Oscillatory systems driven by noise: Frequency and phase synchronization

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The phenomenon of effective phase synchronization in stochastic oscillatory systems can be quantified by an average frequency and a phase diffusion coefficient. A different approach to compute the noise-averaged frequency is put forward. The method is based on a threshold crossing rate pioneered by Rice. After the introduction of the Rice frequency for noisy systems we compare this quantifier with those obtained in the context of other phase concepts, such as the natural and the Hilbert phase, respectively. It is demonstrated that the average Rice frequency $\langle\omega\rangle_R$ typically supersedes the Hilbert frequency $\langle\omega\rangle_H$, i.e. $\langle\omega\rangle_R \geqslant \langle\omega\rangle_H$. We investigate next the Rice frequency for the harmonic and the damped, bistable Kramers oscillator, both without and with external periodic driving. Exact and approximative analytic results are corroborated by numerical simulation results. Our results complement and extend previous findings for the case of noise-driven inertial systems.

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I. INTRODUCTION

The topic of synchronization covers a plethora of phenomena [1] ranging from the entrainment of a system by an external drive over mutual synchronization of two bidirectionally coupled systems to coherent modes of many units with complex coupling patterns [2]. From the conceptual point of view different degrees of synchronization can be distinguished: complete synchronization [3], generalized synchronization [4], lag synchronization [5], phase synchronization [6,7], and burst (or train) synchronization [8]. In the following, we restrict our considerations to phase synchronization that has attracted recent interest for the following reason: In many practical applications the dynamics of a system, though not perfectly periodic, can still be understood as the manifestation of a stochastically modulated limit cycle [9,10]. As examples, we mention neuronal activity [11], the cardiorespiratory system [12], population dynamics [13], or even digital communications (where switching back and forth constitutes a cycle). In all these dynamics, marker events can be used to pinpoint the completion of a cycle, k, and the beginning of a subsequent one $k+1$. It is then possible to define an instantaneous phase $\phi^L(t)$ by linear interpolation, i.e.,

$$\phi^L(t)=\frac{t-t_k}{t_{k+1}-t_k}2\pi+k\,2\pi \qquad (t_k\leqslant t<t_{k+1}), \tag{1}$$

where the times t_k are fixed by the marker events. Introducing a Poincaré section is another way of defining a marker event and was widely used in model studies [1]. Reexpressing the time series $x(t)$ of the system as

$$x(t)=a(t)\cos[\phi^L(t)], \tag{2}$$

then defines an instantaneous amplitude $a(t)$. The benefit of such a treatment is to reveal a synchronization of two or more such signals: whereas the instantaneous amplitudes and, therefore, the time series might look rather different, the phase evolution can display quite some similarity. If the average growth rates of phases match each other (notwithstanding the fact that phases may diffuse rapidly) the result is termed frequency locking. Small phase diffusion, in addition to frequency locking, means that phases are practically locked during long episodes that occasionally are disrupted by phase slips caused by sufficiently large fluctuations. This elucidates the meaning of effective phase synchronization in stochastic systems.

The mentioned marker events that determine the moments when the phase passes multiples of 2π can be either spikey peaks as for the neural activity, pronounced maxima as for population dynamics or switchings from "low" to "high" in digital communications. In a unifying approach the phase evolution can be related to zero crossings since relative maxima of a smooth function are zeros of the derivative and the inclusion of an arbitrary threshold only requires a shift of the coordinate. In this view the average frequency, i.e. the average phase velocity, turns out to be the average rate of zero crossings that is captured by a formula put forward by Rice [14,15].

This elementary observation yields a proposed way (i) to quantify the average frequency of a phase evolution, henceforth termed the "Rice frequency," and (ii) to track down frequency locking in stochastic systems. In this investigation, we will elaborate how the proposed approach contrasts with other definitions, e.g., the one based on the analytic signal [16] and the Hilbert transform [6]. For developing the method and illustrating its purpose, we will employ the damped harmonic oscillator and the damped bistable Kramers oscillator serving as paradigmatic systems. As we will show, the Rice frequency matches, in practice, the mean phase velocity as computed from the linear interpolated

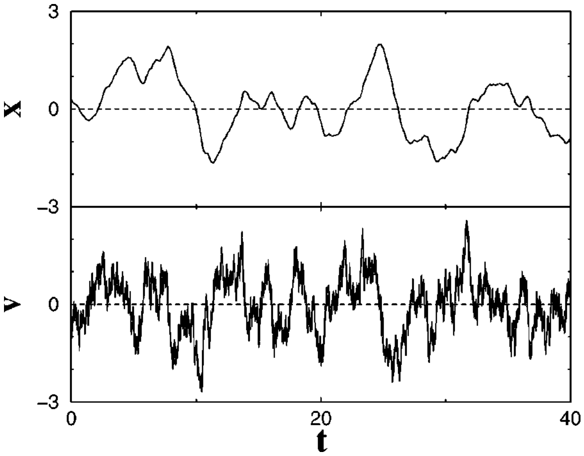

FIG. 1. Position x and velocity v of the undriven noisy harmonic oscillator Eq. (6) with friction coefficient $\gamma=1$, and natural frequency $\omega_0=1$. Whereas the position x is smooth the velocity v is continuous but nowhere differentiable. Counting of zero crossings is, consequently, only possible for the x coordinate.

phase ϕ^L. In theory, it opens a simple way to derive stationary or even time dependent average phase velocities from known probability densities.

II. NOISE-DRIVEN SYSTEMS: DEVELOPING THE RICE FREQUENCY

To detail our derivation of the Rice frequency in this section, we start from the following one-dimensional potential system [17]:

$$\ddot{x}+\gamma\dot{x}+U'(x)=\sqrt{\gamma}\xi+F\cos(\Omega t) \tag{3}$$

subjected to Gaussian white noise ξ of intensity D, i.e.,

$$\langle\xi(t)\rangle=0, \qquad \langle\xi(t)\xi(s)\rangle=2D\delta(t-s), \tag{4}$$

and being driven by the external harmonic force $F\cos(\Omega t)$. In Fig. 1 we show a sample path for the harmonic oscillator specified by the potential

$$U(x)=\omega_0^2\frac{x^2}{2} \tag{5}$$

and the corresponding Langevin equation

$$\ddot{x}+\gamma\dot{x}+\omega_0^2x=\sqrt{\gamma}\xi+F\cos(\Omega t). \tag{6}$$

In Fig. 1 we used the friction coefficient $\gamma=1$, the natural frequency $\omega_0=1$, and a vanishing amplitude $F=0$ of the external drive. As can be read off from Fig. 1, the velocity $v=\dot{x}$ basically undergoes a Brownian motion and, therefore, constitutes a rather jerky continuous, but generally not differentiable signal. In particular, near a zero crossing of v there are many other zero crossings. In contrast to that, the coordinate x is a much smoother signal since it is determined by an integral over a continuous function

$$x(t)=x(0)+\int_0^t v(\tau)d\tau, \tag{7}$$

and, therefore, differentiable. In particular, near a zero crossing of x there are no other zero crossings. In the following, we will take advantage of this remarkable smoothness property of x that is an intrinsic property of the full oscillatory system (3) and disappears when we perform the overdamped limit.

