of the control parameter by telegraphic noise of vanishing mean $F_\nu(t)$,

$$\begin{aligned} &F_\nu(t)=a(-1)^{n(t)} \ ; \\ &\langle F_\nu(t)F_\nu(s)\rangle = D\nu\exp(-\nu|t-s|) \ , \end{aligned} \tag{2}$$

where $n(t)$ is a Poisson counting process with parameter $\nu/2$ and a denotes a random step with density

$$\rho_a=\tfrac{1}{2}[\delta(a-\sqrt{D\nu})+\delta(a+\sqrt{D\nu})] \ .$$

An important property of telegraphic noise $F_\nu(t)$ is the approach to a *Gaussian* white noise $\xi(t)$ in the limit $\nu\rightarrow\infty$.[7] With

$$\lim_{\nu\rightarrow\infty}\nu/2\exp-\nu|t|=\delta(t) \ ,$$

Eq. (2) reduces to

$$\lim_{\nu\rightarrow\infty}\langle F_\nu(t)F_\nu(s)\rangle=\langle\xi(t)\xi(s)\rangle=2D\,\delta(t-s) \ . \tag{3}$$

Figure 1 illustrates qualitatively the random realizations of telegraphic noise for two different correlation strengths ν. The realizations change sign at random Poisson arrival times $\{t_i\}$ with waiting time probability

$$\rho(t_{i+1}-t_i=t)=\tfrac{1}{2}\nu\exp-\tfrac{1}{2}\nu t \ .$$

In the presence of a fluctuating control parameter α, the deterministic flow in (1) changes over into a stochastic flow,

$$\dot{x}=f(x,\alpha)+g(x)F_\nu(t) \ , \tag{4}$$

where the generally state-dependent coupling $g(x)$ (multiplicative noise) represents the linear coupling of α to the order parameter x in the dynamical flow.

The problem of interest can be posed as follows. Given random noises $F_\nu(t)$ with different correlation parameters ν_1 and ν_2, but possessing identical spectral

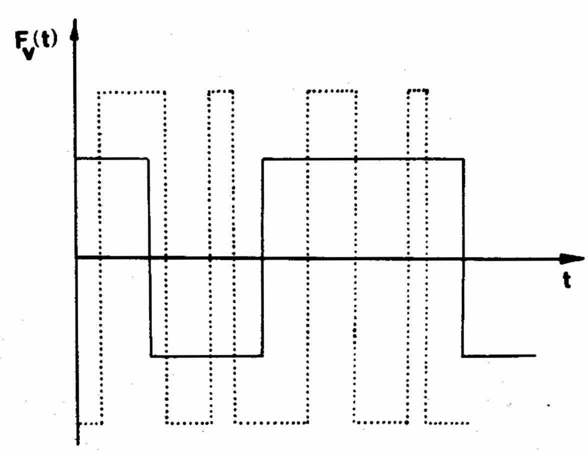

FIG. 1. Sketch of possible realizations of $F_\nu(t)$ for differing correlation times $\tau=1/\nu$. The solid curve is for ν_1 and the dotted curve is for ν_2 where $\nu_2>\nu_1$.

densities $S_F(\omega=0)$ at frequency zero, i.e.,

$$S_F(\omega=0)=\int_{-\infty}^{\infty}\langle F_{\nu_1}(t)F_{\nu_1}(0)\rangle dt$$
$$=\int_{-\infty}^{\infty}\langle F_{\nu_2}(t)F_{\nu_2}(0)\rangle dt$$
$$=2D=S_\xi(\omega)\ , \quad (5)$$

what is the relationship between the corresponding activation rates of metastable states? The system with a smaller correlation time $1/\nu$ is subjected to random forces with larger amplitude (see Fig. 1); this might lead one to conclude that the rate might be enhanced. However, the duration over which the force is constant is decreased; because the random force changes sign more rapidly, one might now expect that the system might not have enough time to reach the point of instability and, consequently, the rate might be suppressed for a larger correlation parameter ν. Thus, it is not obvious *a priori* which of the two random forces will yield a smaller rate (i.e., longer escape time).

