Can Self-Sustaining Currents Be Induced In A System Of Mesoscopic Rings?

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Abstract. Mesoscopic systems exhibit both quantum and classical features. A purely quantum and topological phenomenon is the occurrence of a persistent current. We analyze a collection of coaxial, mesoscopic rings which are coupled via mutual inductances. At temperatures $T > 0$, thermal fluctuations are taken into account. The system is described in terms of a set of Langevin equations. The unsolved problem is to find steady states in the limit of infinitely many rings (i.e. in the thermodynamic limit). The first problem is to evaluate the effective coupling constants which are determined by elements of the inverse matrix of mutual inductances. The second problem is to apply the mean-field type approximation: can it be justified? If yes, then the resulting steady states are determined by a nonlinear Fokker-Planck equation from which it follows that self-sustaining currents can be induced by interactions among rings. If no, then it is an open and much more difficult problem to identify the corresponding effective state equation.

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DESCRIPTION OF MODEL

In mesoscopic systems [1] of ring, toroidal or cylindrical symmetry persistent currents can occur [2] as a result of the phase coherence among electrons, the so-called *coherent electrons*. Experiments [3] have confirmed the existence of such persistent currents. In the ground state, at temperature $T = 0$, the only electrons present in the system are coherent ones, possessing a non-dissipative flow. At non-zero temperature, $T > 0$, some of those electrons become "normal" and their behavior is dissipative. This induces a decrease of the amplitude of the persistent current.

We consider a system of N identical, coaxially composed mesoscopic metal rings (toroids) which are coupled via the mutual inductances. The rings are separated with a distance r between their centers. Magnetic fluxes and currents in the rings are coupled according to the formula [4]

$$\phi_i = \sum_{k=1}^{N} \mathscr{M}_{ik} I_k + \phi_{ext} , \qquad (1)$$

where ϕ_i and I_i are flux and current in the i-th ring, respectively. The flux ϕ_{ext} is induced by an external uniform magnetic field B. The coupling coefficients $\mathscr{M}_{ik} = \mathscr{M}_{ki}$ (forming the matrix $\mathscr{M}$) denote the mutual inductances for $i \neq k$ and identical self-inductances $\mathscr{L} = \mathscr{M}_{ii}$ for $i = k$.

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The current in the k-th ring equals a sum

$$I_k = I_k^{nor} + I_k^{coh} \tag{2}$$

of the Ohmic (dissipative) current I_k^{nor} plus the persistent current I_k^{coh}. The Ohmic current contribution $I_k^{nor} = I_{nor}(\phi_k)$ is determined by Ohm's law and Lenz's rule, i.e.,

$$I_{nor}(\phi_k) = -\frac{1}{R}\frac{d}{dt}\phi_k + \sqrt{\frac{2k_BT}{R}}\Gamma_k(t) \,, \tag{3}$$

where R denotes the resistance of a single ring [5], k_B is the Boltzmann constant and $\Gamma_k(t)$ describes the thermal, Johnson-Nyquist fluctuations of the Ohmic current. This thermal noise is modelled by a set of independent Gaussian white noises of zero average, i.e., $\langle\Gamma_k(t)\rangle = 0$ and δ-correlated correlations $\langle\Gamma_k(t)\Gamma_i(s)\rangle = \delta_{ki}\delta(t-s)$. The noise intensity $D_0 = \sqrt{2k_BT/R}$ is chosen in accordance with the classical fluctuation-dissipation theorem [6].