In 1944, Rice [14] deduced a formula for the average number of zero crossings of a smooth signal like x in the oscillator Eq. (3). In this rate formula enters the probability density $P(x,v;t)$ of x and its time derivative, $v=\dot{x}$, at a given instant t. The Rice rate for passages through zero with positive slope (velocity) is determined by [15]

$$\langle f\rangle(t)=\int_0^\infty vP(x=0,v;t)dv. \tag{8}$$

This time-dependent rate is to be understood as an ensemble average. If the dynamical system is ergodic and mixing the asymptotic stationary rate $\langle f_s\rangle$ can likewise be achieved by the temporal average of a single realization. Let $N([0,t])$ be the number of positive-going zeros of the signal x in the time interval $[0,t]$. Using ergodicity, the relation

$$\langle f_s\rangle=\int_0^\infty vP_s(x=0,v)dv=\lim_{t\to\infty}\frac{N([0,t])}{t} \tag{9}$$

is fulfilled for the process characterized by the stationary density $P_s(x,v)$. In the following we always consider stationary quantities. As explained in the Introduction, the zero crossings can be used as marker events to define an instantaneous phase $\phi^L(t)$ by linear interpolation, cf. Eq. (1). The related average phase velocity is the product of the (stationary) Rice rate and 2π and, hence, called the (stationary) Rice frequency

$$\langle\omega\rangle_R=2\pi\langle f_s\rangle=2\pi\int_0^\infty vP_s(x=0,v)dv. \tag{10}$$

For a dynamics described by a potential $U(x)$ in the absence of an external driving, i.e., Eq. (3) with $F=0$, the stationary density can be calculated explicitly yielding

$$P_s(x,v)=C\exp\left[-\left(\frac{v^2}{2}+U(x)\right)\Big/ D\right], \tag{11}$$

where C is the normalization constant. Note that the independence of the stationary probability density from the friction coefficient will also make the Rice frequency independent from γ. From Eq. (11) and the application of Eq. (10), it is straightforward to derive the exact result

$$\langle\omega\rangle_R=\frac{\sqrt{2\pi D}\exp\left[-\frac{U(0)}{D}\right]}{\int_{-\infty}^{\infty}\exp\left[-\frac{U(x)}{D}\right]dx}. \tag{12}$$

Without loss of generality we can set $U(0)=0$. In the limit $D\to0$, we can perform a saddle-point approximation around the deepest minima x_m (e.g., for symmetric potentials). In this way we find the following expression valid for $D\ll\Delta U=U(0)-U(x_i)$, i.e., the small noise approximation,

$$\langle\omega\rangle_R=\left[\sum_i\frac{\exp\left[-\frac{U(x_i)}{D}\right]}{\sqrt{U''(x_i)}}\right]^{-1}. \tag{13}$$

In the limit $D\to\infty$, we have to consider the asymptotic behavior of the potential, $\lim_{x\to\pm\infty}U(x)$, to estimate the integral in Eq. (12). For potentials that can be expanded in a Taylor series about zero and that, therefore, result in a power series of order $2m$, i.e., $U(|x|\to\infty)\sim x^{2m}$, we can rescale the integration variable by $x=D^{1/2m}\tilde{x}$. For sufficiently large D, the integral is dominated by the power $2m$ term. In this way we find the large noise scaling

$$\langle\omega\rangle_R\sim D^{\alpha}\quad\text{with}\qquad\alpha=\frac{m-1}{2m}. \tag{14}$$

Applying Eqs. (12) and (13) to the harmonic oscillator (5) we immediately find that $\langle\omega\rangle_R=\omega_0$, independent of γ and for all values of $D>0$. This is also in agreement with Eq. (14). It follows because $m=1$ implies that, for large noise, the Rice frequency $\langle\omega\rangle_R$ does not depend on D at all. Note, however, that in the deterministic limit, i.e., for $D=0$, we have the standard result

$$\langle\omega\rangle_R=\begin{cases}\sqrt{\omega_0^2-\gamma^2/4} & \text{for}\quad\gamma<2\omega_0\\ 0 & \text{for}\quad\gamma\geqslant2_{\omega_0}\end{cases} \tag{15}$$

which explicitly does depend on the friction strength $\gamma>0$. Therefore, the limit $D=0$ is discontinuous except in the undamped situation $\gamma=0$.

The similarity of Eqs. (12) and (13) with rates from transition state theory [18] will be addressed below when we discuss the bistable potential.

III. ALTERNATIVE PHASE DEFINITIONS

An alternative phase definition stems from the method of Bogoliubov and Mitropolski [19]. This method starts from the following decomposition of the dynamics:

$$\dot{x}=v, \tag{16}$$

$$\dot{v}=-\omega_0^2x+f(x,v,t,\xi,\ldots), \tag{17}$$

where the function f comprises all terms of higher than first order in x (nonlinearities), velocity dependent terms (friction), and noise. In their work Bogoliubov and Mitropolski considered the function f to be a perturbation of order ϵ. For the subsequent discussion, however, this assumption is by no means necessary. The definition of an instantaneous phase proceeds by expressing the position x and the velocity v in polar coordinates r and ϕ^P,

$$x(t)=r(t)\cos[\phi^P(t)], \tag{18}$$

$$v(t)=-\omega_0r(t)\sin[\phi^P(t)], \tag{19}$$

which yields by inversion [20]

$$r(t)=\sqrt{x^2(t)+[v(t)/\omega_0]^2}, \tag{20}$$

$$\phi^P(t)=\arctan\left[-\frac{v(t)/\omega_0}{x(t)}\right]. \tag{21}$$

Here, it should be noted that a meaningful clockwise rotation in the x,v plane determines angles to be measured in a specific way depending on the sign of ω_0. Using Eqs. (18), (19), (20), and (21) it is straightforward to transform the dynamics in x and v, Eqs. (16) and (17), into the following dynamics for r and ϕ^P [21,22]:

$$\dot{r}=-\frac{f(r\cos(\phi^P),-\omega_0r\sin(\phi^P),t,\xi)}{\omega_0}\sin(\phi^P), \tag{22}$$

$$\dot{\phi}^P=\omega_0-\frac{f(r\cos(\phi^P),-\omega_0r\sin(\phi^P),t,\xi)}{\omega_0r}\cos(\phi^P). \tag{23}$$

