Theory of relativistic Brownian motion: The (1+1)-dimensional case

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We construct a theory for the (1+1)-dimensional Brownian motion in a viscous medium, which is (i) consistent with Einstein's theory of special relativity and (ii) reduces to the standard Brownian motion in the Newtonian limit case. In the first part of this work the classical Langevin equations of motion, governing the nonrelativistic dynamics of a free Brownian particle in the presence of a heat bath (white noise), are generalized in the framework of special relativity. Subsequently, the corresponding relativistic Langevin equations are discussed in the context of the generalized Ito (prepoint discretization rule) versus the Stratonovich (midpoint discretization rule) dilemma: It is found that the relativistic Langevin equation in the Hänggi-Klimontovich interpretation (with the postpoint discretization rule) is the only one that yields agreement with the relativistic Maxwell distribution. Numerical results for the relativistic Langevin equation of a free Brownian particle are presented.

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I. INTRODUCTION

For almost 100 years, Einstein's theory of special relativity [1,2] is serving as the foundation of our most successful physical standard models (apart from gravity). The most prominent and, probably, also the most important feature of this theory is the absolute character of the speed of light c, representing an unsurmountable barrier for the velocity of any (macroscopic) physical process. Due to the great experimental success of the original theory, almost all other physical theories have successfully been adapted to the framework of special relativity over the past decades. Surprisingly, however, the scientific literature provides relatively few publications on the subject of relativistic Brownian motions (classical references are [3–5] and more recent contributions include [6–13]).

Brownian particles are physical objects (e.g., dust grains) that move randomly through a surrounding medium (heat bath). Their stochastic motions are caused by permanent collisions with much lighter constituents of the heat bath (e.g., molecules of a liquid). The classical theory of Brownian motion or *nonrelativistic* diffusion theory, respectively, was developed by Einstein [14] and Einstein and von Smoluchowski [15]. Since the beginning of the last century, when their seminal papers were published, the classical theory has been investigated and generalized by a large number of physicists [16–20] and mathematicians [21–23]. The intense research led, among others, to different mathematical representations of the Brownian motion dynamics [Langevin equations, Fokker-Planck equations (FPE), etc.] [18–20], to the notion of the Wiener processes [21], and to new techniques for solving partial differential equations (Feynman-Kac formula, etc. [22,23]).

With regard to special relativity, standard Brownian motion faces the problem that it permits velocity jumps Δv, that exceed the speed of light c (see also Schay [3]). This is due to the fact that in the nonrelativistic theory the velocity increments Δv have a Gaussian distribution, which always assigns a nonvanishing (though small) probability to events $\Delta v > c$. This problem is also reflected by the Maxwell distribution, which represents the stationary velocity distribution for an ensemble of free Brownian particles and permits absolute velocity values $v > c$ [20].

The first relativistically consistent generalization of Maxwell's velocity distribution was introduced by Jüttner [24] in 1911. Starting from an extremum principle for the entropy, he obtained the probability distribution function of the relativistic ideal Boltzmann gas [see Eq. (67) below]. In principle, however, Jüttner's approach made no contact with the theory of Brownian motion. Fifty years after Jüttner's work, Schay [3] performed the first comprehensive mathematical investigation of relativistic diffusion processes based on Lorentz-invariant transition probabilities. On the mathematical side, Schay's analysis was complemented by Hakim [5] and Dudley [4], who studied in detail the properties of Lorentz-invariant Markov processes in relativistic phase space. After 40 more years, Franchi and Le Jan [13] have presented an extension of Dudley's work to general relativity. In particular, these authors discuss relativistic diffusions in the presence of a Schwarzschild metric [25]. Hence, over the past 100 years there has been steady (though relatively slow) progress in the mathematical analysis of relativistic diffusion processes.

By contrast, one finds in the physical literature only very few publications that directly address the topic of the relativistic Brownian motion (despite the fact that relativistic kinetic theory has been fairly well established for more than 30 years [26–29]). Among the few exceptions are the papers by Boyer [8,9] and Ben-Ya'acov [6], who have studied the interaction between two energy-level particles and electro-

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magnetic radiation in thermal equilibrium, the latter acting as a heat bath. In contrast to their specific microscopic model, we shall adopt a more coarse-grained point of view here by assuming that the heat bath is sufficiently well described by macroscopic friction and diffusion coefficients.

Generally, the objective of the present paper can be summarized as follows: We would like to discuss how one can construct, in a physically straightforward manner, a relativistic theory of Brownian motion for particles moving in a homogeneous, viscous medium. For this purpose it is sufficient to concentrate on the case of $1+1$ dimensions (generalizations to the $1+3$ dimensions are straightforward and will be discussed separately in a forthcoming contribution). As a starting point we choose the nonrelativistic Langevin equations of the free Brownian particle. In Sec. II these equations will be generalized such that they comply with special relativity. As we shall see in Sec. III due to multiplicative noise for the momentum degree of freedom, the resulting relativistic Langevin equations are *not* sufficient in order to uniquely determine the corresponding Fokker-Planck equation (generalized Ito-Stratonovich dilemma). Furthermore, it it is shown that the stationary solution of a particular form for the relativistic Fokker-Planck equation coincides with Jüttner's relativistic Maxwell distribution (Sec. III B 3). Finally, we also discuss numerical results for the mean-square displacement in Sec. IV.

It might be worthwhile to emphasize that the systematic Langevin approach pursued below is methodically different from those in Refs. [3–13] and also from the kinetic theory approach [26–29]. It is therefore satisfactory that our findings are apparently consistent with rigorous mathematical results, obtained by Schay [3] and Dudley [4] for the case of free relativistic diffusion. Moreover, it will become clear in Sec. IV that numerical simulations of the relativistic Langevin equations constitute a very useful tool for the numerical investigation of relativistic diffusion processes, provided that the discretization rule is carefully chosen.

II. LANGEVIN DYNAMICS

First the main properties of the nonrelativistic Langevin equations for free Brownian particles are briefly summarized (Sec. II A). Subsequently, we construct generalized Lorentz-covariant Langevin equations (Sec. II B). Finally, the covariant Langevin equations will be rewritten in laboratory coordinates (Sec. II C).

The following notations will be used throughout the paper. Since we confine ourselves to the $(1+1)$-dimensional case, upper and lower Greek indices $\alpha,\beta,\ldots$ can take values 0, 1, where 0 refers to the time component. The $(1+1)$-dimensional Minkowski metric tensor with respect to Cartesian coordinates is taken as

$$(\eta_{\alpha\beta}) = (\eta^{\alpha\beta}) = \mathrm{diag}(-1,1).$$

Moreover, Einstein's summation convention is invoked throughout.

A. Physical foundations

Consider the nonrelativistic one-dimensional motion of a Brownian particle with mass m that is surrounded by a heat bath (e.g., small liquid particles). In the Langevin approach the nonrelativistic dynamics of the Brownian particle is described by the stochastic dynamical equations (see, e.g., [20] Chap. IX)

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v(t), \tag{1a}$$

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\nu m v(t) + L(t), \tag{1b}$$

where ν is the viscous friction coefficient. The Langevin force $L(t)$ is characterized by

$$\langle L(t)\rangle = 0, \quad \langle L(t)L(s)\rangle = 2D\delta(t-s), \tag{2}$$

with all higher cumulants being zero (Gaussian white noise), and D being constant. More general models may include velocity-dependent parameters ν and D (see, e.g., [19,30–32]), but we shall restrict ourselves to the simplest case here. It is worthwhile to summarize the physical assumptions, implicitly underlying Eqs. (1) as follows:

(i) The heat bath is homogeneous.