The coherent current can be either of paramagnetic nature for an even number N_e of coherent electrons, or of diamagnetic nature for an odd number of coherent electrons. The probability of finding a channel (ring) with an odd number of coherent electrons is denoted by P and the probability of finding a channel with an even number of coherent electrons is equal to $1-P$, respectively. The current of the coherent electrons $I_k^{coh} = I_{coh}(\phi_k, T)$ has been determined in Ref. [7], reading

$$I_{coh}(\phi_k, T) = I^* \left[Pg(\phi_k/\phi_0, T) + (1-P)g(\phi_k/\phi_0 + 1/2, T)\right] \,, \tag{4}$$

where the flux quantum $\phi_0 := h/e$ is the ratio of the Planck constant h and the electron charge e. The characteristic current $I^* = heN_e/(2l_x^2 m_e)$, with N_e being the number of coherent electrons in a single current channel (ring), l_x is the circumference of the ring and m_e is the mass of electron. Moreover, [7]

$$g(x,T) = \sum_{n=1}^{\infty} A_n(T)\sin(2n\pi x) \tag{5}$$

denotes the current in a channel with an even number of coherent electrons. The amplitudes read

$$A_n(T) = \frac{4T}{\pi T^*}\frac{\exp(-nT/T^*)}{1-\exp(-2nT/T^*)}\cos(nk_F l_x) \,. \tag{6}$$

The characteristic temperature T^* is determined from the relation $k_BT^* = \Delta_F/2\pi^2$, where Δ_F marks the energy gap and k_F is the momentum at the Fermi surface.

STOCHASTIC EVOLUTION EQUATIONS

Given eqs. (1)-(4), it follows that the evolution equations for the fluxes assume the form

$$\frac{1}{R}\sum_{k=1}^{N}\mathscr{M}_{ik}\frac{d\phi_k}{dt} = \phi_{ext} - \phi_i + \sum_{k=1}^{N}\mathscr{M}_{ik}I_{coh}(\phi_k, T) + \sqrt{\frac{2k_BT}{R}}\sum_{k=1}^{N}\mathscr{M}_{ik}\Gamma_k(t) \tag{7}$$

for $i = 1 \ldots N$. We like to remark that this set of equations possesses an untypical form: In the language of particles, the left hand side of this equation corresponds to an unusual coupling of 'velocity' degrees of freedom $d\phi_k/dt$. This note is essential because the known literature results for particle or spin systems with a position-position interaction cannot be directly applied to the set (7).

Let us recall that the interaction among the coaxially composed rings is characterized by the mutual inductances $\mathscr{M}_{ik} = f(r_{ik})$, where r_{ik} is the distance between the i-th and the k-th ring and [8]

$$f(r) = \frac{8\pi R}{b(r)} \left[\left(1 - \frac{b^2(r)}{2}\right) \mathrm{K}(b(r)) - \mathrm{E}(b(r)) \right]. \tag{8}$$

The function $b^2(r) = 4R^2/[4R^2 + r^2]$ and R is the radius of the ring. The functions $\mathrm{K}(z) =$ and $\mathrm{E}(z)$ are the complete elliptic integrals [4]. For short distances between the rings $\mathscr{M}_{ik} \propto -\ln r_{ik}$, while $\mathscr{M}_{ik} \propto r_{ik}^{-3}$ for large distances r_{ik}.

The system (7) can be reformulated in terms of an effective 'position-position' interaction. Indeed, multiplying this system of equations by the elements $\left(\mathscr{M}^{-1}\right)_{ni}$ of the inverse matrix $\mathscr{M}^{-1}$ and next summing up over i one obtains the following set of Langevin equations

$$\frac{1}{R}\frac{d\phi_n}{dt} = \sum_{i=1}^{N} (\mathscr{M}^{-1})_{ni} \left[\phi_{ext} - \phi_i\right] + I_{coh}(\phi_n, T) + \sqrt{\frac{2k_BT}{R}}\, \Gamma_n(t) \,. \tag{9}$$

Its dimensionless form reads [9]

$$\frac{dx_n}{ds} = -V'(x_n, T) - \sum_{i(\neq n)}^{N} \lambda_{ni} x_i + \sqrt{2D}\, \tilde{\Gamma}_n(s) \,. \tag{10}$$