For instance, for the harmonic oscillator Eq. (6) with $F=0$ these equations read

$$\dot{r}=-\left[\gamma r\sin(\phi^P)+\frac{\sqrt{\gamma}}{\omega_0}\xi\right]\sin(\phi^P), \tag{24}$$

$$\dot{\phi}^P=\omega_0-\left[\gamma\sin(\phi^P)+\frac{\sqrt{\gamma}}{\omega_0}\frac{\xi}{r}\right]\cos(\phi^P). \tag{25}$$

The line $x=0$ corresponds to angles $\phi^P=\pi/2+n\pi,n\in N$. As can be read off from Eq. (23), the phase velocity always assumes a specific value for $x=0$ [23], i.e.,

$$\dot{\phi}^P(x=0)=\omega_0. \tag{26}$$

This has the following remarkable consequence. We see that even in the presence of noise passages through zero in the upper half plane $v>0$ are only possible from $x<0$ to $x>0$, in the lower half plane only from $x>0$ to $x<0$. This insight becomes even more obvious from a geometrical interpretation: as the noise exclusively acts on the velocity v, cf. Eq.

(17), it can only affect changes in the vertical direction (in x,v space). Along the vertical line $x=0$, however, the angular motion possesses no vertical component while radial motion is solely in the vertical direction and, therefore, only affected by the noise. From this, we conclude that both the linear interpolating and the polar phase increase between two passages through $x=0$ with positive slope by an amount of 2π. Therefore, the (stationary) average phase velocity is identical for the linear interpolating phase ϕ^L, cf. Eq. (1), and the natural phase defined by virtue of polar coordinates ϕ^P, i.e.,

$$\langle \dot{\phi}^P \rangle := \langle \omega \rangle_P = \langle \omega \rangle_L = \langle \omega \rangle_R . \tag{27}$$

The Hilbert phase ϕ^H constitutes yet another phase definition. It is widely used in applications in the context of the analytic signal [10,16,24] to construct a phase for a one-dimensional time series $x(t)$. In this context the Hilbert transform x^H of the signal x is defined by the convolution

$$x^H(t) = H[x](t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau, \tag{28}$$

where the integral in the last equation has to be evaluated in the sense of the Cauchy principal value (P). Rewriting the original signal $x(t) = \langle x \rangle + \hat{x}(t)$, where $\langle x \rangle$ represents the constant mean and, consequently, $\hat{x}(t)$ a zero mean signal, we find that $x^H(t) = H[x](t) = H[\hat{x}](t)$. Hence, we can always subtract the signal mean without changing the result. The Hilbert transform is subsequently used to define the Hilbert phase [20],

$$\phi^H(t) = \arctan\left[\frac{x^H(t)}{x(t)}\right]. \tag{29}$$

The convolution kernel in Eq. (28) has the property of properly reproducing the phase of a harmonic signal. The asymptotic Hilbert phase velocity $\langle \omega \rangle_H$ is then defined in connection with Eq. (29) in a straightforward manner by

$$\langle \omega \rangle_H = \lim_{t\to\infty} \frac{\phi^H(t)}{t}. \tag{30}$$

Again, as a consequence of stationarity and ergodicity we also find

$$\langle \omega \rangle_H = \langle \dot{\phi}^H \rangle_s = \left\langle \frac{v^H x - v x^H}{x^2 + (x^H)^2} \right\rangle_s , \tag{31}$$

where the subscript s is a reminder of the stationary statistics.

To exemplify the relation between the Rice frequency $\langle \omega \rangle_R$ and the Hilbert frequency $\langle \omega \rangle_H$, we consider the damped harmonic oscillator Eq. (6) agitated by noise alone. In Fig. 2 we show a numerically evaluated sample path and the corresponding Hilbert phase modulo 2π using the parameters $\gamma=1$, $D=1$, $\omega_0=1$, $F=0$. An important point to observe here is that at $t\approx 2.6$ and $t\approx 9$ the Hilbert phase ϕ^H does not increase by 2π after two successive passages through zero with positive slope. This illustrates a remarkable difference between the Hilbert phase and the previously discussed phase definitions. This observation is a consequence of the nonlocal character of the Hilbert transform, i.e., the fact that the whole temporal history of $x(t)$ enters in the convolution (28). In particular, short and very small amplitude crossings to positive x are not properly taken into account by the Hilbert phase since they only result in a small reduction of ϕ^H. This leads us to conjecture that quite generally

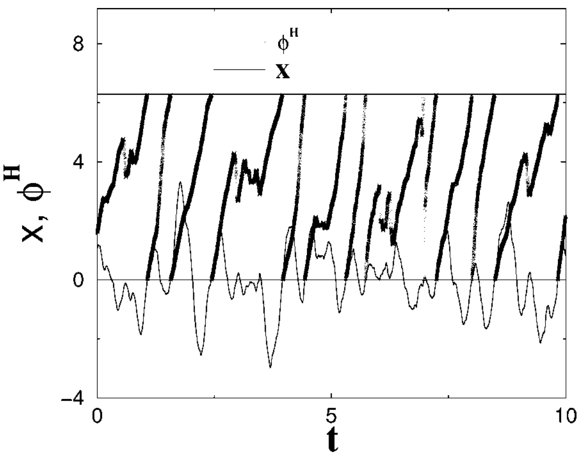

FIG. 2. Hilbert phase $\phi^H(t)$ modulo 2π (thick solid) and corresponding signal $x(t)$ (thin solid) for the undriven harmonic oscillator Eq. (6) with friction strength $\gamma=1$, noise intensity $D=1$, natural frequency $\omega_0=1$, and driving amplitude $F=0$. Note that although there are two successive zero crossings of x with positive slope near $t\approx 2.6$ and $t\approx 9$ the Hilbert phase does not increase by 2π.

$$\langle \omega \rangle_H \leq \langle \omega \rangle_R \tag{32}$$

holds. In fact, for the case of the harmonic oscillator that generates a stationary Gaussian process we even can prove this conjecture by deriving explicit expressions for $\langle \omega \rangle_R$ and $\langle \omega \rangle_H$. As usual, let $S(\omega)$ denote the spectrum of the stationary Gaussian process x. Then the Rice frequency can be recast in the form of [15]

$$\langle \omega \rangle_R = \left[\frac{\int_0^\infty \omega^2 S(\omega)\, d\omega}{\int_0^\infty S(\omega)\, d\omega} \right]^{1/2} . \tag{33}$$

In the Appendix, we show that the Hilbert frequency of the same process x is given by a similar expression [25] (cf. also [24]), namely,

$$\langle \omega \rangle_H = \left[\frac{\int_0^\infty \omega\, S(\omega)\, d\omega}{\int_0^\infty S(\omega)\, d\omega} \right]. \tag{34}$$

Interpreting the quantity $S(\omega)/\int_0^\infty S(\hat{\omega})d\hat{\omega}$ as a probability density $P(\omega)$, $\omega \in (0,\infty)$, we can use the property that the related variance is positive, i.e.,

$$\left[\int_0^\infty \omega P(\omega)d\omega \right]^2 \leqslant \int_0^\infty \omega^2 P(\omega)d\omega. \tag{35}$$

Taking the square root on both sides of the last inequality immediately proves Eq. (32).