(ii) Stochastic impacts between the Brownian particle and the constituents of the heat bath occur virtually uncorrelated.

(iii) On the macroscopic level, the interaction between Brownian particle and heat bath is sufficiently well described by the constant viscous friction coefficient ν and the white noise force L.

(iv) Equations (1) hold in the rest frame Σ_0 of the heat bath (corresponding to the specific inertial system, in which the average velocity of the heat bath vanishes for all times t). In the following Σ_0 will also be referred to as *laboratory frame*.

In the mathematical literature, Eq. (1b) is usually written as

$$\mathrm{d}[mv(t)] = -\nu m v(t)\mathrm{d}t + \mathrm{d}W(t), \tag{3a}$$

where $W(t)$ is a one-dimensional Wiener process [19,22,23], i.e., the density of the increments

$$w(t) \equiv \mathrm{d}W(t) \equiv W(t+\mathrm{d}t) - W(t) \tag{3b}$$

is given by

$$\mathcal{P}^1[w(t)] = \frac{1}{\sqrt{4\pi D\,\mathrm{d}t}}\exp\left[-\frac{w(t)^2}{4D\,\mathrm{d}t}\right]. \tag{3c}$$

Here the abbreviation $w \equiv \mathrm{d}W$ has been introduced to simplify the notation in subsequent formulas. From Eq. (3c) one finds in agreement with (2)

$$\langle w(t)\rangle = 0, \quad \langle w(t)w(s)\rangle = \begin{cases} 0, & t \neq s \\ 2D\,\mathrm{d}t, & t = s. \end{cases} \tag{4}$$

Depending on which notation is more convenient for the current purpose, we shall use below either the physical formulation (1) or the mathematical formulation (3). The two formulations can be connected by (formally) setting

$$w(t) = \mathrm{d}W(t) = L(t)\mathrm{d}t. \tag{5}$$

B. Relativistic generalization

It is well known that in inertial coordinate systems, which are comoving with a particle at a given moment t, the relativistic equations must reduce to the nonrelativistic Newtonian equations (see, e.g., [25] Chap. 2.3). Therefore, our strategy is as follows. Starting from the Langevin equations (1) or (3a), respectively, we construct in the first step the nonrelativistic equations of motion with respect to a coordinate frame Σ_*, comoving with the Brownian particle at a given moment t. In the second step, the general form of the covariant relativistic equation motions are found by applying a Lorentz transformation to the nonrelativistic equations that have been obtained for Σ_*.

It is useful to begin by considering the deterministic (noise-free) limit case, corresponding to a pure damping of the particle's motion. This will be done Sec. II B 1. Subsequently, the stochastic force is separately treated in Sec. II B 2.

1. Viscous friction

Setting the stochastic force term to zero (corresponding to a vanishing temperature of the heat bath), the nonrelativistic Eq. (1b) simplifies to

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\nu m v(t). \tag{6}$$

The energy of the Brownian particle is purely kinetic,

$$E(t) = \frac{mv(t)^2}{2}, \tag{7}$$

and, by virtue of (6), its time derivative is given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = mv\frac{\mathrm{d}v}{\mathrm{d}t} = -\nu m v^2. \tag{8}$$

As stated above, in the nonrelativistic theory the last three equations are assumed to hold in the rest frame Σ_0 of the heat bath. Now consider another inertial coordinate system Σ_*, in which the Brownian particle is temporarily at rest at time t or $t_*=t_*(t)$, respectively, where t_* denotes the Σ_*-time coordinate. That is, in Σ_* we have at time t

$$v_*(t) \equiv v_*(t_*(t)) = 0. \tag{9}$$

[Conventionally, we use throughout the lax notation $g_*(t) \equiv g_*(t_*(t))$, where g_* is originally a function of t_*.] With respect to the comoving frame Σ_*, the heat bath will, in general, have a nonvanishing (average) velocity V_*. Then, using a Galilean transformation we find that Eq. (6) in Σ_* coordinates at time t reads as follows:

$$m\frac{\mathrm{d}v_*}{\mathrm{d}t_*}(t) = -\nu m(v_*(t) - V_*) \overset{(9)}{=} \nu m V_*. \tag{10a}$$

Similarly, in Σ_* coordinates Eq. (8) is given by

$$\frac{\mathrm{d}E_*}{\mathrm{d}t_*}(t) = -\nu m v_*(t)(v_*(t) - V_*) \overset{(9)}{=} 0. \tag{10b}$$

Note that in the nonrelativistic (Newtonian) theory the left equalities in Eqs. (10) are valid for arbitrary time t. By contrast, in the relativistic theory these equations are exact at time t only if Σ_* is comoving at time t. In the latter case, we can use Eqs. (10) to construct relativistically covariant equations of motion. Introducing, as usual, the proper time τ by the definition

$$\mathrm{d}\tau \equiv \mathrm{d}t\sqrt{1-\frac{v^2}{c^2}} = \mathrm{d}t_*\sqrt{1-\frac{v_*^2}{c^2}}, \tag{11}$$

and combining momentum $p_*=mv_*$ and energy into a (1+1)-vector $(p_*^\alpha)=(p^0,p_*)=(E_*/c,p_*)$, we can rewrite Eqs. (10) in the covariant form

$$\frac{\mathrm{d}p_*^\alpha}{\mathrm{d}\tau} = f_*^\alpha, \quad (f_*^\alpha) = -m\nu(0, v_* - V_*). \tag{12}$$

Let (u_*^α) and (U_*^α) denote the (1+1)-velocity components of Brownian particle and heat bath, respectively. Now it is important to realize that the covariant force vector f^α *cannot* be simply proportional to the (1+1)-velocity difference,

$$f_*^\alpha \neq -m\nu(u_*^\alpha - U_*^\alpha), \tag{13}$$

because, in general, at time t in Σ_*

$$u_*^0 - U_*^0 = \frac{c}{\sqrt{1-v_*^2/c^2}} - \frac{c}{\sqrt{1-V_*^2/c^2}} \overset{(9)}{=} c - \frac{c}{\sqrt{1-V_*^2/c^2}} \neq 0. \tag{14}$$

However, we can write f_*^α in a manifestly covariant form, if we introduce the *friction tensor*

$$({\nu_*}^\alpha{}_\beta) = \begin{pmatrix} 0 & 0 \\ 0 & \nu \end{pmatrix}, \tag{15}$$

which allows us to rewrite (12) as

$$\frac{\mathrm{d}p_*^\alpha}{\mathrm{d}\tau} = -m{\nu_*}^\alpha{}_\beta(u_*^\beta - U_*^\beta). \tag{16}$$

This equation is manifestly Lorentz-invariant, and we drop the asterisk from now on, while keeping in mind that the diagonal form of the friction tensor (15) is linked to the rest frame Σ_* of the Brownian particle. In this respect the friction tensor is very similar to the pressure tensor, as known from the relativistic hydrodynamics of perfect fluids (see, e.g., [25] Chap. 2.10). This analogy yields immediately the following representation:

$$\nu^\alpha{}_\beta = \nu\left(\eta^\alpha{}_\beta + \frac{u^\alpha u_\beta}{c^2}\right). \tag{17}$$

