

CHECKING LINEAR RESPONSE THEORY IN DRIVEN BISTABLE SYSTEMS

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The validity of linear response theory to describe the response of a nonlinear stochastic system driven by an external periodical time dependent force is put to a critical test. A variety of numerical and analytical approximations is used to compare its predictions with numerical solutions over an extended parameter regime of driving amplitudes and frequencies and noise strengths. The relevance of the driving frequency and the noise value for the applicability of linear response theory is explored for single and multi-frequency input signals.

Keywords: Stochastic resonance; driven stochastic dynamics; nonlinear response; linear response theory; analytical formulas; comparison: theory vs. numerics.

1. Introduction

The response of dissipative physical systems to small amplitude external perturbations is usually described with the powerful tools of linear response theory (LRT) [1]. This method has found widespread application for the phenomenon of *Stochastic Resonance* which describes the constructive role of noise for the transduction and detection of weak information-carrying signals in systems that possess some sort of threshold [2, 3, 4]. In this context, the stochastic dynamics of a rocked bistable potential has served as an archetype to study many features of this exciting and prominent physical effect [2]. For sufficiently small external forces, one can introduce a smallness parameter proportional to the external amplitude. The evolution equation for the probability density describing the system dynamics can be treated to first order in perturbation theory. Thus, one finds that the effect of the perturbation on the evolution of the relevant system variables can be described

in terms of small deviations from their behaviors in the unperturbed system. In particular, for long times and for systems which in the absence of driving reach an equilibrium distribution, LRT provides an approximate expression for the probability distribution where the deviations with respect to the equilibrium distribution in the absence of driving are proportional to the driving amplitude. Consequently, LRT predicts that the first moment of the probability distribution scales linearly with the driving amplitude.

When the driving force is periodic in time, the linearity of the evolution equation for the probability density allows us to use the Floquet ideas [5]. The main consequence is that the long time probability distribution is periodic in time with the same period as the external driving. This is a general result valid for any value of the input amplitude. Consequently, the first moment will also be periodic in time and it can be expanded in a Fourier series containing the fundamental and higher order harmonics. LRT predicts that the only harmonics that should show up in the system response are those already present in the input.

In recent work [6], we have considered the dynamics of a noisy bistable system subject to the action of an external driving. We have pointed out the relevance of parameters other than the amplitude of the driving force, for the validity of LRT. We showed that for a periodic single-frequency external force, the validity of LRT depends not just on the amplitude of the driving term but also crucially on its frequency. The purpose of this work is to further extend the analysis in two aspects. First, in [6], the 2-mode approximation for the susceptibility described in [7] was used. This analytical expression is not valid when the noise strength is large. In this work, we evaluate the susceptibility using a numerical procedure which is not subject to the limitations of small noise strengths. This allows us to explore the system response in a wider range of parameter values. Limitations of the LRT description can not be ascribed to an invalid estimate of the system susceptibility. Second, we consider the response of the system when the external driving contains many frequencies [8]. In particular, we analyze an external driving of the form depicted in Fig. 1. Driving forces of this kind have been recently considered by Gingl et al. [9] in their studies of the signal-to-noise ratio in double wells.

After the introduction of the model in Sec. 2 we study analytically the system response in Sec. 3 within LRT and check our predictions of LRT against the numerically obtained behavior for a sinusoidal driving, see Sec. 4, and a multifrequency input in Sec. 5. Our conclusions are presented in Sec. 5.

