Activation rates in bistable systems in the presence of correlated noise

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In the case of telegraphic noise \( F_{t}(t) \), the activation rates for bistable flows of the type
\[ \dot{x} = f(x) + \sigma(x) F_{t}(t) \]
can be calculated exactly. For a noise \( F_{t}(t) \) of constant intensity, the activation rates are enhanced exponentially with decreasing correlation time \( \tau = \langle |D(t) - D(t - \tau)| \rangle \). \( \tag{3} \)

\[ \lim_{\tau \to 0} \left( F_{t}(t) F_{t}(s) \right) = \left( \xi(t) \xi(s) \right) - 2D \delta(t - s) \] \( \tag{4} \)

In the presence of a fluctuating control parameter \( a \), the stochastic flow in (1) changes over into a stochastic flow,
\[ \dot{x} = f(x, a) + g(x) F_{t}(t) \] \( \tag{5} \)

where the generally state-dependent coupling \( g(x) \) (multiplicative noise) represents the linear coupling of the order parameter \( x \) in the dynamical flow. The problem of interest can be posed as follows. Given random noises \( F_{t}(t) \) with different correlation parameters \( \tau_{1} \) and \( \tau_{2} \), but possessing identical spectral

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**FIG. 1.** Sketch of possible realizations of \( F_{t}(t) \) for different correlation times \( \tau = \tau_{1} < \tau_{2} \). The bold curve is for \( \tau_{1} \) and the dotted curve is for \( \tau_{2} \) where \( \tau_{2} > \tau_{1} \).

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\[ \xi(t) = x(0) \exp \left( \int_{0}^{t} D(t') \, dt' \right) \]

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densities \( S_r(u=0) \) at frequency zero, i.e.,

\[
S_r(u=0) = \frac{1}{2} \left( \int_0^\infty \left( F_r(x,t) + F_r(-x,t) \right) dt \right)
\]

\[
- \frac{1}{2} \left( \int_0^\infty \left( F_r(x,t) + F_r(-x,t) \right) dt \right) = 2 \pi \delta(0).
\]

what is the relationship between the corresponding activation rates of metastable states? The system with a smaller correlation time \( \tau \) is subjected to random forces with larger amplitude (see Fig. 1); this might lead one to conclude that the rate might be

\[
\dot{p}(x) = \frac{\partial}{\partial x} f(x, t) - \frac{\partial}{\partial x} \int_0^\infty \left[ f(x, t) \delta(x - \frac{x}{\tau}) - D \frac{\partial}{\partial x} \right] dt.
\]

By setting the current \( f(x, t) = 0 \), we readily find the normalizable stationary probability \( \dot{p}(x) \) (Refs. 8 and 9)

\[
\dot{p}(x) = \frac{Z_{eff}(\dot{p}(x))}{\dot{p}(x)} \left[ \exp \int_0^\infty \left( f(x, t) \delta(x - \frac{x}{\tau}) - D \frac{\partial}{\partial x} \right) dt \right] \frac{\dot{p}(x)}{D \frac{\partial}{\partial x} \dot{p}(x)}
\]

The support of \( \dot{p}(x) \) is given by the Heaviside step function expression \( \theta(\cdot - \cdot) \) and the extrema \( [\dot{p}] \) are located at the prime denotes differentiation with respect to \( x \)

\[
\left( f + 2 f' \dot{r} - f' \dot{y} + v - v \dot{g} \right) \dot{p} = 0.
\]

In the following, we assume a "effective diffusion",

\[
\tilde{D}(x) = D \frac{\partial}{\partial x} \dot{p}(x) \frac{\partial}{\partial x}
\]

for \( x \in [\dot{r}_1, \dot{r}_2] \), thereby guaranteeing a nonzero support of \( \tilde{D}(x) \) over \( [\dot{r}_1, \dot{r}_2] \). The forward rate, \( \dot{r}_1 \), is defined as the inverse of the escape time \( T \) of the metastable state \( \dot{r}_1 \). At low noise \( D \) (without this assumption the problem of escape is not well defined anyhow), can be evaluated for the general non-Markovian master equation in the following way. We inject particles at the locally stable state \( \dot{r}_1 \) and remove them the moment they reach the

\[
r = \frac{1}{2T} \left[ \int_0^\infty \dot{r}_1 \left[ 1 - g + \frac{z^2}{2} \left( \frac{\partial^2}{\partial x^2} + f(x) \delta(x - \frac{x_1}{\tau}) \right) \right] \dot{p}(x) dx \right]
\]