The rate of change of the probability $p_t(x)$, corresponding to the stochastic flow in (4), has been studied in previous work in a different context.[8,9] The exact non-Markovian master equation reads

$$\dot p_t(x)=-\frac{\partial}{\partial x}j(x,t)=-\frac{\partial}{\partial x}\left\{f(x,\lambda)p_t(x)-D\nu g(x)\int_0^t d\tau\exp\left[-\left\{\nu+\frac{\partial}{\partial x}f\right\}(t-\tau)\right]\frac{\partial}{\partial x}g(x)p_\tau(x)\right\}\ . \quad (6)$$

By setting the current $j(x,t=\infty)=0$, we readily find the normalizable stationary probability $\bar p(x)$ (Refs. 8 and 9) (Z is the normalization constant),

$$\bar p(x)=\frac{Z^{-1}|g(x)|}{g^2(x)-f^2(x)/D\nu}\left\{\exp\int^x\frac{dy\,f(y)/[g^2(y)-f^2(y)/D\nu]}{D}\right\}\theta(D\nu g^2(x)-f^2(x))\ . \quad (7)$$

The support of $\bar p(x)$ is given by the Heaviside step function expression $\theta(\cdots)$ and the extrema $\{\hat x\}$ are located at (the prime denotes differentiation with respect to x)

$$(f+2ff'/\nu-f^2g'/\nu g-Dgg')|_{\hat x}=0\ . \quad (8)$$

In the following, we assume a positive "effective diffusion,"

$$\bar D(x)\equiv Dg^2(x)-f^2(x)/\nu>0$$

for $x\in[\bar x_1,\bar x_2]$, thereby guaranteeing a nonzero support of $\bar p(x)$ over the bistable region.

The forward rate, $r:\bar x_1\rightarrow\bar x_2$, defined as the inverse of the escape time T of the metastable state $\bar x_1$ at low noise D (without this assumption the problem of escape is not well defined anyhow), can be evaluated for the general non-Markovian master equation in the following way. We inject particles at the locally stable state $\bar x_1$ and remove them the moment they reach the locally unstable state $\bar x_u$. The resulting constant *nonequilibrium* current j_0 will build up a total integrated probability p_0 proportional to the escape time T,[2] i.e.,

$$j_0T=\int_{-\infty}^{\bar x_u}p_0(x)\,dx\ . \quad (9)$$

Solving for the nonequilibrium probability p_0 by setting $j(x,\infty)=j_0$, one finds the exact expression

$$p_0(x)=\left\{\frac{-j_0}{D}\int_{\bar x_u}^{x}\frac{[1+g(f/g)'/\nu]}{\bar p(g^2-f^2/D\nu)}\bigg|_y dy\right\}\bar p(x)\ . \quad (10)$$

If one takes into account that a particle reaching the unstable state $\bar x_u$ has equal probability to either fall back or to continue on into the new locally stable state at $\bar x_2$, one obtains by virtue of (9) the exact rate r,

$$r=\frac{1}{2T}=\frac{D}{2}\left[\int_{-\infty}^{\bar x_u}\bar p(x)\,dx\int_x^{\bar x_u}\frac{[1+g(f/g)'/\nu]}{\bar p[g^2-f^2/D\nu]}\bigg|_y dy\right]^{-1}\ . \quad (11)$$

This is the main result of this paper. Equation (11) can be simplified considerably under the following general conditions. For the sake of clarity we assume additive noise with $g(x)=\text{const}=g$. Moreover, the correlation rate ν entering (11) is subject to the inequalities

$$g^2\nu>f^2(x)/D\ ,\quad \nu>-2f'(x)\ ,\quad x\in[\bar x_1,\bar x_2]\ , \quad (12)$$

thereby guaranteeing that $\bar p(x)$ has a nonvanishing support over the bistable region $[\bar x_1,\bar x_2]$ with the extrema $\{\hat x\}$, $f(1+2f'/\nu)|_{\hat x}=0$, being the deterministic steady states $\{\hat x\}=\{\bar x_1,\bar x_u,\bar x_2\}$. With low noise, i.e., $D\leq Q/5$, $Q=-\int_{\bar x_1}^{\bar x_u}f(y)/g^2dy$, and the fact that the maxima of $\bar p(\bar x_1)$ and $1/\bar p(\bar x_u)$ are strongly peaked, it is justified to approximate the integrals in (11) by the method of steepest descent. The rate $r(\nu)$ then takes the form

