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RATE OF PHASE SLIPS OF A DRIVEN VAN DER POL OSCILLATOR AT LOW NOISE

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The rate of noise-induced desynchronizations (phase slips) of a driven Van der Pol oscillator is determined in the limit of weak noise. This is accomplished by a newly developed theory for the lifetime of metastable states, whereas Kramers' standard method is not applicable.

The Van der Pol oscillator driven by a periodic force,

$$\ddot{x} + \omega_0^2 x - \gamma(1 - x^2)\dot{x} = \gamma E \sin \omega t \quad (\gamma > 0), \tag{1}$$

is a standard model for many nonlinear phenomena in mechanics [1–3], optics [4,5], radio engineering [6] and chemistry [4]. An additional stochastic driving force $\xi(t)$, which is supposed to be gaussian and "white":

$$\langle \xi(t_1)\xi(t_2)\rangle = 2\omega^2 \epsilon \delta(t_2 - t_1), \tag{2}$$

is often included to describe environmental influences; in laser theory this term accounts for spontaneously emitted light [4,5].

In the steady state, and for small detuning $\omega - \omega_0$, $x(t)$ essentially oscillates with the driving frequency ω, but due to the stochastic force a random motion is superimposed, which leads to occasional losses of synchronization even when ϵ is arbitrarily small. More explicitly, the phase of $x(t)$ occasionally departs by more than $\pm\pi$ from that of the unperturbed motion and then acquires a shift of $\pm 2\pi$. Such an event is called a "phase slip". The aim of this paper is to evaluate the rate of these phase slips in the limit of low noise ($\epsilon \to 0$). From the theoretical point of view this problem has its own interest, due to the fact that a treatment according to Kramers' ideas [7,8] is not really possible. This aspect will be discussed in greater detail.

The basic analysis of the oscillator's motion was given e.g. in ref. [9], and we briefly mention the essential points: First, it is convenient to introduce two variables $y_1(t), y_2(t)$ referring to a frame in $(x, \dot{x})$ space, rotating with frequency ω:

$$y_1 = x \cos \omega t - (\dot{x}/\omega) \sin \omega t, \quad y_2 = x \sin \omega t + (\dot{x}/\omega) \cos \omega t, \tag{3}$$

from which the original variables may be reobtained by

$$x = y_1 \cos \omega t + y_2 \sin \omega t, \quad \dot{x} = \omega(-y_1 \sin \omega t + y_2 \cos \omega t). \tag{4}$$

Differentiating (3) and using (4) and (1) we arrive at

$$\dot{y}_1 = -B(t)\omega^{-1} \sin \omega t, \quad \dot{y}_2 = B(t)\omega^{-1} \cos \omega t, \tag{5}$$

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with

$$B(t) = (\omega^2 - \omega_0^2)(y_1 \cos \omega t + y_2 \sin \omega t) - \omega\gamma[1 - (y_1 \cos \omega t + y_2 \sin \omega t)^2](y_1 \sin \omega t - y_2 \cos \omega t)$$
$$+ \gamma E \sin \omega t + \xi(t).$$

From the assumption that the friction is not too large ($\gamma \ll \omega$) it follows that y_1 and y_2 do not change appreciably during one period $2\pi/\omega$. It is therefore reasonable to perform the according time average in (5), which corresponds to the Krylow–Bogoliubov method in first order [10]. For small detuning ($\omega + \omega_0 \approx 2\omega$) the result is

$$\dot{y}_1 = (\omega_0 - \omega)y_2 + (\gamma/2)[1 - (y_1^2 + y_2^2)/4]y_1 - \gamma E/2\omega - \bar{\xi}_1(t),$$
$$\dot{y}_2 = (\omega - \omega_0)y_1 + (\gamma/2)[1 - (y_1^2 + y_2^2)/4]y_2 + \bar{\xi}_2(t). \tag{6}$$

For the noise sources $\bar{\xi}_{1,2}$ this procedure yields with (2):

$$\langle\bar{\xi}_1(t_1)\bar{\xi}_1(t_2)\rangle = \omega^{-2}\,\overline{\sin \omega t_1 \sin \omega t_2}\langle\xi(t_1)\xi(t_2)\rangle = \epsilon\delta(t_2 - t_1) = \langle\bar{\xi}_2(t_1)\bar{\xi}_2(t_2)\rangle,$$
$$\langle\bar{\xi}_1(t_1)\bar{\xi}_2(t_2)\rangle = \omega^{-2}\,\overline{\sin \omega t_1 \cos \omega t_2}\langle\xi(t_1)\xi(t_2)\rangle = 0. \tag{7}$$