This is the main result of this paper. Equation (11) can be simplified considerably under the following general conditions. For the sake of clarity we assume additive noise with \( g(x) = \text{const} = g \). Moreover, the correlation rate \( v \) entering (11) is subject to the inequalities

\[
\dot{r}_0 > f(x) / D \quad , \quad v > 2 f(x) \quad , \quad x \in [\dot{r}_1, \dot{r}_2].
\]

enhanced. However, the duration over which the force is constant is decreased; because the random force changes sign more rapidly, one might now expect that the system might not have enough time to reach the point of instability and, consequently, the rate might be suppressed for a larger correlation parameter \( v \). Thus, it is not obvious a priori which of the two random forces will yield a smaller rate (i.e., longer escape time).

The rate of change of the probability \( \dot{p}(x) \), corresponding to the stochastic flow in (4), has been studied in previous work in a different context.\(^{15}\) The exact non-Markovian master equation reads

\[
\dot{p}(x) = \frac{Z_{eff}(\dot{p}(x))}{\dot{p}(x)} \left[ \exp \int_0^\infty \left( f(x, t) \delta(x - \frac{x}{\tau}) - D \frac{\partial}{\partial x} \right) dt \right] \frac{\dot{p}(x)}{D \frac{\partial}{\partial x} \dot{p}(x)}
\]

locally unstable state \( \dot{r}_0 \). The resulting constant non-equilibrium current \( j_0 \) will build up a total integrated probability \( p_0 \) proportional to the escape time \( T \) (i.e.,

\[
\int_0^\infty \int_0^\infty \dot{p}(x) dx.
\]

Solving for the non-equilibrium probability \( p_0 \) by setting \( \dot{r}_1 = 0 \), one finds the exact expression

\[
p_0(x) = \frac{1}{E} \int_0^\infty \left[ 1 - g(x) / \dot{r}_1 - v / \dot{r}_1 \right] \dot{p}(x).
\]

If one takes into account that a particle reaching the unstable state \( \dot{r}_0 \) has equal probability to either fall back or continue on into the new locally stable state at \( \dot{r}_1 \), one obtains by virtue of (9) the exact rate \( r \)

\[
r = \frac{1}{2T} \left[ \int_0^\infty \dot{r}_1 \left[ 1 - g + \frac{z^2}{2} \left( \frac{\partial^2}{\partial x^2} + f(x) \delta(x - \frac{x_1}{\tau}) \right) \right] \dot{p}(x) dx \right]
\]

thereby guaranteeing that \( \dot{p}(x) \) has a nonvanishing support over the bistable region \( [\dot{r}_1, \dot{r}_2] \) with the extrema \( [\dot{r}_1, \dot{r}_2 - f'(x) \delta(x - \dot{r}_1) - f'(x) \delta(x - \dot{r}_2) \] being the deterministic steady states \( \{ \dot{r}_1, \dot{r}_2 \} \). With low noise, i.e.,

\[
D \equiv 0/\tau \delta = \frac{1}{2} f'(x) \delta(x - \dot{r}_1) \dot{p}(x) dx \delta(x - \dot{r}_2) \dot{p}(x) dx.
\]

and the fact that the maxima of \( \dot{p}(x) \) and \( 1 / \dot{p}(x) \) are strongly peaked, is justified to approximate the integrals in (11) by the method of steepest descent. The rate \( r \) then
take the form

$$
\sigma_T = \frac{(a^2 \lambda^2_{at})^{1/2}}{2\pi \Gamma(1 + \lambda_{at}/\alpha)} \exp(-\Delta \Phi(x)/D) .
$$