It is now interesting to consider Eq. (16) in the laboratory frame Σ_0, defined above as the rest frame of the heat bath. There we have

$$(U^\beta) = (c,0), \quad (u^\beta) = (\gamma c, \gamma v), \quad \mathrm{d}\tau = \frac{\mathrm{d}t}{\gamma}, \quad \gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}}. \tag{18}$$

Combining (16)–(18) we find that the relativistic equations of motion in Σ_0 are given by

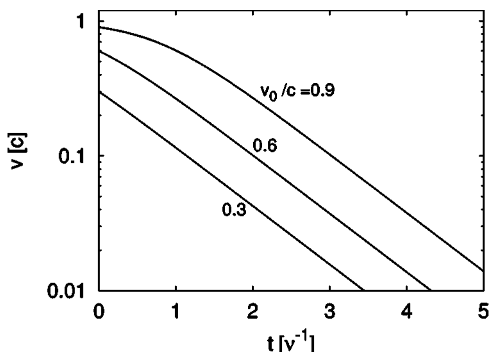

FIG. 1. Velocity curves $v(t)$, corresponding to Eq. (24), for the purely damped motion of a relativistic particle in the rest frame of the heat bath (laboratory frame). Especially at high velocities $|v| \lesssim c$, the relativistic velocity curves deviate from the exponential decay, predicted by the Newtonian theory.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\nu\frac{mv}{\sqrt{1-v^2/c^2}} \tag{19a}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\nu\frac{mv^2}{\sqrt{1-v^2/c^2}}. \tag{19b}$$

On comparing (19a) with (6) and (19b) with (8), one readily observes that the relativistic equations (19) do indeed reduce to the known Newtonian laws in the limit case $v^2/c^2 \ll 1$.

Using the relativistic definitions

$$E = \gamma mc^2, \quad p = \gamma mv, \tag{20}$$

Eqs. (19) can also be rewritten as

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\nu p, \tag{21a}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\nu pv = -\nu E\frac{v^2}{c^2}. \tag{21b}$$

In fact, only one of the two Eqs. (19) or (21), respectively, must be solved due to the fixed relation between relativistic energy and momentum:

$$p_\alpha p^\alpha = -E^2/c^2 + p^2 = -m^2c^2 \quad \Rightarrow \quad E(t) = \frac{mc^2}{\sqrt{1-v^2/c^2}}. \tag{22}$$

The solution of (21a) reads

$$p(t) = p_0 \exp(-\nu t), \quad p(0) = p_0, \tag{23}$$

and, by using (20), one thus obtains for the velocity of the particle in the laboratory frame Σ_0 (rest frame of the heat bath)

$$v(t) = v_0\left[\left(1-\frac{v_0^2}{c^2}\right)e^{2\nu t} + \frac{v_0^2}{c^2}\right]^{-1/2}. \tag{24}$$

Figure 1 depicts a semi-logarithmic representation of the velocity $v(t)$ for different values of the initial velocity v_0. As one can see in the diagram, at high velocities $|v| \lesssim c$ the relativistic velocity curves, given by Eq. (24), exhibit essential deviations from the purely exponential decay, predicted by the Newtonian theory.

2. Stochastic force

We now construct a *relativistic generalization of the stochastic force*. To this end, we consider Eq. (11) as an *operational definition* for the proper time parameter τ. The generalization procedure will be based on the standard assumption (postulate) that, in temporarily comoving inertial frames Σ_*, the relativistic equations of motions must reduce to the Newtonian equations of motions. According to this assumption, for frames Σ_*, comoving with the particle at laboratory time t, the relativistic stochastic differential equation must reduce to

$$\mathrm{d}p_*(t) = -\nu(p_*(t) - mV_*)\mathrm{d}t_* + w_*(t), \tag{25}$$

where the momentum increments $w_* \equiv dW_*$ represent a Wiener-process with parameter D, i.e., the increments $w_*(t)$ have a Gaussian distribution

$$\mathcal{P}_*^1[w_*(t)] = \frac{1}{\sqrt{4\pi D\,\mathrm{d}t_*}}\exp\left(-\frac{w_*(t)^2}{4D\,\mathrm{d}t_*}\right). \tag{26}$$

Note that also in the relativistic theory the *momentum* increments $w(t) = dW(t)$ may tend to infinity, as long as the related *velocity* increments remain bounded. In other words, in the relativistic theory one must carefully distinguish between stochastic momentum and velocity increments (this is not necessary in the nonrelativistic theory, because Newtonian momenta are simply proportional to their velocities).

The next step is now to define the *increment (1+1)-vector* by

$$(w_*^\alpha) = (0, w_*). \tag{27}$$

This definition is in agreement with the requirement that in a comoving inertial system Σ_* the 0-component of the (1+1)-force vector must vanish (see, e.g., [25] Chap. 2.3, and also compare Eqs. (12), (31), and (32) of the present paper). Moreover, if the Lorentz frame Σ_* is comoving with the Brownian particle at given time t, then the (equal-time) white-noise relations (4) generalize to

$$\langle w_*^\alpha(t)\rangle = 0, \quad \langle w_*^\alpha(t)w_*^\beta(t)\rangle = \begin{cases} 0, & \alpha = 0 \text{ and/or } \beta = 0, \\ 2D\,\mathrm{d}t_*, & \text{otherwise}. \end{cases} \tag{28}$$

The rhs. of the second equation in (28) makes it plausible to introduce a *correlation tensor* by

$$(D_{*\alpha\beta}) = \begin{pmatrix} 0 & 0 \\ 0 & 2D\,\mathrm{d}t_* \end{pmatrix}, \tag{29a}$$

thus,

$$\langle w_*^\alpha(t)w_*^\beta(t)\rangle = D_*^{\ \alpha\beta}. \tag{29b}$$

Additionally defining an *"inverse" correlation tensor* by

$$(\hat{D}_{*\alpha\beta}) = \begin{pmatrix} 0 & 0 \\ 0 & (2D\,\mathrm{d}t_*)^{-1} \end{pmatrix}, \tag{29c}$$

allows us to generalize the distribution of the increments from Eq. (26) as follows:

$$\mathcal{P}_*^{1+1}(w_*^\alpha(t)) = \frac{1}{\sqrt{4\pi D\,\mathrm{d}t*}} \exp\left[-\frac{1}{2}\hat{D}_{*\alpha\beta}w_*^\alpha(t)w_*^\beta(t)\right]\delta(w_*^0(t)). \tag{30}$$

Here, the Dirac δ-function on the right-hand side accounts for the fact that the 0 component of the stochastic force must vanish in every inertial frame, comoving with the Brownian particle at time t; compare Eq. (27). This also follows more generally from the identity

$$0 \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}(-mc^2) = m\frac{\mathrm{d}}{\mathrm{d}\tau}(u_\alpha u^\alpha) = 2u_\alpha f^\alpha, \tag{31}$$

which, in the case of the stochastic force, translates to

$$0 = u_\alpha w^\alpha. \tag{32}$$

Hence, we can rewrite the probability distribution (30) as

$$\mathcal{P}_*^{1+1}(w_*^\alpha(t)) = \frac{c}{\sqrt{4\pi D\,\mathrm{d}t_*}} \exp\left(-\frac{1}{2}\hat{D}_{*\alpha\beta}w_*^\alpha(t)w_*^\beta(t)\right) \times \delta(u_{*\alpha}w_*^\alpha(t)), \tag{33}$$

where $(u_{*\alpha})=(-c,0)$ is the covariant $(1+1)$-velocity of the particle a the comoving rest frame. It should be stressed that, because of the constraint (32), only one of the two increments $w^\alpha \equiv dW^\alpha$ is to be regarded as "independent," which is reflected by the appearance of the δ function in (33). Also note that, due to the prefactor c, the normalization condition takes the simple form

$$1 = \left\{\prod_{\alpha=0}^{1}\int_{-\infty}^{\infty}\mathrm{d}(w_*^\alpha(t))\right\}\mathcal{P}_*^{1+1}(w_*^\alpha(t)). \tag{34}$$