Using the spectrum of the undriven noisy harmonic oscillator

$$S(\omega) = \frac{4\gamma D}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \tag{36}$$

and employing Eqs. (33) and (34), it is easy to see that both $\langle\omega\rangle_R$ and $\langle\omega\rangle_H$ do not vary with D. We have already shown above that $\langle\omega\rangle_R = \omega_0$. In contrast to this, $\langle\omega\rangle_H$ is a monotonically decreasing function of γ that approaches ω_0 from below in the limit $\gamma \to 0^+$.

IV. RICE FREQUENCY FOR THE PERIODICALLY DRIVEN NOISY HARMONIC OSCILLATOR

The probability density of the periodically driven noisy harmonic oscillator can be determined analytically by taking advantage of the linearity of the problem. Introducing the mean values of the coordinate and the velocity, $\langle x(t)\rangle$ and $\langle v(t)\rangle$, the variables

$$\tilde{x} = x - \langle x\rangle, \qquad \tilde{v} = v - \langle v\rangle \tag{37}$$

obey the differential equation of the undriven noisy harmonic oscillator. In the asymptotic limit $t \to \infty$ the mean values converge to the well known deterministic solution

$$\langle x(t)\rangle = \frac{F}{\sqrt{(\omega_0^2 - \Omega^2)^2 + \gamma^2\Omega^2}} \cos(\Omega t - \delta), \tag{38}$$

$$\langle v(t)\rangle = -\Omega \frac{F}{\sqrt{(\omega_0^2 - \Omega^2)^2 + \gamma^2\Omega^2}} \sin(\Omega t - \delta), \tag{39}$$

$$\delta = \arctan\left[\frac{\gamma\Omega}{\omega_0^2 - \Omega^2} \right] \tag{40}$$

with the common phase lag δ. Therefore, after deterministic transients have settled the *cyclostationary* probability density of the driven oscillator reads

$$P_{cs}(x,v;t) = P_s(x - \langle x(t)\rangle, v - \langle v(t)\rangle) \tag{41}$$

with the Gaussian density

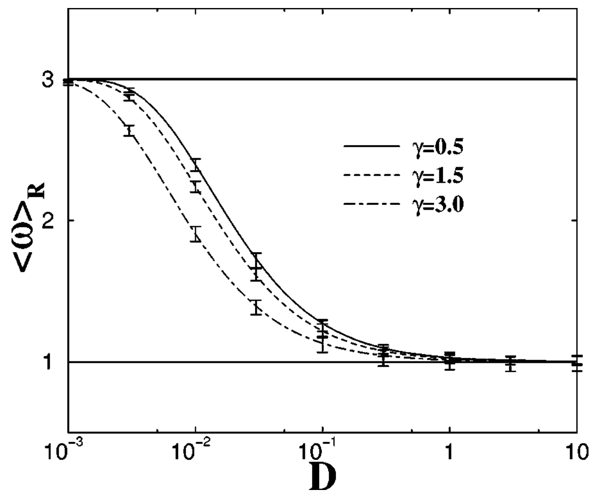

FIG. 3. Rice frequencies for the driven harmonic oscillator Eq. (6) with natural frequency $\omega_0 = 1$, driving amplitude $F = 1$, and driving frequency $\Omega = 3$ for different values of the friction strength γ. The numerically achieved values (symbols with error bars) nicely match the analytical curves determined using Eq. (44).

$$P_s(x,v) = \frac{\omega_0}{2\pi D} \exp\left[-\left(\frac{v^2}{2} + \frac{\omega_0^2 x^2}{2} \right) \Big/ D \right]. \tag{42}$$

Using Eq. (10) the cyclostationary probability density (41) yields an oscillating expression for the Rice frequency $\langle\omega\rangle_R(t)$. The time dependence of this stochastic average can be removed by an initial phase average, i.e., a subsequent average over one external driving period $2\pi/\Omega$,

$$\langle\omega\rangle_R = \int_0^{2\pi/\Omega} \langle\omega\rangle_R(t) \frac{\Omega dt}{2\pi} \tag{43}$$

$$= \int_0^{2\pi/\Omega} \int_0^\infty v\, P_{cs}(0,v;t) dv\, \Omega dt. \tag{44}$$

The resulting analytical and numerically achieved values of the Rice frequency as a function of the noise intensity D are shown in Fig. 3 for fixed $\omega_0 = 1, F = 1, \Omega = 3$ and various values of γ. For small noise intensities D the Rice frequency $\langle\omega\rangle_R$ is identical to the external driving frequency Ω, whereas for large noise intensities the external drive becomes inessential and the Rice frequency approaches $\langle\omega\rangle_R = \omega_0$.

Further insight into the analytic expression (44) is gained from performing the following scale transformations:

$$\tilde{t} = \Omega t - \delta \quad \text{and} \quad \tilde{x} = \frac{x}{\sqrt{2D/\Omega}}, \tag{45}$$

from which we immediately find the rescaled velocity

$$\tilde{v} = \frac{d\tilde{x}}{d\tilde{t}} = \frac{\sqrt{2D}/\Omega}{1/\Omega} \frac{dx}{dt} = \sqrt{2D}\, v. \tag{46}$$