The dimensionless flux is $x_n = \phi_n/\phi_0$ and the dimensionless time reads $s = t/\tau_0$, where $\tau_0 = \mathscr{L}/R$ is the relaxation time of the averaged Ohmic current. The prime denotes the derivative with respect to the first argument of the generalized potential $V(x_n, T)$, i.e. here with respect to x_n. The generalized potential itself is given by

$$V(x_n, T) = \frac{1}{2} a_n x_n^2 - b_n x_n - I_0 \int^{x_n} f(y, T)\, dy \,. \tag{11}$$

The coupling constants are $\lambda_{ni} = \mathscr{L}(\mathscr{M}^{-1})_{ni}$, and the parameter $a_n = \mathscr{L}(\mathscr{M}^{-1})_{nn}$ corresponds to the n-th diagonal element of the inverse matrix $\mathscr{M}^{-1}$. The re-scaled, externally induced fluxes read $b_n = \gamma_n \phi_{ext}/\phi_0$, where $\gamma_n = \mathscr{L} \sum_{i=1}^{N} (\mathscr{M}^{-1})_{ni}$. The re-scaled characteristic current is given by $I_0 = \mathscr{L} I^*/\phi_0$ and

$$f(y, T) = P g(y, T) + (1 - P) g(y + 1/2, T) \,. \tag{12}$$

The zero-mean re-scaled noise reads $\tilde{\Gamma}_n(s) = \sqrt{\tau_0}\, \Gamma_n(\tau_0 s)$ with the correlations $\langle \tilde{\Gamma}_n(s_1) \tilde{\Gamma}_m(s_2) \rangle = \delta_{nm} \delta(s_1 - s_2)$. Its intensity is $D = k_BT/2\varepsilon_0$, where $\varepsilon_0 = \phi_0^2/2\mathscr{L}$ [10].

The generalized potential (11) can either be multistable, specifically assume a symmetric monostable or also a symmetric bistable shape. The dynamics is thus equivalent to a dynamic model of a collection of one-dimensional anharmonic oscillators moving in a multistable on-site potential that are coupled among each other with bi-linear interactions.

STEADY-STATE EQUATION

From the corresponding Fokker-Planck equation for the joint probability density $p(\{x_n\},s)$ of the N-ring system described by (10) one can derive, – after integration over all variables except x_k, – the nonlinear steady-state equation for the one-dimensional stationary probability density $p_s(x_k)$, which has the form [9]

$$\frac{\partial}{\partial x_k}\left[V'(x_k,T)+\sum_{i\neq k}^{N}\lambda_{ik}\langle x_i|x_k\rangle\right]p_s(x_k)+D\frac{\partial^2}{\partial x_k^2}p_s(x_k)=0, \tag{13}$$

where $\langle x_i|x_k\rangle=\int x_i p_s(x_i|x_k)dx_i$ is a stationary, conditional mean value of x_i with respect to the conditional probability density $p_s(x_i|x_k)$. This equation is formally exact, however it is clearly not closed: it contains the unknown quantity $\langle x_i|x_k\rangle$ which can be determined only via an approximation scheme. The most popular mean-field approximation can be formulated as follows. Rewrite the conditional mean value as $\langle x_i|x_k\rangle=\langle x_i\rangle+c_{ik}$, wherein c_{ik} accounts for correlations between i-th and k-th ring. In the thermodynamic limit, when $N\longrightarrow\infty$, the system becomes statistically homogeneous so that the stationary average $\langle x_k\rangle=\langle x\rangle$ no longer depends on the index k. If in this limit we shall neglect the correlations, i.e. if we put $c_{ik}=0$ then (13) renders a closed but non-linear equation. The crucial problem is whether this constitutes an approximation or whether in fact it presents an exact, limiting result. Let us recall that the coupling constants λ_{ik} are expressed by elements of the inverse matrix $\left(\mathscr{M}^{-1}\right)_{ik}$. In the thermodynamic limit, it is an infinite-dimensional matrix. How does its elements behave with respect to the dependence on the distance r_{ik} between the rings? Presently, we are not able to answer to this question. As a consequence we lack a rigorous proof that $c_{ik}\longrightarrow 0$ as $N\longrightarrow\infty$. If the mean-field approximation can indeed be applied, then the stationary probability density for $x=x_k$ would satisfy the non-linear Fokker-Planck equation