Furthermore, analogous to (17), we have the following more general representation of the correlation tensors:

$$D_{\alpha\beta} = 2D\,\mathrm{d}\tau\left(\eta_{\alpha\beta} + \frac{u_\alpha u_\beta}{c^2}\right) \tag{35a}$$

$$\hat{D}_{\alpha\beta} = \frac{1}{2D\,\mathrm{d}\tau}\left(\eta_{\alpha\beta} + \frac{u_\alpha u_\beta}{c^2}\right). \tag{35b}$$

Then, in an arbitrary Lorentz frame, the density (33) can be written as

$$\begin{aligned}\mathcal{P}^{1+1}(w^\alpha(\tau)) &= \frac{c}{\sqrt{4\pi D\,\mathrm{d}\tau}} \exp\left[-\frac{1}{2}\hat{D}_{\alpha\beta}w^\alpha(\tau)w^\beta(\tau)\right] \times \delta(u_\alpha w^\alpha(\tau)) \\ &= \frac{c}{\sqrt{4\pi D\,\mathrm{d}\tau}} \exp\left[-\frac{w_\alpha(\tau)w^\alpha(\tau)}{4D\,\mathrm{d}\tau}\right]\delta(u_\alpha w^\alpha(\tau)).\end{aligned} \tag{35c}$$

To obtain the last line from the first, we have inserted $\hat{D}_{\alpha\beta}$ from (35b) and then used that $u_\alpha w^\alpha = 0$, see Eq. (32).

By virtue of the above results, we are now in the position to write down the covariant Langevin equations with respect to an arbitrary inertial system: If a Brownian particle with rest mass m, proper time τ and $(1+1)$-velocity u^β is surrounded by an isotropic, homogeneous heat bath with constant $1+1$ velocity U^β, then the relativistic Langevin equations of motions read

$$\mathrm{d}x^\alpha(\tau) = \frac{p^\alpha(\tau)}{m}\mathrm{d}\tau \tag{36a}$$

$$\mathrm{d}p^\alpha(\tau) = -\nu^\alpha{}_\beta(p^\beta(\tau) - mU^\beta)\mathrm{d}\tau + w^\alpha(\tau), \tag{36b}$$

where, according to Eq. (17), the friction tensor is given by

$$\nu^\alpha{}_\beta = \nu\left(\eta^\alpha{}_\beta + \frac{u^\alpha u_\beta}{c^2}\right), \tag{36c}$$

with ν denoting the viscous friction coefficient measured in the rest frame of the particle. This is a first main result of this work. The stochastic increments $w^\alpha(\tau) \equiv dW^\alpha(\tau)$ are distributed according to (35c) and, therefore, characterized by

$$\langle w^\alpha(\tau)\rangle = 0, \tag{36d}$$

$$\langle w^\alpha(\tau)w^\beta(\tau')\rangle = \begin{cases} 0, & \tau \neq \tau'; \\ D^{\alpha\beta}, & \tau = \tau', \end{cases} \tag{36e}$$

with $D^{\alpha\beta}$ given by (35a). Note that in each comoving Lorentz frame, in which, at a given moment t, the particle is at rest, the marginal distribution of the spatial momentum increments, defined by

$$\mathcal{P}^1(w(t)) = \int_{-\infty}^{\infty}\mathrm{d}(w^0(t))\mathcal{P}^{1+1}(w^\alpha(t)), \tag{37}$$

reduces to a Gaussian. In the Newtonian limit case, corresponding to $v^2 \ll c^2$, one thus recovers from Eqs. (35) and (36) the usual nonrelativistic Brownian motion.

C. Langevin dynamics in the laboratory frame

A laboratory frame Σ_0 is, by definition, an inertial system, in which the heat bath is at rest, i.e., in Σ_0 we have $(U^\beta) = (c,0)$ for all times t. Hence, with respect to Σ_0 coordinates, the two stochastic differential Eqs. (36b) assume with (36c) the form

$$\mathrm{d}p = -\nu p\mathrm{d}t + w(t), \tag{38a}$$

$$\mathrm{d}E = -\nu pv\mathrm{d}t + cw^0(t). \tag{38b}$$

Here it is important to note that the stochastic increments $w^\alpha(t)$, appearing on the right-hand side. of (38), are *not* of simple Gaussian type anymore. Instead, their distribution now also depends on the particle velocity v. This becomes immediately evident, when we rewrite the increment density (35c) in terms of Σ_0 coordinates. Using

$$(u_\alpha) = (-\gamma c, \gamma v), \quad \gamma^{-1} = \sqrt{1 - \frac{v^2}{c^2}}, \quad (w^\alpha) = (w^0, w), \tag{39}$$

we find

$$\mathcal{P}^{1+1}(w^\alpha(t)) = c\left(\frac{\gamma}{4\pi D\,\mathrm{d}t}\right)^{1/2} \exp\left(-\frac{w(t)^2 - w^0(t)^2}{4D\,\mathrm{d}t/\gamma}\right) \times \delta(c\gamma w^0(t) - \gamma v w(t)). \tag{40}$$

As we already pointed out earlier, the δ function in (40) reflects the fact that the energy increment w^0 is coupled to the spatial (momentum) increment w *via*

$$0 = u_\alpha w^\alpha = -c\gamma w^0 + \gamma v w \quad \Rightarrow \quad w^0 = \frac{vw}{c}. \tag{41}$$

Hence, w^0 can be eliminated from the Langevin equations (38b), yielding

$$\mathrm{d}E = -\nu p v \mathrm{d}t + v w(t) = v \mathrm{d}p. \tag{42}$$

Using the identity

$$v = \frac{cp}{\sqrt{m^2c^2 + p^2}}, \tag{43}$$

we can further rewrite (42) as

$$\mathrm{d}E = \frac{cp}{\sqrt{m^2c^2 + p^2}}\mathrm{d}p \quad \Rightarrow \quad E(t) = \sqrt{m^2c^4 + p(t)^2c^2}. \tag{44}$$

Thus, in the laboratory frame Σ_0 the relativistic Brownian motion is completely described by the Langevin equation (38a) already. If we assume that the Brownian particle has fixed initial momentum $p(0)=p_0$ or initial velocity $v(0)=v_0$, respectively, then the formal solution of (38a) reads ([20] Chap. IX.1)

$$p(t) = p_0 e^{-\nu t} + e^{-\nu t}\int_0^t e^{\nu s} w(s). \tag{45}$$

The stochastic process (45) is determined by the marginal distribution $\mathcal{P}^1(w(t))$, defined in Eq. (37). Performing the integration over the δ function in (40), we find

$$\mathcal{P}^1(w(t)) = \left(\frac{1}{4\pi D\gamma\,\mathrm{d}t}\right)^{1/2} \exp\left(-\frac{w(t)^2}{4D\gamma\,\mathrm{d}t}\right), \tag{46a}$$

where

$$\gamma = \left[1 - \frac{v^2}{c^2}\right]^{-1/2} = \left[1 + \frac{p^2}{m^2c^2}\right]^{1/2}. \tag{46b}$$

On the basis of Eqs. (38a) and (46) one can immediately perform computer simulations, provided one still specifies the rules of stochastic calculus, i.e., which value of p is to be taken to determine γ in (46). In Sec. IV several numerical results are presented. Before, it is useful to consider in more detail the Fokker-Planck equations of the relativistic Brownian motion in the laboratory frame Σ_0. By doing so in Sec. III, it will become clear that, for example, choosing $p=p(t)$ in Eqs. (46) would be consistent with an Ito-interpretation [20,33,34] of the stochastic differential equation (38a). However, we will also see that alternative interpretations lead to reasonable results as well.