The Fokker–Planck equation associated with (6) is now readily found to be

$$\partial p/\partial t = -\partial(A_i p)/\partial y_i + \epsilon(\partial^2 p/\partial y_1^2 + \partial^2 p/\partial y_2^2),$$
$$A_1 = (\omega_0 - \omega)y_2 + (\gamma/2)[1 - (y_1^2 + y_2^2)/4]y_1 - \gamma E/2\omega, \quad A_2 = (\omega - \omega_0)y_1 + (\gamma/2)[1 - (y_1^2 + y_2^2)/4]y_2. \tag{8}$$

Without detuning ($\omega = \omega_0$) detailed balance holds, and the stationary solution of (8) is

$$p_s(y_1, y_2) = N \exp[-\phi(y_1, y_2)\gamma/\epsilon], \tag{9}$$

with

$$\phi(y_1, y_2) = -(y_1^2 + y_2^2)/4 + (y_1^2 + y_2^2)^2/32 + E(2\omega)^{-1}y_1. \tag{10}$$

The function ϕ has the shape of a Mexican hat, with an inclination depending on E/ω. Since (6) with $\bar{\xi}_1 = 0 = \bar{\xi}_2$ can be rewritten as

$$\dot{y}_1 = -\gamma\partial\phi/\partial y_1, \quad \dot{y}_2 = -\gamma\partial\phi/\partial y_2, \tag{11}$$

ϕ may be viewed as the "potential" for the purely frictional motion in the (y_1, y_2) plane. Therefore the stationary points of ϕ coincide with the fixed points of the unperturbed motion, i.e. with those of (11). For $E \neq 0$ all these points lie on the y_1 axis (see fig. 1), and their y_1 coordinates follow from

$$4y_1 - y_1^3 - 4E/\omega = 0. \tag{12}$$

With $\cos\varphi \stackrel{\Delta}{=} -3^{3/2} E/4\omega$ the roots of (12) are

$$y_1^{\mathrm{a}} = -(4/3^{1/2})\cos(\varphi/3 - 60°), \quad y_1^{\mathrm{b}} = -(4/3^{1/2})\cos(\varphi/3 + 60°), \quad y_1^{\mathrm{c}} = (4/3^{1/2})\cos\varphi/3. \tag{13}$$

Here a denotes the minimum of ϕ, which is the stable point of (11), b the saddle of ϕ being the hyperbolic point of (11), and c the maximum of ϕ being the unstable point of (11). We mention that $E = 0$ implies $\varphi = 90°$ and thus

$$y_1^{\mathrm{a}} = -2, \quad y_1^{\mathrm{b}} = 2, \quad y_1^{\mathrm{c}} = 0,$$

furthermore that for the largest E admitting real roots, i.e. for $E = 4\omega/3^{3/2}$), $\varphi = 180°$ and thus

$$y_1^{\mathrm{a}} = -4/3^{1/2}, \quad y_1^{\mathrm{b}} = y_1^{\mathrm{c}} = 2/3^{1/2}.$$

E is supposed to be contained within this range, excluding the limits.

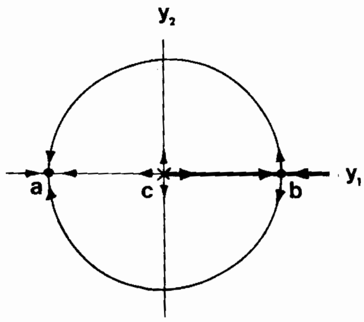

Fig. 1. Fixpoints of the noiseless motion in the rotating frame of reference: a stable, c unstable, b hyperbolic.

In what follows it is assumed that the noise is weak, i.e. that ϵ is small. Then the stationary distribution p_s is concentrated in a small neighbourhood of a. A phase slip can now be characterized in the following way: while the system stays most of the time near a, it is occasionally driven to the saddle b by the noise. If then it leaves the region of b on the other side than it had approached it, so that a full surrounding of c is achieved, a phase slip is performed. At first glance one would expect that the rate of these events can be determined by Kramers' method. The failure in applying his idea arises from the fact that both before and after a phase slip the system stays in the same state (i.e. near a), and a current-carrying solution of (8) vanishing at the final state, but not at the initial state, does not make sense. A way out of this problem is provided by the following argument: a phase slip is a crossing of the part $y_1 > y_1^c$ of the y_1 axis. Instead of considering the mean time elapsed between two such events, one can artificially assume that this half-line is absorbing and calculate the mean time T until "absorption" occurs [11,12]. One merely has to take into account that an arrival on this half-line only results in an actual crossing with probability $\frac{1}{2}$, since for the departure both sides of the line are equally probable. The total slipping rate (in either direction) is therefore $\frac{1}{2}$ of the absorbing rate. The absorbing rate itself can readily be determined by the result of ref. [12]. There the mean time T until "absorption" was given by the general expression