2. The Model

Let us consider a system characterized by a single degree of freedom, x , whose time evolution is governed by the nonlinear Langevin equation (in dimensionless form),

$$\dot{x}(t) = x(t) - x^3(t) + F(t) + \eta(t), \quad (1)$$

where $F(t)$ represents an external periodic signal with period T and $\eta(t)$ is a Gaussian white noise with zero average and $\langle \eta(t)\eta(s) \rangle = 2D\delta(t-s)$. The corresponding linear Fokker-Planck equation (FPE) for the probability density $P(x, t)$ reads

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x} \{(-x + x^3 - F(t))P\} + D \frac{\partial^2 P}{\partial x^2}. \quad (2)$$

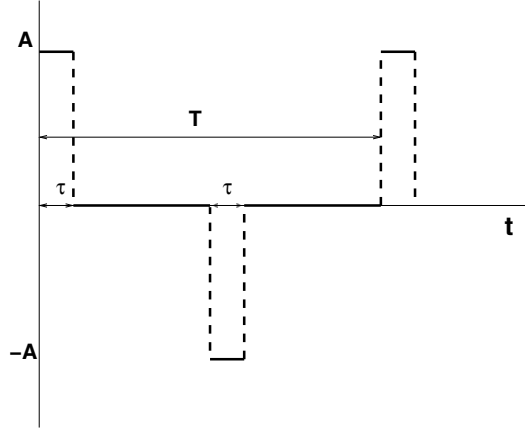


Fig. 1. Sketch of a periodic external force containing many harmonics of the fundamental frequency.

The unperturbed system has an equilibrium distribution of the form

$$P_{eq}(x) = N \exp\left(-\frac{U_0(x)}{D}\right), \quad (3)$$

where N is a normalization constant and $U_0(x)$ is the unperturbed potential

$$U_0(x) = -\frac{x^2}{2} + \frac{x^4}{4}. \quad (4)$$

This potential has two minima located at $x_m = \pm 1$ and a maximum at $x_M = 0$, with a barrier height of 0.25.

The analysis of the dynamics is simplified by making use of two important theorems: the H-theorem, which ensures the existence of a unique long time distribution function $P_\infty(x, t)$ [10, 11] and the Floquet theorem, which guarantees that $P_\infty(x, t)$ is periodic in time with the same period as the external force [5]. Then, the long time distribution can be expanded in the Fourier series

$$P_\infty(x, t) = \sum_{n=0}^{\infty} [H_n(x) \cos(n\Omega t) + I_n(x) \sin(n\Omega t)], \quad (5)$$

where the quantity $\Omega = 2\pi/T$ denotes the fundamental angular frequency of the external driving signal.

The first moment of the probability distribution can be used to characterize the system response to the external driving. In the long time limit, $\langle x(t) \rangle_\infty$ becomes a periodic function of time. It can be written as a Fourier series,

$$\langle x(t) \rangle_\infty = \sum_{n=1}^{\infty} [M_n \cos(n\Omega t) + N_n \sin(n\Omega t)]. \quad (6)$$

Unfortunately, the two powerful theorems cited above equation (5) are not enough for a detailed analysis of the system behavior under the action of the external

driving. The nonlinearity of the Langevin dynamics precludes the exact analytical knowledge of the probability distribution and, therefore, the coefficients M_n and N_n in the Fourier expansion are unknown. In order to gain quantitative information, one needs to turn to approximations of analytical or numerical character.

3. The Linear Response Theory Description of the System Response

The basic quantity of LRT is the system response function, $\chi(t)$. It is related to the equilibrium time correlation function of the system in the absence of external driving, $K(t)$, via the fluctuation-dissipation theorem (FDT) [7], i.e.,

$$\chi(t) = \begin{cases} 0 & : t \leq 0 \\ -\frac{1}{D}\dot{K}(t) & : t > 0. \end{cases} \quad (7)$$

In LRT the long time average value $\langle x(t) \rangle_\infty^{LRT}$ is given by

$$\langle x(t) \rangle_\infty^{LRT} = \int_0^\infty d\tau \chi(\tau) F(t - \tau). \quad (8)$$