III. DERIVATION OF CORRESPONDING FOKKER-PLANCK EQUATIONS

The objective in this part is to derive relativistic Fokker-Planck equations (FPE) for the momentum density $f(t,p)$ of a free particle in the laboratory frame Σ_0. Before we deal with this problem in Sec. III B, it is useful to briefly recall the nonrelativistic case.

A. Nonrelativistic case

Consider the nonrelativistic Langevin equation (1b)

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\nu p + L(t), \tag{47a}$$

where $p(t)=mv(t)$ denotes the nonrelativistic momentum, and, in agreement with (3c), the Langevin force $L(t)$ is distributed according to

$$\mathcal{P}(L(t)) = \left(\frac{\mathrm{d}t}{4\pi D}\right)^{1/2} \exp\left(-\frac{\mathrm{d}t}{4D}L(t)^2\right). \tag{47b}$$

As is well known [20,35], the related momentum probability density $f(t,p)$ is governed by the Fokker-Planck equation

$$\frac{\partial}{\partial t}f = \frac{\partial}{\partial p}\left(\nu p f + D\frac{\partial}{\partial p}f\right), \tag{48}$$

whose stationary solution is the Maxwell distribution

$$f(p) = \left(\frac{\nu}{2\pi D}\right)^{1/2} \exp\left(-\frac{\nu p^2}{2D}\right). \tag{49}$$

B. Relativistic case

We next discuss three different relativistic Fokker-Planck equations for the momentum density $f(t,p)$, related to the stochastic processes defined by (38a) and (46).

Our starting point is the relativistic Langevin equation (38a), which holds in the laboratory frame Σ_0 (i.e., in the rest frame of the heat bath). Next we define a stochastic process by

$$y(t) = \frac{w(t)}{\sqrt{\gamma}}, \tag{50}$$

and using (46b), we can rewrite (38a) as

$$\mathrm{d}p = -\nu p \mathrm{d}t + \sqrt{\gamma} y(t), \tag{51a}$$

where $y(t)$ is distributed according to the momentum-independent density

$$\mathcal{P}^1_y[y(t)] = \left(\frac{1}{4\pi D\,\mathrm{d}t}\right)^{1/2} \exp\left(-\frac{y(t)^2}{4D\,\mathrm{d}t}\right). \tag{51b}$$

Thus, instead of the increments $w(t)$, which implicitly depend on the stochastic process p *via* Eqs. (46), we consider ordinary p-independent white noise $y(t)$, determined by (51b), from now on. Due to the multiplicative coupling of $y(t)$ in (51a), we must next specify rules for the "multiplication with white noise" [note that, on viewing Eqs. (11), (35), and (36) as postulates of the relativistic Brownian motion, all above considerations remain valid, independent of this specification].

In Secs. III B 1, III B 2, and III B 3, we shall discuss three popular multiplication rules, which go back to proposals made by Hänggi and Thomas [19,42], Van Kampen [20], Ito [33,34], Stratonovich [36,37], Fisk [38,39], Hänggi [40,41], and Klimontovich [31]. As it is well-known from [19,20,30], these different interpretations of the stochastic process (51) result in different Fokker-Planck equations, i.e., the Langevin equation (51) *per se* does *not* uniquely determine the corresponding Fokker-Planck equation; it is the stochastic interpretation of the multiplicative noise that matters from a physical point of view.

Nevertheless, the three approaches discussed below have in common that, formally, the related Fokker-Planck equation can be written as a continuity equation [42]

$$\frac{\partial}{\partial t} f(t,p) + \frac{\partial}{\partial p} j(t,p) = 0, \tag{52}$$

but with different expressions for the probability current $j(t,p)$. It is worthwhile to anticipate that only for the Hänggi-Klimontovich approach (see Sec. III B 3) the current $j(t,p)$ takes such a form that the stationary distribution of (52) can be identified with Jüttner's relativistic Maxwell distribution [24].

1. Ito approach

According to Ito's interpretation of the Langevin equation (51a), the coefficient before $y(t)$ is to be evaluated at the lower boundary of the interval $[t, t+\mathrm{d}t]$, i.e., we use the prepoint discretization rule

$$\gamma = \gamma(p(t)), \tag{53}$$

where as before

$$\gamma(p) = \left(1 + \frac{p^2}{m^2c^2}\right)^{1/2}.$$

Ito's choice leads to the following expression for the current [19,20,33,34]:

$$j_{\mathrm{I}}(p,t) = -\left[\nu p f + D \frac{\partial}{\partial p} \gamma(p) f\right]. \tag{54}$$

The related relativistic Fokker-Planck equation is obtained by inserting this current into the conservation law (52). The current (54) vanishes identically for

$$f_{\mathrm{I}}(p) = \frac{C_I}{\gamma(p)} \exp\left[-\frac{\nu}{D}\int \mathrm{d}p \frac{p}{\gamma(p)}\right], \tag{55}$$

where C_{I} is the normalization constant. Consequently, $f_{\mathrm{I}}(p)$ is a stationary solution of the Fokker-Planck equation. In view of the fact that

$$\int \mathrm{d}p \frac{p}{\gamma(p)} = c^2m^2\sqrt{1 + \frac{p^2}{c^2m^2}}, \tag{56}$$

we find the following explicit representation of (55):

$$f_{\mathrm{I}}(p) = C_{\mathrm{I}}\left(1 + \frac{p^2}{m^2c^2}\right)^{-1/2} \exp\left(-\beta\sqrt{1 + \frac{p^2}{c^2m^2}}\right), \tag{57}$$

where

$$\beta = \frac{\nu m^2c^2}{D}. \tag{58}$$

The dimensionless parameter β can be used to define the scalar temperature T of the heat bath via the Einstein relation

$$k_BT \equiv \frac{mc^2}{\beta} = \frac{D}{m\nu}, \tag{59}$$

with k_B denoting the Boltzmann constant. Put differently, the parameter $\beta = mc^2/(k_BT)$ measures the ratio between rest mass and thermal energy of the Brownian particle.