Inserting these dimensionless quantities into Eq. (44) yields

$$\langle \omega \rangle_R = \omega_0 I(\tilde{A}, \tilde{\omega}_0), \tag{47}$$

$$I(\tilde{A}, \tilde{\omega}_0) = \frac{1}{\pi} \int_{-\delta}^{2\pi-\delta} \int_0^\infty \tilde{v} \exp[-(\tilde{v} + \tilde{A}\sin\tilde{t})^2 - (\tilde{\omega}_0 \tilde{A}\cos\tilde{t})^2] d\tilde{v}\, d\tilde{t}, \tag{48}$$

where we have defined further dimensionless quantities

$$\tilde{A} = \frac{\Omega}{\sqrt{2D}} \frac{F}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (\gamma\Omega)^2}}, \tag{49}$$

$$\tilde{\omega}_0 = \frac{\omega_0}{\Omega}. \tag{50}$$

Due to the 2π periodicity of the trigonometric functions, the integral (48) does not change when shifting the interval for the integration with respect to $\tilde{t}$ back to $[0,2\pi]$. Hence, I is only a function of $\tilde{A}$ and $\tilde{\omega}_0$. An expansion for small $\tilde{A}$ yields

$$\langle \omega \rangle_R = \omega_0 \left[1 + \frac{1 - \tilde{\omega}_0^2}{2} \tilde{A}^2 + O(\tilde{A}^4) \right], \tag{51}$$

which implies for large D

$$\langle \omega \rangle_R - \omega_0 \sim \frac{1}{D}. \tag{52}$$

The opposite extreme, $\tilde{A} \to \infty$ or $D \to 0$, can be extracted from a saddle point approximation around $\tilde{v} = \tilde{A}$ and $\tilde{t} = 3\pi/2$. Following this procedure, the integral (48) gives the constant $1/\tilde{\omega}_0$. This directly implies $\langle \omega \rangle_R = \Omega$.

The crossover between these two extremes occurs when the first correction term in Eq. (51) is no longer negligible, i.e., for

$$\frac{|1 - \tilde{\omega}_0^2|}{2} \tilde{A}^2 \approx 1. \tag{53}$$

When solved for the crossover noise intensity D_{co}, this yields

$$D_{\mathrm{co}} \approx \frac{F^2 |\Omega^2 - \omega_0^2|}{4[(\omega_0^2 - \Omega^2)^2 + (\gamma\Omega)^2]}, \tag{54}$$

which, for the parameters used in Fig. 3, correctly gives values between 10^{-2} and 10^{-1}.

In Fig. 3 the parameters F, Ω, and ω_0 and, hence, $\tilde{\omega}_0$ are identical for all curves. Solving $\tilde{A}(\gamma_1, D_1) = \tilde{A}(\gamma_2, D_2)$ with respect to D_2 shows that the curves become shifted horizontally as in the log-linear plot in Fig. 3. Another way to explain this shift is by noting that $dD_{\mathrm{co}}/d\gamma < 0$.

V. THE ROLE OF COLORED NOISE

As noted already above, see also in Fig. 1, the Rice frequency—strictly speaking —cannot be defined for stochastic realizations that are directly driven by Gaussian white noise (i.e., the "derivative" of the Wiener process). From Eq. (3) this holds true for the velocity degree of freedom $\dot{v} = \ddot{x}$. This is so, because the stochastic trajectories of degrees of freedom being subjected to white Gaussian noise forces are continuous but are of *unbounded* variation and nowhere differentiable [26]. This fact implies that such stochastic realizations cross a given threshold within a fixed time interval infinitely often if only the numerical resolution is increased *ad infinitum*. This drawback, which is rooted in the mathematical peculiarities of idealized Gaussian white noise, can be overcome if we consider instead a noise source possessing a finite correlation time, i.e., colored noise, see Ref. [27]. To this end, we consider here an oscillatory noisy harmonic dynamics driven by Gaussian exponentially correlated noise $z(t)$, i.e.,

$$\dot{x} = v, \tag{55}$$

$$\dot{v} = -\gamma v - \omega_0^2 x + \sqrt{\gamma} z(t), \tag{56}$$

$$\dot{z} = -\frac{z}{\tau} + \frac{1}{\tau}\xi, \tag{57}$$

with $z(t)$ obeying $\langle z(t) \rangle = 0$ and

$$\langle z(t) z(s) \rangle = \frac{D}{\tau} \exp\left(-\frac{|t-s|}{\tau} \right). \tag{58}$$

Following the reasoning in Sec. II, we find for the Rice frequency of $x(t)$ as before

$$\langle \omega \rangle_x = \int_0^\infty dv \int_{-\infty}^\infty dz v\, P_s(0,v,z) \tag{59}$$

$$= \frac{\omega_0}{\sqrt{1+\gamma\tau}}. \tag{60}$$

Likewise, upon noting that within a time interval Δt, $-\Delta t(-\gamma\dot{x} - \omega_0^2 x + \sqrt{\gamma} z) < v < 0$, or $-\Delta t(-\omega_0^2 x + \sqrt{\gamma} z) + O(\Delta t)^2 < v < 0$, respectively, the Rice frequency of the zero crossings with positive slope of the process $v(t)$ is given by

$$\langle \omega \rangle_v = \int_{-\infty}^\infty dx \int_x^\infty dz (\sqrt{\gamma} z - \omega_0^2 x) P_s(x,0,z), \tag{61}$$

which is evaluated to read

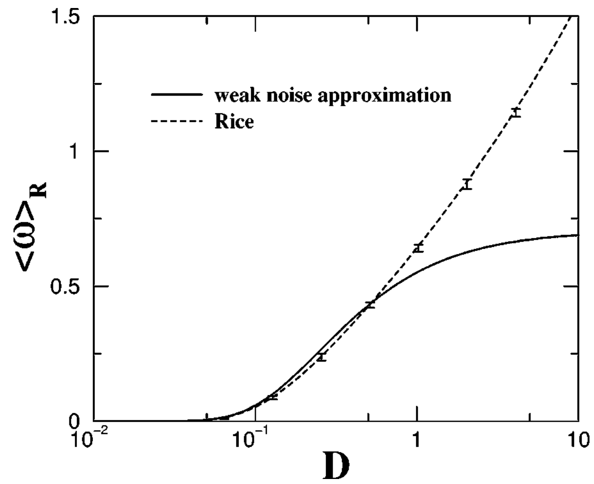

FIG. 4. Rice frequencies for the undriven bistable oscillator Eq. (64) with friction strength $\gamma=1$. Numerical values with error bars match the analytically determined values (dotted line) using Eq. (10) with Eq. (65). As expected, for large values of D the Rice frequency scales like $D^{1/4}$. The solid line presents the leading weak noise approximation in Eq. (70).

$$\langle\omega\rangle_v=\sqrt{\omega_0^2+\frac{\gamma}{\tau}}. \tag{62}$$

The result in Eq. (60) shows that for small noise color τ the Rice frequency for $\langle\omega\rangle_x$ assumes a correction $\langle\omega\rangle_x\sim\omega_0(1-\tau/2\gamma)$, as $\tau\to0^+$. In clear contrast, the finite Rice frequency for the velocity process $v(t)$ (56) diverges in the limit of vanishing noise color proportional to $\tau^{-1/2}$.