$$T = [2/\epsilon(\alpha+1)]^{\alpha/(\alpha+1)}\Gamma[1/(\alpha+1)] \int \mathrm{d}y_1\, \mathrm{d}y_2\, w \left(\int_{\partial\Omega} (-\mathrm{d}S_r) w (D^{rr})^{\alpha/(\alpha+1)} g^{1/(\alpha+1)} \right)^{-1} . \tag{14}$$

To apply this formula we note that here r denotes the y_2 direction, and $-\mathrm{d}S_r = \mathrm{d}y_1$; the drift in the y_2 direction near the absorbing line is $-y_2\gamma(\partial^2\phi/\partial y_2^2)_{y_2=0}$, which gives both $\alpha = 1$ and $g = \gamma|\partial^2\phi/\partial y_2^2|_{y_2=0}$. Furthermore $D^{rr} = 2$ and $w = p_s$. Thus (14) is now reduced to

$$T^{-1} = (2\gamma\epsilon/\pi)^{1/2} \int_{y_1^c}^{\infty} \mathrm{d}y_1\, p_s(y_1, 0)(|\partial^2\phi/\partial y_2^2|_{y_2=0})^{1/2} . \tag{15}$$

For small ϵ, $p_s(y_1, 0)$ only contributes near the saddle point b, and there it can be approximated by

$N \exp\{(-\gamma/\epsilon)[\phi_b + (\partial^2\phi/\partial y_1^2)_b (y_1 - y_1^b)^2/2]\}$,

while $|\partial^2\phi/\partial y_2^2|_{y_2=0}$ may be replaced by its value at the saddle itself. Therefore (15) becomes

$T^{-1} = 2\epsilon[|\partial^2\phi/\partial y_2^2|_b (\partial^2\phi/\partial y_1^2)_b^{-1}]^{1/2} N \exp[(-\gamma/\epsilon)\phi_b]$.

It remains to evaluate the normalizing factor N of (9). For this an expansion of p_s around a is sufficient:

$p_s \approx N \exp\{(-\gamma/\epsilon)[\phi_a + (\partial^2\phi/\partial y_1^2)_a (y_1 - y_1^a)^2/2 + (\partial^2\phi/\partial y_2^2)_a (y_2 - y_2^a)^2/2]\}$,

which leads to

$$N = \gamma(2\pi\epsilon)^{-1}\,[(\partial^2\phi/\partial y_1^2)_a(\partial^2\phi/\partial y_2^2)_a]^{1/2}\,\exp[(\gamma/\epsilon)\phi_a]\,,$$

so that finally

$$T^{-1} = \gamma\pi^{-1}\,\exp[(-\gamma/\epsilon)(\phi_b - \phi_a)]\,[(\partial^2\phi/\partial y_1^2)_a(\partial^2\phi/\partial y_2^2)_a|\partial^2\phi/\partial y_2^2|_b/(\partial^2\phi/\partial y_1^2)_b]^{1/2}. \tag{16}$$

In ref. [12] it is understood that the absorbing line is approached from one side only. Therefore, $(2T)^{-1}$ is the rate of phase slips of one direction, and the total slipping rate is just given by (16).

The second derivatives of ϕ at a and b may be expressed in terms of (13):

$$2(\partial^2\phi/\partial y_1^2)_{a,b} = \tfrac{3}{2}(y_1^{a,b}/2)^2 - 1, \quad 2|\partial^2\phi/\partial y_2^2|_{a,b} = (y_1^{a,b}/2)^2 - 1. \tag{17}$$

Eqs. (16), (17), (13) and (10) give the phase-slipping rate in leading order as $\epsilon \to 0$. The method presented here can be extended to include a detuning, but since this case requires a more detailed discussion of the solutions of the stationary Fokker–Planck equation at low noise, this will be presented separately. If the noise is not really weak, the phase slip can be defined and calculated according to ref. [13]; a simpler method, which for the present system gives the exact slipping rate (see ref. [14]), is presented in ref. [15].

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