2. Stratonovich approach

According to Stratonovich, the coefficient before $y(t)$ in (51a) is to be evaluated with the midpoint discretization rule, i.e.,

$$\gamma = \gamma\left(\frac{p(t) + p(t+\mathrm{d}t)}{2}\right). \tag{60}$$

This choice leads to a different expression for the current [19,36–38], namely,

$$j_{\mathrm{S}}(p,t) = -\left[\nu p f + D\sqrt{\gamma(p)}\frac{\partial}{\partial p}\sqrt{\gamma(p)} f\right]. \tag{61}$$

This Stratonovich-Fisk current j_{S} vanishes identically for

$$f_{\mathrm{S}}(p) = \frac{C_{\mathrm{S}}}{\sqrt{\gamma(p)}} \exp\left[-\frac{\nu}{D}\int \mathrm{d}p \frac{p}{\gamma(p)}\right], \tag{62}$$

and, by virtue of (56), the explicit stationary solution of Stratonovich's Fokker-Planck equation reads

$$f_{\mathrm{S}}(p) = C_{\mathrm{S}}\left(1 + \frac{p^2}{m^2c^2}\right)^{-1/4} \exp\left(-\beta\sqrt{1 + \frac{p^2}{c^2m^2}}\right). \tag{63}$$

3. Hänggi-Klimontovich approach

Now let us still consider the Hänggi-Klimontovich stochastic integral interpretation, sometimes referred to as the transport form [40–42] or also as the kinetic form [31]. According to this interpretation, the coefficient in front of $y(t)$ in (51a) is to be evaluated at the upper boundary value of the interval $[t, t+\mathrm{d}t]$; i.e., within the postpoint discretization we set

$$\gamma = \gamma(p(t+\mathrm{d}t)). \tag{64}$$

This choice leads to the following expression for the current [31,41,42]:

$$j_{\mathrm{HK}}(p,t) = -\left[\nu p f + D\gamma(p)\frac{\partial}{\partial p} f\right]. \tag{65}$$

The current j_{HK} vanishes identically for

$$f_{\mathrm{HK}}(p) = C_{\mathrm{HK}} \exp\left[-\frac{\nu}{D}\int \mathrm{d}p \frac{p}{\gamma(p)}\right], \tag{66}$$

and, by virtue of (56), the stationary solution explicitly reads

$$f_{\mathrm{HK}}(p) = C_{\mathrm{HK}} \exp\left(-\beta\sqrt{1+\frac{p^2}{c^2m^2}}\right). \tag{67a}$$

Using the temperature definition in (59) and the relativistic kinetic energy formula $E=\sqrt{m^2c^4+p^2c^2}$, one can further rewrite (67a) in a more concise form as

$$f_{\mathrm{HK}}(p) = C_{\mathrm{HK}} \exp\left(-\frac{E}{k_B T}\right). \tag{67b}$$

The distribution function (67) is known as the relativistic Maxwell distribution. It was first obtained by Jüttner [24] back in 1911. Pursuing a completely different line of reasoning, he found that (67) describes the velocity distribution of the noninteracting relativistic gas (see also [43]). In contrast to our approach, which started out with constructing the relativistic generalization of the Langevin equations, Jüttner's derivation started from a maximum-entropy-principle for the gas.

By comparing (55), (63), and (67a) one readily observes that the stationary solutions $f_{\mathrm{I/S}}$ differ from the Jüttner function f_{HK} through additional p-dependent prefactors. In order to illustrate the differences between the different stationary solutions, it useful to consider the related velocity probability density functions $\phi_{\mathrm{I/S/HK}}(v)$, which can be obtained by applying the general transformation law

$$\phi(v) \equiv f(p(v))\left|\frac{\partial p}{\partial v}\right| \tag{68}$$

in combination with

$$p = \frac{mv}{\sqrt{1-v^2/c^2}}.$$

The determinant factor $|\partial p/\partial v|$ in (68) is responsible for the fact that the velocity density functions $\phi_{\mathrm{I/S/HK}}(v)$ are, in fact, zero if $v^2 > c^2$.

In Fig. 2 we have plotted the probability density functions $\phi_{\mathrm{I/S/HK}}(v)$ for different values of the parameter β. The normalization constants were determined by numerically integrating $\phi(v)$ over the interval $[-c,c]$. As one can observe in Fig. 2(a), for large values of β, corresponding to low-temperature values $k_BT \ll mc^2$, the density functions $\phi_{\mathrm{I/S/HK}}(v)$ approach a common Gaussian shape. On the other hand, for high-temperature values $k_BT \gg mc^2$ the deviations from the Gaussian shape become essential. The reason is that, for a (virtual) Brownian ensemble in the high-temperature regime, the majority of particles assumes velocities that are close to the speed of light. It is also clear that in other Lorentz frames Σ', which are not rest frames of the heat bath, the stationary distributions will no longer stay symmetric around $v=0$. Instead, they will be centered around the nonvanishing Σ' velocity V' of the heat bath.

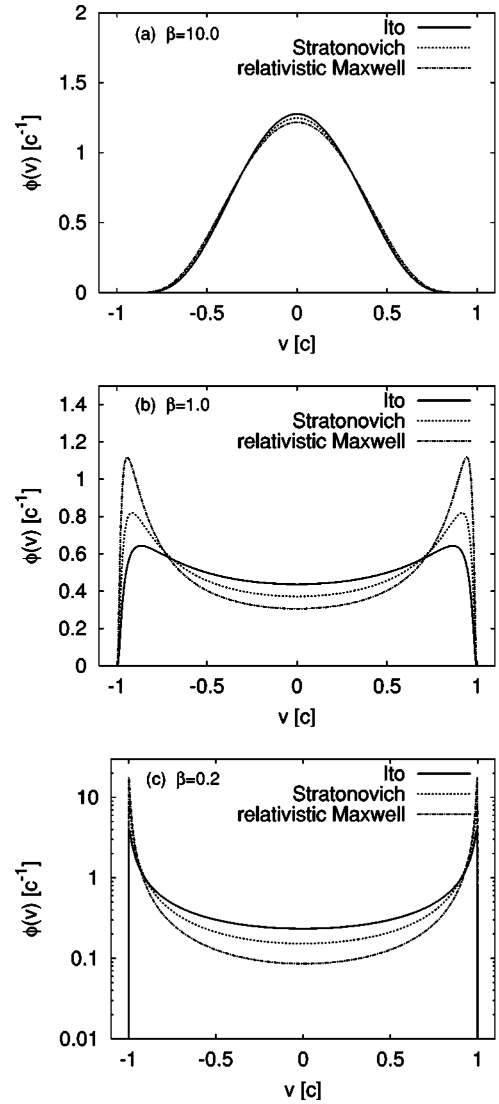

FIG. 2. Stationary solutions $\phi_{\mathrm{I/S/HK}}(v)$ of the relativistic Fokker-Planck equations, according to Ito (I:solid line), Stratonovich (S:dotted), and Hänggi-Klimontovich (HK:dashed-dotted). For low temperatures, i.e., for $\beta \gg 1$, a Gaussian shape is approached [see (a)]. On the other hand, for very high-temperature values, corresponding to $\beta \ll 1$, the distributions exhibit a bistable shape, and the quantitative deviations between $\phi_{\mathrm{I/S/HK}}(v)$ increase significantly as $\beta \to 0$.

An obvious question then arises, which of the above approaches (Ito, Stratonovich, or Hänggi-Klimontovich) is the physically correct one. We believe that, at this level of analysis, it is impossible to provide a definite answer to this question. Most likely, the answer to this problem requires additional information about the microscopic structure of the heat bath (see, e.g., the discussion of Ito-Stratonovich dilemma in the context of "internal and external" noise as given in Chap. IX.5 of van Kampen's textbook [20]). At this point, it might be worthwhile to mention that the relativistic Maxwell distribution (67) is also obtained via the transfer probability method used by Schay, see Eq. (3.63) and (3.64) in Ref. [3], and that this distribution also results in the relativistic kinetic theory [29]. By contrast, the recent work of Franchi and Le Jan [13] is based on the Stratonovich approach. From physical insight, however, it is the transport form interpretation of Hänggi and Klimontovich that is expected to provide the physically correct description.