VI. RICE FREQUENCY OF THE BISTABLE KRAMERS OSCILLATOR

The bistable Kramers oscillator, i.e., Eq. (3) with the double well potential,

$$U(x)=\frac{x^4}{4}-\frac{x^2}{2}, \tag{63}$$

is often used as a paradigm for nonlinear systems. With reference to Eq. (3) the corresponding Langevin equation is given by

$$\ddot{x}+\gamma\dot{x}+x^3-x=\sqrt{\gamma}\xi+F\cos(\Omega t), \tag{64}$$

which, in the absence of the external signal, $F=0$, generates the stationary probability distribution

$$P_s(x,v)=C\exp\left\{-\left(\frac{v^2}{2}+\frac{x^4}{4}-\frac{x^2}{2}\right)\Big/ D\right\} \tag{65}$$

with the normalization constant C. Using this stationary probability density and Eq. (10), we can determine the Rice frequency analytically. In Fig. 4 we depict this analytic result together with numerical simulation data including error bars. The simulation points perfectly match the analytically determined curve. As expected for the asymptotically dominant quartic term, i.e., $m=2$ [cf. Sec.II, especially. Eq. (14)], the Rice frequency scales as $\langle\omega\rangle_R\sim D^{1/4}$ for large values of D .

Comparing the Rice frequency formula, Eq. (10), with the forward jumping rate k^+_{TST} from the transition state theory [18],

$$k^+_{\mathrm{TST}}=\hat{Z}_0^{-1}\int dx\,dv\quad\theta(v)\,\delta(x)v\,\exp[-H(x,v)/D], \tag{66}$$

where

$$\hat{Z}_0=\int_{x<0}dx\,dv\,\exp[-H(x,v)/D], \tag{67}$$

and $H(x,v)=(1/2)v^2+(1/4)x^4-(1/2)x^2$ represents the corresponding Hamiltonian, one can see that the difference between both solely rests upon normalizing prefactors. Whereas the rate k^+_{TST} is determined by the division of the integral Eq. (66) by the "semipartition" function $\hat{Z}_0$, the rate $\langle\omega\rangle_R/2\pi$ is established by dividing the same integral Eq. (66) by the complete partition function Z_0

$$Z_0=\int dx\,dv\,\exp[-H(x,v)/D]. \tag{68}$$

Particularly for symmetric (unbiased) potentials, i.e., $V(-x)=V(x)$, this amounts to the relation $Z_0=2\hat{Z}_0$, hence,

$$\langle\omega\rangle_R=\pi\,k^+_{\mathrm{TST}}. \tag{69}$$

At weak noise, $E_b/D\gg1$, this relation simplifies to

$$\langle\omega\rangle_R\approx\frac{\omega_0}{2}\exp[-E_b/D], \tag{70}$$

wherein E_b denotes the barrier height and ω_0 the angular frequency inside the well ($\omega_0=\sqrt{2}$). Indeed, in the small-to-moderate regime of weak noise this estimate nicely predicts the exact Rice frequency, cf. Fig. 4.

The periodically driven bistable Kramers oscillator was the first model considered to explain the phenomenon of stochastic resonance (SR) [28] and it still serves as one of the major paradigms of SR [29,30]. In its overdamped form it was used to support experimental data (from the Schmitt trigger) displaying the effect of stochastic frequency locking [9] observed for sufficiently large, albeit subthreshold signal amplitudes, i.e., for $F_{\min}<F<2/\sqrt{27}$. From a numerical simulation of the overdamped Kramers oscillator it was also found that noise-induced frequency locking for large signal amplitudes was accompanied by noise-induced phase coherence, the latter implies a pronounced minimum of the effective phase diffusion coefficient [31]

$$\tilde{D}_{\mathrm{eff}}=\frac{d}{dt}[\langle(\phi(t))^2\rangle-\langle\phi(t)\rangle^2] \tag{71}$$

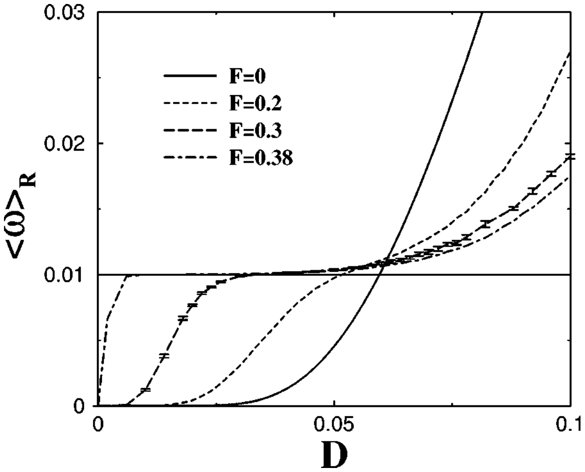

FIG. 5. Numerically determined Rice frequencies of the periodically driven bistable Kramers oscillator Eq. (64) computed with the friction coefficient $\gamma=0.5$ and the angular driving frequency $\Omega=0.01$ and plotted as a function of the noise intensity D. Different curves correspond to various amplitudes of the harmonic drive F. For larger values of F wider regions appear where the Rice frequency is locked to the external driving frequency Ω.

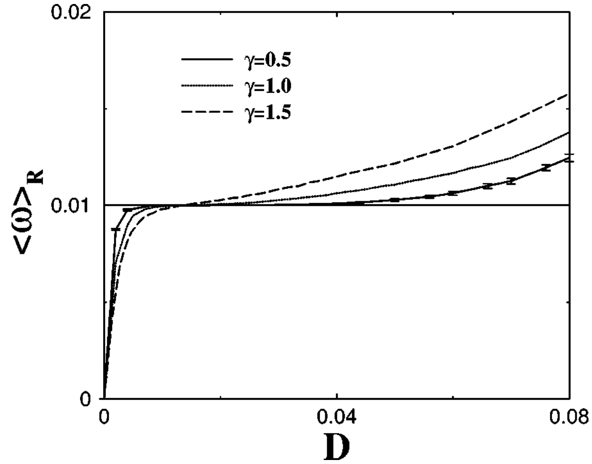

FIG. 6. Numerically determined Rice frequency as a function of the noise intensity D for the periodically driven Kramers oscillator Eq. (64) with the angular driving frequency $\Omega=0.01$ and driving amplitude $F=0.384$ for different values of the friction coefficient γ. For smaller values of γ wider regions of frequency locking appear.

occurring for optimal noise intensity. Based on a discrete model [32], analytic expressions for the frequency and phase diffusion coefficient were derived that correctly reflect the conditions for noise-induced phase synchronization [33] for both periodic and aperiodic input signals.