The external driving $F(t)$ can be expanded in a Fourier series as

$$F(t) = \sum_{n=1}^\infty (f_n \cos(n\Omega t) + g_n \sin(n\Omega t)), \quad (9)$$

with the Fourier coefficients, f_n and g_n , given by

$$\begin{aligned} f_n &= \frac{2}{T} \int_0^T dt F(t) \cos(n\Omega t), \\ g_n &= \frac{2}{T} \int_0^T dt F(t) \sin(n\Omega t). \end{aligned} \quad (10)$$

Here, we are assuming that the cycle average of the external driving over its period is zero. Insertion of the Fourier expansion into Eq. (8) leads to

$$\langle x(t) \rangle_\infty^{LRT} = \sum_{n=1}^\infty [M_n^{LRT} \cos(n\Omega t) + N_n^{LRT} \sin(n\Omega t)], \quad (11)$$

where the coefficients M_n^{LRT} and N_n^{LRT} are given by

$$M_n^{LRT} = f_n \chi_n^r - g_n \chi_n^i; \quad N_n^{LRT} = f_n \chi_n^i + g_n \chi_n^r. \quad (12)$$

In these formulas, we have introduced χ_n^r and χ_n^i defined as

$$\chi_n^r = \int_0^\infty d\tau \chi(\tau) \cos(n\Omega\tau), \quad (13)$$

$$\chi_n^i = \int_0^\infty d\tau \chi(\tau) \sin(n\Omega\tau). \quad (14)$$

Use of the FDT in the above expressions allows us to write immediately

$$\chi_n^r = \frac{\langle x^2 \rangle_{eq} - n\Omega \int_0^\infty dt K(t) \sin(n\Omega t)}{D}, \quad (15)$$

$$\chi_n^i = \frac{n\Omega}{D} \int_0^\infty dt K(t) \cos(n\Omega t). \quad (16)$$

For nonlinear problems, explicit expressions for $K(t)$ are unknown, but useful approximations have been presented in the literature [12]. For the bistable potential in Eq. (4), we can use the two-mode approximation of Jung and Hänggi [7]. It is based on a large difference in the time scales associated to inter-well and intra-well motions, and it is expected to be valid for small values of the noise strength D . With this model, one gets

$$K(\tau) = g_1 \exp(-\lambda_1 \tau) + g_2 \exp(-\alpha \tau), \quad (17)$$

where [13]

$$\lambda_1 \approx \frac{\sqrt{2}}{\pi} \left(1 - \frac{3}{2} D \right) \exp[-1/(4D)] \quad (18)$$

and $\alpha = 2$. The weights, g_1 and g_2 , can be obtained from the moments of the equilibrium distribution in the absence of driving using the expressions

$$g_2 = \frac{\lambda_1 \langle x^2 \rangle_{eq}}{\lambda_1 - \alpha} + \frac{\langle x^2 \rangle_{eq} - \langle x^4 \rangle_{eq}}{\lambda_1 - \alpha}, \quad (19)$$

$$g_1 = \langle x^2 \rangle_{eq} - g_2. \quad (20)$$

If the noise strength D is small, so that the time scales of inter and intra-well motions are well separated, the 2-mode approximation to $K(t)$ (Eq. (17)) can be used to evaluate the coefficients χ_n^r , χ_n^i needed to describe the behavior of $\langle x(t) \rangle_\infty^{LRT}$. When the separation between the time scales associated to inter- and intra-well motions is not clear, the 2-mode approximation to $K(t)$ is unreliable. Fortunately, the correlation function in the absence of driving can also be evaluated numerically from the FPE. (See [14] for details). Comparison of the numerical results with those obtained using Eq. (17) indicates that for noise strengths $D < 0.2$ the 2-mode formula gives an adequate approximation.

The LRT leads to explicit predictions about the behavior of the system response to an external driving force. Clearly, if the amplitude of the driver is infinitesimal, i.e., smaller than any other parameter in the problem, there is no doubt about its applicability. But for finite values of the driving amplitude, the reliability of the LRT description for different regions of the parameter values has to be investigated. Differences in the observed response with respect to the LRT predictions in a range of parameter values indicate a failure of LRT.