IV. NUMERICAL INVESTIGATIONS

The numerical results presented in this section were obtained on the basis of the relativistic Langevin equation (51), which holds in the laboratory frame Σ_0. For simplicity, we confined ourselves here to considering the Ito-discretization scheme with fixed time step dt (see Sec. III B 1). In all simulations we have used an ensemble size of $N=10\,000$ particles. Moreover, a characteristic unit system was fixed by setting $m=c=\nu=1$. Formally, this corresponds to using rescaled dimensionless quantities, such as $\tilde{p}=p/mc$, $\tilde{x}=x\nu/c$, $\tilde{t}=t\nu$, $\tilde{v}=v/c$, etc. The simulation time-step was always chosen as $\mathrm{d}t=0.001\nu^{-1}$, and the Gaussian random variables $y(t)$ were generated by using a standard random number generator.

A. Distribution functions

In our simulations we have numerically measured the cumulative velocity distribution function $F(t,v)$ in the laboratory frame Σ_0. Given the probability density $\phi(t,v)$, the cumulative velocity distribution function is defined by

$$F(t,v) = \int_{-c}^{v} \mathrm{d}u\,\phi(t,u). \tag{69}$$

In order to obtain $F(t,v)$ from numerical simulations, one simply measures the relative fraction of particles with velocities in the interval $[-c,v]$. Figure 3 shows the numerically determined *stationary* distribution functions (squares), taken at time $t=100\ \nu^{-1}$ and also the corresponding analytical curves $F_{\mathrm{I/S/HK}}(v)$. The latter were obtained by numerically integrating Eq. (69) using the three different stationary density functions $\phi_{\mathrm{I/S/HK}}(v)$ from Sec. III.

As one can see in Fig. 3(a), for low-temperature values corresponding to $\beta \gg 1$, the three stationary distribution functions are nearly indistinguishable. For high temperatures corresponding to $\beta \leqslant 1$, the stationary solutions exhibit significant quantitative differences, [see Figs. 3(b) and 3(c)]. Because our simulations are based on an Ito-discretization scheme the numerical values (squares) are best fitted by the Ito solution (solid line). Also note that the quality of the fit is very good for the parameters chosen in the simulations, and that this property is conserved over several magnitudes of β. This suggests that numerical simulations of the Langevin equations provide a very useful tool if one wishes to study relativistic Brownian motions in more complicated settings (e.g., in higher dimensions or in the presence of additional external fields and interactions). In this context, it should again be stressed that the appropriate choice of the discretization rule is especially important in applications to realistic systems.

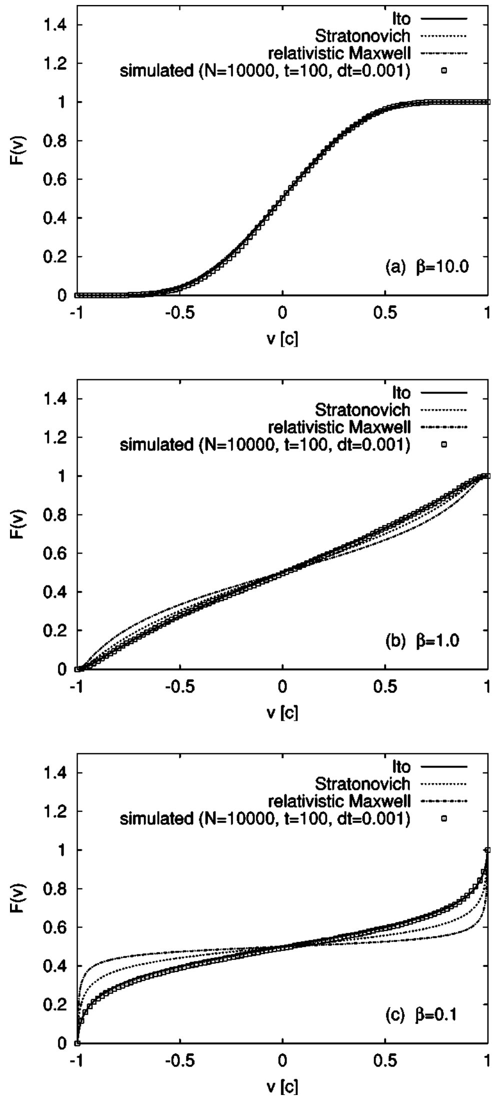

FIG. 3. These diagrams show a comparison between numerical and analytical results for the stationary cumulative distribution function $F(v)$ in the laboratory frame Σ_0. (a) In the nonrelativistic limit $\beta \gg 1$ the stationary solutions of the three different FPE are nearly indistinguishable. (b)–(c) In the relativistic limit case $\beta \leqslant 1$, however, the stationary solutions exhibit deviations from each other. Because our simulations are based on an Ito-discretization scheme, the numerical values (squares) are best fitted by the Ito solution (solid line).

B. Mean-square displacement

In this section we consider the spatial mean-square displacement of the free *relativistic* Brownian motion. Because this quantity is easily accessible in experiments, it has played an important role in the verification of the nonrelativistic theory.

As before, we consider an ensemble of N-independent Brownian particles with coordinates $x_{(i)}(t)$ in Σ_0 and initial conditions $x_{(i)}(0)=0, v_{(i)}(0)=0$ for $i=1,2,\ldots,N$. The position mean value is defined as

$$\overline{x(t)} \equiv \frac{1}{N}\sum_{i=1}^{N} x_{(i)}(t), \tag{70}$$

and the related second moment is given by

$$\overline{x^2}(t) \equiv \frac{1}{N}\sum_{i=1}^{N} [x_{(i)}(t)]^2. \tag{71}$$

The empirical mean-square displacement can then be defined as follows:

$$\sigma^2(t) \equiv \overline{x^2}(t) - (\overline{x(t)})^2. \tag{72}$$

Cornerstone results in the nonrelativistic theory of the one-dimensional Brownian motion are

$$\lim_{t\to+\infty} \overline{x(t)} \to 0, \tag{73a}$$

$$\lim_{t\to+\infty} \frac{\sigma^2(t)}{t} \to 2D^x, \tag{73b}$$

where the constant

$$D^x = \frac{k_B T}{m\nu} = \frac{D}{m^2\nu^2} \tag{74}$$

is the nonrelativistic coefficient of diffusion in coordinate space (not to be confused with noise parameter D).

It is therefore interesting to consider the asymptotic behavior of the quantity $\sigma^2(t)/t$ for relativistic Brownian motions, using again the Ito-relativistic Langevin dynamics from Sec. III B 1. In Fig. 4(a) one can see the corresponding numerical results for different values of β. As one can observe in this diagram, for each value of β, the quantity $\sigma^2(t)/t$ converges to a constant value. This means that, at least in the laboratory frame Σ_0, the asymptotic mean-square displacement of the free relativistic Brownian motions increases linearly with t. For completeness, we mention that according to our simulations the asymptotic relation (73a) holds in the relativistic case, too.

In spite of these similarities between nonrelativistic and relativistic theory, an essential difference consists of the explicit temperature dependence of the limit value $2D^x$. As illustrated in Fig. 4(b), the numerical limit values $2D^x_{100}$, measured at time $t=100\nu^{-1}$, are well fitted by the empirical formula

$$D^x = \frac{c^2}{\nu(\beta+2)}, \tag{75}$$

which reduces to the nonrelativistic result (74) in the limit case $\beta \gg 2$.