(13a)

where the Arhennius factor $\Delta \Phi(x)$ is given by

$$
\Delta \Phi(x) = \int_0^x \frac{f(y)}{\alpha + f(y)} dy
$$

and

$$
\lambda_{at} = -\frac{d}{d \bar{x}} (\lambda_{at}) > 0, \quad \lambda_\infty = -\frac{d}{d \bar{x}} (\lambda_\infty) < 0 .
$$

(13b)

For a more general multiplicative noise $g(x)/f(y)$, the equation simplifies to

$$
\sigma_T = \frac{(a^2 \lambda^2_{at})^{1/2}}{2\pi \Gamma(1 + \lambda_{at}/\alpha)} \exp(-\Delta \Phi(x)/D) .
$$

(13c)

with

$$
\Delta \Phi(x) = \int_0^x \frac{f(y)}{\alpha + f(y)} dy - \int_0^x \frac{f(y)}{\alpha} dy = \Delta \Phi(x) / \alpha .
$$

(13d)

In conclusion, when the noise intensity $\sigma(x) = 0$ is constant, the rates are exponentially enhanced with decreasing correlation time $\tau = \frac{1}{\alpha}$ and this is independent of the specific form of the nonlinear transfer function $f(x, u)$ and independent of whether the random noise is additive or multiplicative. The conclusion remains true for the reverse transition from $\lambda_{at} \rightarrow \lambda_\infty$. Our result approaches the well-known Smoluchowski rate $k_{\infty}$ in the limit $\tau \rightarrow 0$ (i.e., Gaussian white noise). Because

$$
\Delta \Phi(x) \approx \ln \sigma_T
$$

and the prefactors also increase with decreasing correlation time $\tau$, the rates are maximal in the Smoluchowski limit.

A physical interpretation of the results of Eqs. (13) can be visualized as follows:

- If one considers the mean-square deviation from zero force field, i.e.,

$$
\sigma_T = \left[ \int_0^\infty f(x)/\tau^2 dx \right]^{1/2}
$$

(14a)

which with $\tau \gg 0$ reduces to

$$
\sigma_T = \sqrt{\frac{2}{\pi \tau}}
$$

(14b)

one observes that $D_{eff}$ is enhanced for decreasing correlation time $\tau$. Thus, over a smaller rate time $T$, the "parabolic" is more likely to be pushed forward for a smaller correlation time of the random noise, yielding a rate enhancement or equivalently a smaller escape time.

All of the above calculations can be carried through for more general telegraphic noise which is a random variable with vanishing mean with density $p_x = 0$. For random forces $f(x)$ of finite correlation, which are not of telegraphic type, it is not possible to derive a closed, exact master equation for the order parameter $x$ of a truly bistable flow.\(^{11}\) In particular, for Gaussian noise $f(x)$ with a correlation given by \(\tau\), one is unable to derive exact expressions for activation rates.\(^{12}\) It should be noted also that a calculation of the activated escape according to Eq. (5), which follows the reasoning advocated by the authors of Ref. 2, carries forward in the present situation, Eq. (6), with memory and nonlocal transition probabilities; a direct evaluation of the escape time via the concept of the mean first passage time is complicated by the fact that integral operators, as well as nonlocal boundary conditions which account for the zero backflow of probability into the domain of attraction, must be considered.\(^{13}\)

Finally, we point out that the discussed "intrinsically enhanced" of the activation rates should be distinguished from an "external enhancement" of activation rates induced by additional parametric noise.\(^{14}\)

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24. For intensive variables, the intrinsic noise scales as $F(\tau)/\gamma$, where $\gamma$ is a dimensionless parameter. For example, for an Eke diode with a capacitance $C \cdot 10^{-11}$ F at 0.5 V and junction area of $10^{-2} \text{ cm}^2$, $D$ is of the order $10^3$, thus the influence of intrinsic noise on the activation rate is extremely small (see Ref. 1, Sec. 6.3).
26. K. Kiyohara, W. Hystinshenke, R. Lejeune, and Y. Inaba,
[References]