To link the mentioned results to the Rice frequency introduced above we next investigate the behavior of the Kramers oscillator with nonvanishing inertia. We show numerical simulations for Eq. (64) with the parameters $\Omega=0.01,\gamma=0.5$ and diverse values of F in Fig. 5. For larger values of F, a region around $D\approx 0.05$ appears where the Rice frequency is locked to the external driving frequency Ω. Since for larger values of the external driving F smaller values of the noise parameter D are needed to obtain the same rate for switching events, the entry into the locking region shifts to smaller values of D for increasing F.

In Fig. 6 we present numerical simulations for fixed $F=0.384$, $\Omega=0.01$ and different values of the damping coefficient γ. Note that the value of F is slightly smaller than the critical value $F_c=2/\sqrt{27}\approx 0.3849\ldots$. For smaller values of γ wider coupling regions appear, since it is easier for the particle to follow the external driving for smaller damping. To check whether frequency synchronization is accompanied by effective phase synchronization, we have also computed the effective average phase diffusion coefficient, this time defined by the following asymptotic expression:

$$D_{\mathrm{eff}}=\lim_{t\to\infty}\frac{1}{t}\langle[\phi(t)-\langle\phi(t)\rangle]^2\rangle. \tag{72}$$

The connection to the instantaneous diffusion coefficient defined in Eq. (71) is established by applying the limit $t\to\infty$

$$D_{\mathrm{eff}}=\lim_{t\to\infty}\frac{1}{t}\int_0^t\tilde{D}_{\mathrm{eff}}(\tilde{t})\,d\tilde{t}. \tag{73}$$

In Fig. 7 we show numerical simulations of the effective phase diffusion coefficient D_{eff} as function of D for the linear interpolating phase ϕ^L. The phase diffusion coefficient displays a local minimum that gets more pronounced if the damping coefficient γ is decreased. Indeed, phase synchronization reveals itself through this local minimum of the average phase diffusion coefficient D_{eff} in the very region of the noise intensity D where we also observe frequency synchronization, cf. Fig. 5.

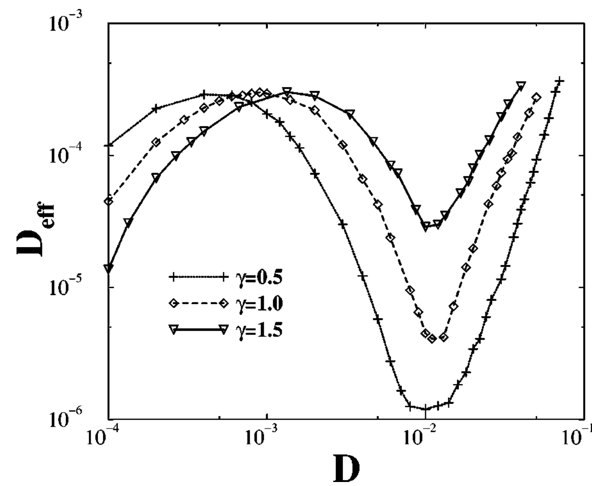

FIG. 7. Effective phase diffusion coefficient vs noise intensity for the periodically driven bistable Kramers oscillator Eq. (64) with angular driving frequency $\Omega=0.01$, driving amplitude $F=0.384$, which is close-to-threshold driving, and for different values of γ. For smaller values of the friction coefficient γ phase diffusion is diminished.

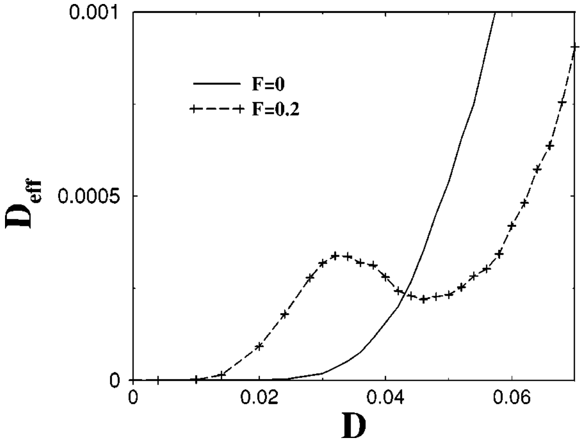

FIG. 8. Effective phase diffusion coefficient *vs* noise intensity for the periodically driven bistable Kramers oscillator Eq. (64) with friction coefficient $\gamma=0.5$, angular driving frequency $\Omega=0.01$, plotted for the undriven case $F=0$ and for driving with an amplitude $F=0.2$.

The qualitative behavior of the diffusion coefficient agrees also with a recently found result related to diffusion of Brownian particles in biased periodic potentials [34]. A necessary condition for the occurrence of a minimum was an anharmonic potential in which the motion takes place. In this biased anharmonic potential the motion over one period consists of a sequence of two events. Every escape over a barrier (Arrhenius-like activation) is followed by a time scale induced by the bias and describing the relaxation to the next minimum. The second step is weakly dependent on the noise intensity and the relaxation time may be even larger then the escape time as a result of the anharmonicity. For such potentials the diffusion coefficient exhibits a minimum for optimal noise, similar to the one presented in Figs. 7 and 8.

The average duration of locking episodes $\langle T_{\mathrm{lock}}\rangle$ can be computed by equating the second moment of the phase difference (between the driving signal and the oscillator) to π^2 [36]. A rough estimate, valid for the regions where frequency synchronization occurs, i.e. where the dynamics of the phase difference is dominated by diffusion, thus reads $\langle T_{\mathrm{lock}}\rangle = \pi^2/D_{\mathrm{eff}}$ or, when expressed by the number of driving periods [37]

$$\langle n_{\mathrm{lock}}\rangle = \frac{\Omega\pi}{2D_{\mathrm{eff}}}. \tag{74}$$

In this way we estimate from Figs. 7 and 8 $\langle n_{\mathrm{lock}}\rangle \sim 150\ldots 15\,000$ for $\Omega=0.01$ and relevant D_{eff} varying between $10^{-4}\cdots 10^{-6}$.

VII. SUMMARY

Positive-going threshold crossings are suitable marker events to define an instantaneous phase by linear interpolation in the case of smooth zero mean signals. The smoothness of the signal (coordinate) and, hence, the applicability of the method, relies on a dynamics with non-negligible inertia (nonoverdamped dynamics). In combination with the linear interpolated phase, the Rice formula can be used to define an asymptotic average phase velocity, namely, the Rice frequency. This frequency is identical to the one obtained in the context of the natural polar phase. The frequency based on the Hilbert phase never exceeds the Rice frequency. This has been proven analytically by the derivation of explicit expressions for the undriven noisy harmonic oscillator (stationary Gaussian process).