4. The Case of Monochromatic Driving

A very large fraction of the papers devoted to the study of *Stochastic Resonance* consider the system response to an external force of the type $F(t) = A \cos \Omega t$, with

$\Omega = 2\pi/T$. In this case, the general formulas obtained within the LRT simplifies considerably. All the coefficients in the expansion in Eq. (9) vanish except one, $f_1 = A$. The average value can be written as

$$\langle x(t) \rangle_\infty^{LRT} = A \int_0^\infty d\tau \chi(\tau) \cos \Omega(t - \tau) = a \cos(\Omega t - \phi), \quad (21)$$

where the response amplitude a is related to the driver amplitude by

$$a = A \sqrt{(\chi_1^r)^2 + (\chi_1^i)^2}, \quad (22)$$

and the phase lag of the response with respect to the input signal, ϕ , lies in the interval $0 \leq \phi \leq \pi/2$. It is given by

$$\phi = \arctan \frac{\chi_1^i}{\chi_1^r}. \quad (23)$$

Within the 2-mode approximation to $K(t)$, the response amplitude and phase lag are given by

$$a = \frac{A}{D} \left[\frac{g_1^2 \lambda_1^2}{\lambda_1^2 + \Omega^2} + \frac{g_2^2 \alpha^2}{\alpha^2 + \Omega^2} + \frac{2g_1 g_2 \lambda_1 \alpha (\lambda_1 \alpha + \Omega^2)}{(\lambda_1^2 + \Omega^2)(\alpha^2 + \Omega^2)} \right]^{\frac{1}{2}}, \quad (24)$$

and

$$\phi = \arctan \frac{\frac{g_1 \lambda_1 \Omega}{\lambda_1^2 + \Omega^2} + \frac{g_2 \alpha \Omega}{\alpha^2 + \Omega^2}}{\frac{g_1 \lambda_1^2}{\lambda_1^2 + \Omega^2} + \frac{g_2 \alpha^2}{\alpha^2 + \Omega^2}}. \quad (25)$$

The result for the amplitude in Eq. (24) reveals already the limiting validity of linear response theory: Note that for small noise intensity, i.e. $D \ll A$, (weak noise regime) the amplitude a assumes large values. In this latter regime the weak noise theory must be invoked, see in Refs. [15, 16]. Likewise, linear response theory does break down for small angular driving frequency Ω at weak noise obeying $A < D$, see also Ref. [6].

Regardless of the way that $K(t)$ is evaluated, it is clear that the LRT description of the response of the system to a monochromatic driving predicts that: i) the first moment $\langle x(t) \rangle_\infty$ should contain a single harmonics with the frequency of the driving force; ii) the output amplitude should behave linearly with A . In order to check these predictions, we have resorted to numerical solutions of the FPE, Eq. (2). The technique is based on the use of the split propagator method of Feit *et al.* [17] and detailed in [18]. From the numerical solution of the FPE we can easily obtain the time dependence of $\langle x(t) \rangle_\infty$. As this is a periodic function of time, its Fourier components can be obtained by numerical quadrature.

In Fig. 2, we show the amplitudes of the harmonics appearing in the Fourier analysis of $\langle x(t) \rangle_\infty$ obtained from the numerical solution of the FPE for several sets of parameter values. In all cases we take for the input amplitude the value $A = 0.35$ that is just slightly below its threshold value ($A_T \approx 0.37$). For the two values of the external frequency, $\Omega = 0.1$ and $\Omega = 0.001$, we consider two illustrative noise strengths: $D = 0.1$ and $D = 1.0$. For both frequencies, increasing the value of D