$$\frac{d}{dx}[V'(x,T)-\lambda\mu]p_s(x)+D\frac{d^2}{dx^2}p_s(x)=0, \tag{14}$$

where

$$\mu\equiv\langle x\rangle=\int_{-\infty}^{\infty}x\,p_s(x)\,dx \tag{15}$$

is the order parameter of the system and $\lambda=-\sum_{i\neq k}\mathscr{L}(\mathscr{M}^{-1})_{ik}$ denotes an effective coupling constant. In this sum, the index k is fixed and $i\in(-\infty,\infty)$. However, for the system composed of infinitely many rings, the result does not depend on k. The next problem involves the sign of λ. If $\lambda>0$, then we can expect a "ferromagnetic" state of the system, characterized by the parallel alignment of the magnetic moments induced by the currents flowing in the neighboring rings. The *flux state* is characterized by the non-vanishing mean flux $\mu = \langle x \rangle \neq 0$. If an external magnetic field is applied, then trivially $\mu \neq 0$. The non-trivial case emerges when the external flux ϕ_{ext} is zero but $\mu \neq 0$. Then, via Eq. (1), currents may flow without any external magnetic field and are solely induced by the interaction among the various rings. This phenomenon is what we call term self-sustaining currents. Can this situation be realized experimentally?

INTERACTION MATRICES

In Eq. (7), the coupling among the $(N+1)$-rings is described by the matrix

$$\mathscr{M} = \begin{pmatrix} \mathscr{L} & f(r) & f(2r) & f(3r) & \dots & \dots & f(Nr) \\ f(r) & \mathscr{L} & f(r) & f(2r) & \dots & \dots & f((N-1)r) \\ f(2r) & f(r) & \mathscr{L} & f(r) & \dots & \dots & f((N-2)r) \\ \vdots & & & & & & \\ f(Nr/2) & \dots & f(r) & \mathscr{L} & f(r) & \dots & f(Nr/2) \\ \vdots & & & & & & \\ f(Nr) & \dots & \dots & f(3r) & f(2r) & f(r) & \mathscr{L} \end{pmatrix} \tag{16}$$

where the function $f(r)$ is defined in Eq. (8). In Eq. (9), the 'position-position' interaction is determined by the inverse matrix $\mathscr{M}^{-1}$. As mentioned above, we are interested in the thermodynamic limit $N \to \infty$. Then the inverse matrix $\mathscr{M}^{-1}$ is infinite-dimensional. In this limit, each ring has infinitely many symmetric neighbors from above and from below. The crucial quantities are the elements $\left(\mathscr{M}^{-1}\right)_{ik}$ of the inverse matrix and their dependence on the distance r_{ik} between rings.

Mean-field theory has played a seminal role for understanding the behavior of complex and cooperative systems, in particular phase transitions. A number of rigorous results have been obtained answering the question: under what conditions is a mean-field procedure yielding a qualitatively correct prediction. Much is known for lattice models and spin systems [11, 12, 13]. There exists a widespread opinion that, for example, there are (at finite temperatures) no phase transitions occurring in one-dimensional (1D) systems possessing *short range* interactions. Van Hove's result [14], and the extension by Ruelle [15], as well as the collection of well-known, exactly solvable models (Ising or Potts models) seem to support this view. However, there are examples of 1D models with short range interactions, and very important – in presence of on-site potentials – that indeed do exhibit a true thermodynamic phase transition [11, 16]. Our reduced description of the real 3D system of coupled mesoscopic rings is based on the classical Langevin stochastic equations (10), which equivalently describes a 1D system of anharmonic, multistable oscillators. For the case of a global interaction, such systems can undergo a well-defined phase transition [17]. The interaction in (10) is not global but local and we do not know of any rigorous results for systems of the type studied in this paper. So, it is left a open question whether we have a phase transition in a one-dimensional system of coupled mesoscopic rings.

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