$$r(\nu)=\frac{(\lambda_1|\lambda_u|)^{1/2}}{2\pi(1+|\lambda_u|/\nu)}\exp[-\Delta\Phi(\nu)/D] \ , \quad (13a)$$

where the Arrhenius factor $\Delta\Phi(\nu)$ is given by

$$\Delta\Phi(\nu)=-\int_{\bar{x}_1}^{\bar{x}_u}\frac{f(y)}{g^2-f^2(y)/\nu D}dy$$

$$>\Delta\Phi(\nu=\infty)=Q \ , \quad (13b)$$

and

$$\lambda_1=-f'(\bar{x}_1)>0, \ \ \lambda_u=-f'(\bar{x}_u)<0 \ . \quad (13c)$$

For a more general multiplicative noise $g(x)F_\nu(t)$, $g^2(x)>0$, the corresponding simplification reads

$$r(\nu)=\frac{(\lambda_1|\lambda_u|)^{1/2}}{2\pi(1+|\lambda_u|/\nu)}\exp[-\Delta\Phi_g(\nu)/D] \ , \quad (13d)$$

with

$$\Delta\Phi_g(\nu)=-\int_{\bar{x}_1}^{\bar{x}_u}\frac{f(y)}{g^2(y)-f^2(y)/D\nu}dy$$

$$>-\int_{\bar{x}_1}^{\bar{x}_u}\frac{f(y)}{g^2(y)}dy=Q_g \ . \quad (13e)$$

In conclusion, when the noise intensity $S_F(\omega=0)$ is constant, the rates are exponentially enhanced with decreasing correlation time $\tau=1/\nu$ and this is independent of the specific form of the nonlinear bistable flow $f(x,\alpha)$ and independent of whether the random noise is additive or multiplicative. The conclusion remains true for the reverse transition from $\bar{x}_2\rightarrow\bar{x}_1$; the equations are only subjected to the trivial replacement $\bar{x}_1\rightarrow\bar{x}_2$. Our result approaches the well-known Smoluchowski rate r_s,[2,3] in the limit $\nu\rightarrow\infty$ (i.e., Gaussian white noise). Because

$$\Delta\Phi(\nu)>\lim_{\nu\rightarrow\infty}\Delta\Phi(\nu)=Q \ ,$$

and the prefactors also increase with decreasing correlation time τ, the rates are maximal in the Smoluchowski limit.

A physical interpretation of the results in Eqs. (13) can be visualized as follows[10]: If one considers the mean-squared displacement σ_T over a time scale T of the escape time in zero force field, i.e.,

$$\sigma_T=\left\langle\left(\int_0^T F_\nu(s)\,ds\right)^2\right\rangle$$

$$=2DT[1-(1-\exp-\nu T)/\nu T] \ , \quad (14a)$$

which with $\nu T>>1$ reduces to

$$\sigma_T=2D\left(1-\frac{\tau}{T}\right)T\equiv 2D_{\mathrm{eff}}T \ , \quad (14b)$$

one observes that D_{eff} is enhanced for decreasing correlation time τ. Thus, over a time scale T, the "particle" is more likely to be pushed forward for a smaller correlation time of the random noise, yielding a rate enhancement or equivalently a smaller escape time.

All of the above calculations can be carried through for more general telegraphic noise in which a is a random variable of vanishing mean with density $\rho_a>0$. For random forces $F(t)$ of finite correlation, which are not of telegraphic type, it is not possible to derive a closed, exact master equation for the order parameter x of a truly bistable flow.[11] In particular, for Gaussian noise $F(t)$ with a correlation given by (2), one is unable to derive exact expressions for activation rates.[11–13] It should be noted also that a calculation of the activated escape according to Eq. (9), which follows the reasoning advocated by the authors of Ref. 2, carries through in the present situation, Eq. (6), with memory and nonlocal transition probabilities; a direct evaluation of the escape time via the concept of the mean first passage time is complicated by the fact that integral operators, as well as nonlocal boundary conditions which account for the zero backflow of probability into the domain of attraction, must be considered.[14]

Finally, we point out that the discussed "intrinsic enhancement" of the activation rates should be distinguished from an "external enhancement" of activation rates induced by *additional* parametric noise.[15–17]

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