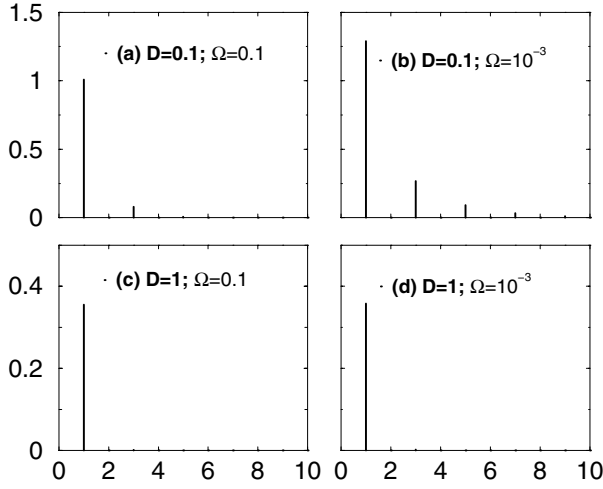


Fig. 2. Amplitudes of the Fourier modes of $\langle x(t) \rangle_\infty$ for a single frequency input signal with amplitude $A = 0.35$ and several values of D and Ω . The order of the harmonics is indicated on the horizontal axis.

reduces the number of harmonics contributing to the overall behavior of $\langle x(t) \rangle_\infty$. Actually, for the value of A considered, and $D = 1$, the response shows essentially the fundamental harmonics, regardless of the frequency value. (See panels (c) and (d) in Fig. (2)). On the other hand, for $D = 0.1$, the importance of the frequency is manifested. From inspection of panels (a) and (b) in Fig. (2) it is clear that as the frequency gets smaller, the system response contains more harmonics than the input. This is an indication of the breakdown of LRT to describe the system response at low external frequencies and finite driving amplitudes, in agreement with the idea that LRT requires that $A/D < 1$.

As mentioned above, a signature of the validity of LRT is that the amplitude of the response should scale linearly with that of the input. In Fig. 3, we show the behavior of the output amplitude, A_{out} , provided by the numerics (circles and the connecting dashed lines), and the behavior of the output amplitude obtained within LRT, a (solid line), for a wide range of input amplitudes A , and different values of D and Ω . The results shown here further corroborate the comments in the above paragraph. For large values of the driving frequency, $\Omega = 0.1$, (panels (a) and (c)) the difference between the numerical amplitude, A_{out} , and the LRT amplitude, a , is very small even for quite large values of A . Large values of D favor the extension of the linear behavior to regions of higher driving amplitudes. As seen in panel (c), the linear behavior is approximately valid even for input amplitudes with values very close to its threshold value ($A = 0.37$). Panels (b) and (d) show the results for $\Omega = 0.001$. At this small frequency and small noise ($D = 0.1$, panel (b)), deviations of the numerical results from the LRT behavior are clearly detectable at $A \approx 0.05$. On the other hand, as D increases (panel (d)), even for A values higher than its threshold value, the linearity of the system amplitude with respect to the driving amplitude is still approximately valid.

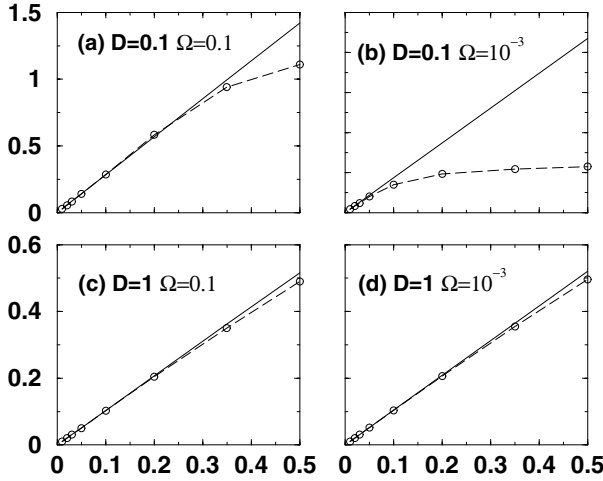


Fig. 3. Behavior of the output amplitude (vertical axes) obtained with the LRT formula Eq. (22) (solid line) and that obtained from the numerics (circles and dashed lines). The input force is sinusoidal with a frequency indicated in the panels and amplitude, A , in the range indicated on the horizontal axes.