When turning on the noise, the Rice frequency of the periodically driven harmonic oscillator shows a crossover from the frequency of the external drive Ω (no noise) to the natural frequency ω_0 of the oscillator (large noise). Scaling of the Rice frequency as a function of the noise intensity in the large noise limit can be derived from a suitable expansion and the crossover region can be estimated quantitatively.

The Rice frequency of the bistable undriven Kramers oscillator (with inertia) is essentially identical with the forward hopping rate within transition state theory [18]. Noise-induced frequency locking regions between the Kramers oscillator and an external periodic driving can be detected employing the Rice frequency. Numerical data are similar to simulations results obtained earlier for the overdamped equation and the Hilbert phase [35]. In the frequency locking region, stochastic phase synchronization, characterized by a drastically diminished effective phase diffusion coefficient, is observed as well. These findings are in agreement with previous analytical results [32,33].

The Rice frequency as well as the Hilbert frequency can both be used to characterize complex physical, biophysical, and physiological noisy processes. They also carry the potential to serve as indicators–and possibly even as useful diagnostic predictors–for such complex processes as cardiorespiratory diseases, epileptic seizures, earthquake, and traffic dynamics, to mention but a few.

APPENDIX: HILBERT FREQUENCY OF THE HARMONIC OSCILLATOR

Here, we determine the Hilbert frequency of the undriven harmonic oscillator Eq. (6) using the stationary probability distribution of (x,x^H) and their temporal derivatives (v,v^H). The variables (x,v) in Eq. (6) describe Gaussian processes and the same is true for their corresponding Hilbert transforms. The correlation matrix of (x,v,x^H,v^H) is given by

$$\begin{pmatrix} \langle 1\rangle & 0 & 0 & \langle\omega\rangle \\ 0 & \langle\omega^2\rangle & -\langle\omega\rangle & 0 \\ 0 & -\langle\omega\rangle & \langle 1\rangle & 0 \\ \langle\omega\rangle & 0 & 0 & \langle\omega^2\rangle \end{pmatrix}, \tag{A1}$$

where we have used the abbreviations

$$\langle 1\rangle = \frac{1}{2\pi}\int_0^\infty S(\omega)d\omega, \tag{A2}$$

$$\langle\omega\rangle=\frac{1}{2\pi}\int_0^\infty \omega S(\omega)d\omega, \tag{A3}$$

$$\langle\omega^2\rangle=\frac{1}{2\pi}\int_0^\infty \omega^2 S(\omega)d\omega, \tag{A4}$$

with the spectral density of the damped harmonic oscillator

$$S(\omega)=\frac{4\gamma D}{[(\omega_0^2-\omega^2)^2+(\gamma\omega)^2]}. \tag{A5}$$

The covariance matrix Eq. (A1) determines the probability density $P(x,v,x^H,v^H)$ and the average of the velocity of the Hilbert phase Eq. (29) is determined by

$$\langle\dot{\phi}^H\rangle=\int dx\, dv\, dx^H\, dv^H \frac{v^H x - v x^H}{x^2+(x^H)^2}P(x,v,x^H,v^H). \tag{A6}$$

This integral can be evaluated using first the transformations,

$$v=r_1\sin(\alpha_1), \tag{A7}$$

$$v^H=r_1\cos(\alpha_1), \tag{A8}$$

$$x=r_2\cos(\alpha_2), \tag{A9}$$

$$x^H=r_2\sin(\alpha_2), \tag{A10}$$

and, after applying a trigonometric theorem, the subsequent transformations $\alpha=\alpha_1+\alpha_2$, $\beta=\alpha_1-\alpha_2$. Since the integrand does not depend on β the related integration can be readily done yielding

$$\langle\dot{\phi}^H\rangle=\int_0^\infty dr_1\int_0^\infty dr_2\int_0^{2\pi} d\alpha \frac{r_1^2\cos(\alpha)}{2\pi[\langle 1\rangle\langle\omega^2\rangle-\langle\omega\rangle^2]} \times\exp\left[-\frac{\langle 1\rangle r_1^2+\langle\omega^2\rangle r_2^2-2r_1r_2\langle\omega\rangle\cos(\alpha)}{2[\langle 1\rangle\langle\omega^2\rangle-\langle\omega\rangle^2]}\right]. \tag{A11}$$

Further evaluation proceeds by using the Bessel function of first kind and first order J_1 together with its series expansions, i.e.,

$$\int_0^{2\pi}\cos(\alpha)\exp[c\cos(\alpha)]\,d\alpha=2\pi i\, J_1\quad(-ic) \tag{A12}$$

$$=2\pi i\sum_{m=0}^\infty \frac{(-1)^m}{m!(m+1)!}\left(\frac{-ic}{2}\right)^{2m+1} \tag{A13}$$

$$=2\pi\sum_{m=0}^\infty \frac{1}{m!(m+1)!}\left(\frac{c}{2}\right)^{2m+1} \tag{A14}$$

leading to

$$\langle\dot{\phi}^H\rangle=\int_0^\infty dr_1\int_0^\infty dr_2\frac{r_1^2}{[\langle 1\rangle\langle\omega^2\rangle-\langle\omega\rangle^2]} \times\exp\left[-\frac{\langle 1\rangle^2 r_1^2+\langle\omega^2\rangle^2 r_2^2}{2[\langle 1\rangle\langle\omega^2\rangle-\langle\omega\rangle^2]}\right] \times\sum_{m=0}^\infty\frac{1}{m!(m+1)!}\left(\frac{r_1r_2\langle\omega\rangle}{2[\langle 1\rangle\langle\omega^2\rangle-\langle\omega\rangle^2]}\right)^{2m+1}. \tag{A15}$$

Upon changing summation and integration and performing the integration we arrive at

$$\langle\dot{\phi}^H\rangle=\sum_{m=0}^\infty\frac{\langle\omega\rangle[\langle 1\rangle\langle\omega^2\rangle-\langle\omega\rangle^2]}{\langle 1\rangle^2\langle\omega^2\rangle}\left(\frac{\langle\omega\rangle^2}{\langle 1\rangle\langle\omega^2\rangle}\right)^m \tag{A16}$$

$$=\frac{\langle\omega\rangle(\langle 1\rangle\langle\omega^2\rangle-\langle\omega\rangle^2)}{\langle 1\rangle^2\langle\omega^2\rangle}\frac{1}{1-\dfrac{\langle\omega\rangle^2}{\langle 1\rangle\langle\omega^2\rangle}} \tag{A17}$$

$$=\frac{\langle\omega\rangle}{\langle 1\rangle}, \tag{A18}$$

which is exactly the desired expression (34).

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