We will leave it as an open problem here, to find an analytical justification for the empirically determined formula (75). Instead we merely mention that, on noting (43), the relativistic Fokker-Planck equations for the full-phase space density reads

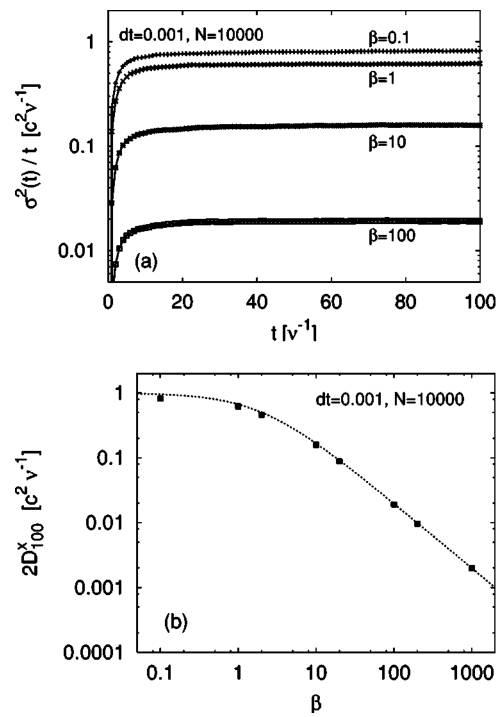

FIG. 4. (a) Mean-square displacement divided by time t as numerically calculated for different β-values in the laboratory frame Σ_0 (rest frame of the heat bath). As evident from this figure, for the relativistic Brownian motion the related asymptotic mean-square displacement grows linearly with t. (b) The coordinate space diffusion constant $D^x_{100}(\beta)$ was numerically determined at time $t=100\nu^{-1}$. The dashed line corresponds to the empirical fitting formula $D^x(\beta)=c^2\nu^{-1}(\beta+2)^{-1}$, which reduces to the classical nonrelativistic result $D^x \simeq c^2/(\nu\beta)=kT/(m\nu)$ for $\beta \gg 2$.

$$\frac{\partial}{\partial t}f(t,p,x) + \frac{cp}{\sqrt{m^2c^2+p^2}}\frac{\partial}{\partial x}f(t,p,x) = -\frac{\partial}{\partial p}j_{\mathrm{I/S/HK}}(t,p,x), \tag{76}$$

which might serve as a suitable starting point for such an analysis. Compared to the relativistic Fokker-Planck equations from Sec. IV, the second term on the left-hand side of (76) is new. In particular, we recover the relativistic Fokker-Planck equations for the marginal density $f(t,p)$, see Sec. III by integrating Eq. (76) over a spatial volume with appropriate boundary conditions. Finally, we mention once again that also (76), as well as all the other results that have been presented in this section, exclusively refer to the laboratory frame Σ_0.

V. CONCLUSION

Concentrating on the simplest case of 1+1 dimensions, we have put forward the Langevin dynamics for the stochastic motion of free relativistic Brownian particles in a viscous medium (heat bath). Analogous to the nonrelativistic Ornstein-Uhlenbeck theory of Brownian motion [17,19,20,44], it was assumed that the heat bath can, in good approximation, be regarded as homogenous. Based on this assumption, a covariant generalization of the Langevin equations has been constructed in Sec. II. According to these generalized stochastic differential equations, the viscous friction between Brownian particle and heat bath is modeled by a friction tensor $\nu_{\alpha\beta}$. For a homogeneous heat bath this friction tensor has the same structure as the pressure tensor of a perfect fluid [25]. In particular, it is uniquely determined by the value of the (scalar) viscous friction coefficient ν, measured in the instantaneous rest frame of the particle (Sec. II B 1). Similarly, the amplitude of the stochastic force is also governed by a single parameter D, specifying the Gaussian fluctuations of the heat bath, as seen in the instantaneous rest frame of the particle (Sec. II B 2).

In Sec. II C the relativistic Langevin equations have been derived in special laboratory coordinates, corresponding to a specific class of Lorentz frames, in which the heat bath is assumed to be at rest (at all times). One finds that the corresponding relativistic distribution of the momentum increments now also depends on the momentum coordinate. This fact is in contrast with the properties of ordinary Wiener processes [21,23], underlying nonrelativistic standard Brownian motions with "additive" Gaussian white noise. However, as shown in Sec. III it is possible to find an equivalent Langevin equation, containing "multiplicative" Gaussian white noise.

In order to achieve a more complete picture of the relativistic Brownian motion, the corresponding relativistic Fokker-Planck equations (FPE) have been discussed in Sec. III (again with respect to the laboratory coordinates with the heat bath at rest). Analogous to nonrelativistic processes with multiplicative noise, one can opt for different interpretations of the stochastic differential equation, which result in different FPE. In this paper, we concentrated on the three most popular cases, namely, the Ito, the Stratonovich-Fisk, and the Hänggi-Klimontovich interpretations. We discussed and compared the corresponding stationary solutions for a free Brownian particle. It could be established that only the Hänggi-Klimontovich interpretation is consistent with the relativistic Maxwell distribution. This very distribution was derived by Jüttner [24] as the equilibrium velocity distribution of the relativistic ideal gas. Later on, it was also discussed by Schay in the context of relativistic diffusions [3] and by de Groot *et al.* in the framework of the relativistic kinetic theory [29].

In Sec. IV we presented numerical results, obtained on the basis of an Ito prepoint discretization rule. The simulations indicate that—analogous to the nonrelativistic case—the relativistic mean-square displacement grows linearly with the laboratory coordinate time; the temperature dependence of the related spatial diffusion constant, however, becomes more intricate. In principle, the numerical results suggest that simulations of the Langevin equations may provide a very useful tool for studying the dynamics of relativistic Brownian particles. In this context it has to be stressed that an appropriate choice of the discretization rule is especially important in applications to realistic physical systems. If, for example, agreement with the kinetic theory [29] is desirable, then a postpoint discretization rule should be used.

From the methodical point of view, the systematic relativistic Langevin approach of the present paper differs from Schay's transition probability approach [3] and also from the techniques applied by other authors [4,6,7]. As we shall discuss in a forthcoming contribution, the above approach can easily be generalized to settings that are more relevant with regard to experiments [such as the (1+3)-dimensional case, the presence of additional external force fields, etc.].

With regard to future work, several challenges remain to be solved. For example, one should try to derive an analytic expression for the temperature dependence of the spatial diffusion constant. A suitable starting point for such studies might be the FPE for the full-phase space density given in Eq. (76). Another possible task consists of finding explicit exact or at least approximate time-dependent solutions of the relativistic FPE. Furthermore, it seems also interesting to consider extensions to general relativity, as, to some extent, recently discussed in the mathematical literature [13]. In this context, the physical consequences of the different interpretations (Ito versus Stratonovich versus Hänggi-Klimontovich) become particularly interesting.

Note added in proof. Recently, we have been informed by F. Debbasch about two interesting recent papers [45,46] on a relativistic generalization of the Ornstein-Uhlenbeck process. These two items are related in spirit to the present work: The authors of those references have postulated a relativistic Langevin equation with additive noise and a drift term that differs from ours; but which also yields the correct relativistic Jüttner distribution. Thus, our HK-approach and theirs possess the same stationary solution, but notably do exhibit a different relaxation dynamics.

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