5. Multiple Frequency Periodic Driving Force

As an example of the response of the system to a periodic driving term containing several harmonics, let us now consider a periodic driver with period T of the form given in Fig. (1). The force vanishes during most of the period except for small time intervals of duration τ where it is kept constant with (alternating) values $\pm A$. The duty cycle of the input is defined as $2\tau/T$. The Fourier coefficients in Eq. (9) are given by,

$$f_n = \begin{cases} 0 & : \text{ n even,} \\ \frac{2A}{n\pi} \sin(n\Omega\tau) & : \text{ n odd,} \end{cases} \tag{26}$$

$$g_n = \begin{cases} 0 & : \text{ n even,} \\ \frac{2A}{n\pi} (1 - \cos(n\Omega\tau)) & : \text{ n odd.} \end{cases} \tag{27}$$

It then follows from Eqs. (9)–(12) that $\langle x(t) \rangle_\infty^{LRT}$ contains just odd harmonics. This feature is more general, valid even if the LRT description of the system response fails, as it is a consequence of the spatial and temporal symmetries in the problem. The driving force changes sign every half period, i.e., $F(t+T/2) = -F(t)$. Then, the symmetry of $U_0(x)$ implies that $P_\infty(-x, t) = P_\infty(x, t+T/2)$. Therefore, the coefficients in the expansion given in Eq. (5) satisfy the relations, $H_n(x) = (-1)^n H_n(-x)$, $I_n(x) = (-1)^n I_n(-x)$ and, consequently, the series expansion in Eq. (6) contains just odd terms.

Figure 4 shows the amplitude of the first 40 harmonics of the fundamental frequency appearing in the output for an input signal with a period of $2\pi/\Omega$ with $\Omega = 0.1$, and a duty cycle of 10%. The solid bars indicate the values of the

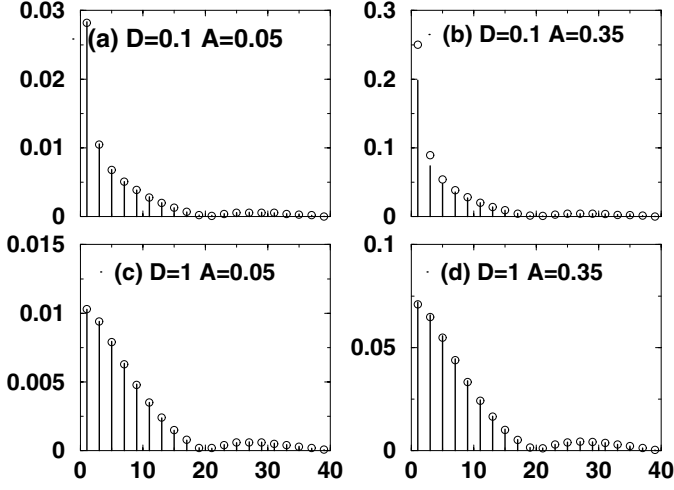


Fig. 4. Amplitude of the harmonics of $\langle x(t) \rangle_\infty^{LRT}$ (bars), and $\langle x(t) \rangle_\infty$ (circles) for an input force of the type sketched in Fig. (1) with a period of $2\pi/0.1$, and a duty cycle of 10%. The order of the harmonics are indicated on the horizontal axes.

amplitudes, a_n , of each harmonics of $\langle x(t) \rangle_\infty^{LRT}$ obtained from

$$a_n = \sqrt{(M_n^{LRT})^2 + (N_n^{LRT})^2}, \quad (28)$$

where M_n^{LRT} and N_n^{LRT} are given by Eqs. (12)–(16) and $K(t)$ is obtained numerically from the FPE in the absence of driving. The circles indicate the amplitudes of the harmonics obtained with the formula $\sqrt{(M_n)^2 + (N_n)^2}$, where M_n and N_n are the Fourier coefficients in Eq. (6) of the numerical result, $\langle x(t) \rangle_\infty$. Except for $A = 0.35$ and $D = 0.1$, the LRT predictions and the numerically obtained values coincide.

Notice that, within LRT, even though the system response contains the same harmonics as the driving term, the degree of amplification of each harmonics depends on its frequency. Therefore, the shape of the output $\langle x(t) \rangle_\infty^{LRT}$ will, in general, be different from the shape of the input $F(t)$. Nonetheless, the maximum value of $\langle x(t) \rangle_\infty^{LRT}$ should scale linearly with the maximum of $F(t)$.

In Fig. 5, we show the time evolution of $\langle x(t) \rangle_\infty^{LRT}$ in the long-time regime given by Eqs. (11)–(16), and the entire time evolution of $\langle x(t) \rangle$ including the transient from some initial condition, obtained by numerically solving the corresponding FPE. We show the results for two driving amplitudes: a small value, $A = 0.05$, and $A = 0.35$, that is slightly below its threshold value. The noise values considered are $D = 0.1$ and $D = 1.0$, while the period of the driving term is $T = 2\pi/\Omega$, with $\Omega = 0.1$. The duty cycle of the input is set at 10%. It is clear from inspection of panels (c) and (d) that for large noise intensities ($D = 1$), the LRT gives an excellent description of the long time dynamics, even for large driving amplitudes. After a very short transient, the numerically obtained $\langle x(t) \rangle_\infty$ coincides with $\langle x(t) \rangle_\infty^{LRT}$. On the other hand, for $D = 0.1$ and $A = 0.35$, the deviations between the two long time results are noticeable. The peaks of the numerical solution are slightly larger

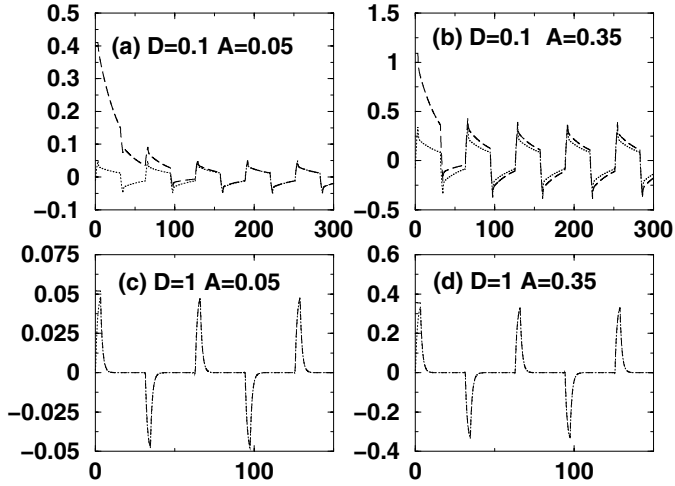


Fig. 5. Time evolution of $\langle x(t) \rangle_{\infty}^{LRT}$ (dotted lines), and $\langle x(t) \rangle$ (dashed lines) for an input force of the type sketched in Fig. (1) with a period of $2\pi/0.1$, and a duty cycle of 10%.

than the peaks obtained with the LRT formulas. These plots are in total agreement with the Fourier spectra depicted in Fig. 4. The deviations of LRT predictions with respect to the numerical results are noticeable only for parameter values such that $A/D > 1$ [15, 16].

The peaks of the output correspond to the impulses of the driver. It is interesting to observe the relevance of the noise strength in the time evolution of the output after each peak. When D is large, after each impulse of the driving force, very quickly the system adapts to a situation corresponding to no forcing and the output decays to zero as expected for a bistable system in the absence of driving. On the other hand, when D is small, the decay time is much longer. Diffusion is not fast enough to make the amplitude decays to zero before the next impulse of the driving term appears. This influence of the noise strength on the decay time after each impulse is clearly observed in Fig. 6, where we show the time evolution during several periods of $\langle x(t) \rangle_{\infty}$ for an input signal with $A = 0.35$ and the same duty cycle as in Figs. 4 and 5, for a sequence of noise strengths. From top to bottom, the D values are: $D = 0.05$ (a), $D = 0.1$ (b), $D = 0.5$ (c) and $D = 1.0$ (d). We also note that the heights of the peaks do not depict a monotonic behavior versus the noise strength D . The amplitude first increases as D increases (panels (a) to (c)), and then, it decreases when D is further augmented going to panel (d). This non-monotonic behavior of the maximum amplitude of the output is typical of the phenomenon of *Stochastic Resonance* [2, 3, 4].

6. Conclusions

In this work, we have further extended the analysis carried out in [6] to check the validity of the LRT description of the dynamics of driven nonlinear stochastic systems. We have compared the predictions of LRT with the results of the numerical solution of the FPE for a wide range of values of the noise strength D . As it is well

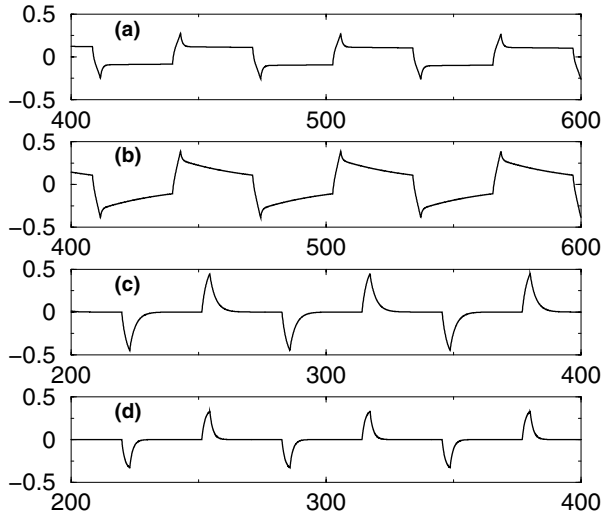


Fig. 6. Time evolution of $\langle x(t) \rangle_\infty$ for an input signal with a duty cycle of 10% and $A = 0.35$. The noise values are: $D = 0.05$ (panel a), $D = 0.1$ (panel b), $D = 0.5$ (panel c) and $D = 1.0$ (panel d).

known, LRT formulas are based on the knowledge of the linear susceptibility, which in turns, depends on the knowledge of the equilibrium time correlation function in the absence of driving. In [6], we restricted ourselves to cases with small values of D so that the 2-mode approximation to the linear susceptibility could be used. Here, we present results without this restriction. The equilibrium time correlation function in the absence of driving is obtained from the numerical solution of the FP equation for the conditional probability density. Then, the linear susceptibility is obtained by numerical quadrature.

As a further extension of the analysis, we have considered the case of periodic input signals with a more complicated structure than a pure sinusoidal signal. In particular, we have studied the behavior under the influence of a multiple frequency input signal of the type considered recently in [9]. The interest of this extension lies in the fact that according to the authors of [9], gains in the signal-to-noise ratio larger than unity can be obtained for subthreshold input signals of this kind.

We have tested the predictions of LRT by concentrating on the analysis of the Fourier spectrum of the output and the scaling of the output amplitude with the input amplitude. Our numerical calculations indicate that, in general terms, LRT provides an adequate approximation to the dynamics even for input amplitudes which are a substantial fraction of its threshold value. For fixed values of the input parameters (amplitude and frequency), the quality of LRT predictions improves as the noise strength increases. On the other hand, for small values of the input frequency, and noise values D such that A/D is large, the deviations of the numerical findings with respect to the LRT predictions are manifested. Thus, in order to decide on the validity of LRT when dealing with input signals with finite amplitude, it is also necessary to take into account the input frequency. These general conclusions extend to the case of multifrequency input signals as well.

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