DYNAMICS OF NONLINEAR OSCILLATORS WITH FLUCTUATING PARAMETERS ☆

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We derive the (integro-differential) master equation of an oscillator in a thermal environment which is driven by a nonlinear randomly varying force. The thermal noise is assumed to be δ-correlated gaussian noise and the parameter fluctuations are assumed to be multiplicative white Poisson noise. For the case of a large viscosity we derive a generalized Smoluchowski equation and sketch the modification of Kramers' reaction rate. The rate is shown to contain a temperature-independent "tunneling" contribution.

In this paper we study aspects of the nonlinear dynamics of oscillators with fluctuating parameters. In contrast to the deterministic dynamics of nonlinear oscillators [1] there are relatively few discussions available of the case where the parameters characterizing the oscillator systems are random functions of time. Most of the sparse recent literature has been dealing with the case of linear systems with randomly varying parameters [2–4]. The nonlinear oscillator problem with fluctuating parameters is relevant for the description of parametric oscillations in electric networks with nonlinear dissipative elements and fluctuating circuit parameters, for the propagation of waves in nonlinear (turbulent) random media or, generally, for the case of stochastic frequency modulations in simple classical oscillatory systems. In particular we confine the study in this paper to the case of an oscillator in a random potential $V(x,t)$,

$$V(x,t) = \tfrac{1}{2}\epsilon\,\omega^2(t)x^2 + \tfrac{1}{4}bx^4, \quad \epsilon = \pm 1, \quad b > 0, \tag{1}$$

with $\omega^2(t)$ denoting a random frequency parameter with fluctuation $\zeta(t)$,

$$\zeta(t) = \omega^2(t) - \omega_0^2. \tag{2}$$

The case with $\epsilon = +1$ refers to the (random) Duffing oscillator and $\epsilon = -1$ to the double-well potential which is of importance for the model description of instabilities. In presence of thermal white gaussian noise fluctuations $f(t)$ the dynamics of the oscillator (mass $m = 1$) is described by the set of Langevin equations for displacement x and velocity v

$$\dot{x} = v, \quad \dot{v} = -\epsilon\,\omega_0^2 x - bx^3 - \gamma v - \epsilon\zeta(t)x + f(t), \tag{3a,b}$$

where the damping constant γ (viscosity) accounts for the coupling to other internal degrees of freedom. The thermal white gaussian noise $f(t)$ will be assumed to obey the thermal fluctuation–dissipation theorem $\langle f(t)\,f(s)\rangle = 2kT\gamma\delta(t-s)$ (T: temperature). Moreover, the parameter noise $\zeta(t)$, emerging from external random perturbations, will be assumed to be of δ-correlated nature. Under these assumptions the pair $(x(t), v(t))$ will describe a Markov process. As a specific example we choose for the δ-correlated noise $\zeta(t)$ white Poisson impulses [5] with Poisson parameter λ and jump length S, i.e.

$$\zeta(t) = -\lambda S + \sum_i S\delta(t-t_i), \tag{4a}$$

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with cumulant averages $\langle\ \rangle_c$

$$\langle\zeta(t_1)\dots\zeta(t_n)\rangle_c = \lambda S^n\delta(t_1-t_2)\cdots\delta(t_{n-1}-t_n), \quad n\geqslant 2. \tag{4b}$$

The master equation for this Markov process in eq. (3) is obtained by use of the method of functional calculus developed in ref. [5]. The crucial part in the derivation of the master equation for the probability $p(x,v,t)$ is given by the term $\zeta(t)x$. If we note that $\dot{x}(t)$ is finite, the displacement $x(t)$ can be assumed to be constant during the time of a Poisson impulse [$\mathrm{d}x$-effects are of order $O(\mathrm{d}t^2)$] giving for the velocity displacement in virtue of eqs. (3) and (4a)

$$v_f = v_i - \epsilon Sx. \tag{5}$$

Now, recalling that the probability for one impulse in $\mathrm{d}t$ is $\lambda\mathrm{d}t$ and that a particle in $(v, v+\mathrm{d}v)$ after an impulse was in $(v+\epsilon Sx, v+\epsilon Sx+\mathrm{d}v)$ *before* we ultimately obtain for the rate of change of $p(x,v,t)$ the *(integro-differential) master equation*

$$\begin{aligned}\dot{p}(x,v,t) =& -\frac{\partial}{\partial x}[vp(x,v,t)] + \gamma\frac{\partial}{\partial v}[vp(x,v,t)] + bx^3\frac{\partial}{\partial v}p(x,v,t) + \epsilon\omega_0^2 x\frac{\partial}{\partial v}p(x,v,t)\\ &- \epsilon\lambda Sx\frac{\partial}{\partial v}p(x,v,t) + kT\gamma\frac{\partial^2}{\partial v^2}p(x,v,t) - \lambda p(x,v,t) + \lambda p(x,v+\epsilon Sx,t).\end{aligned} \tag{6}$$

Here it is worth emphasizing the following: The fact that $x(t)$ can be assumed to be constant during a Poisson impulse implies that in eq. (6) there is *no* fluctuation induced drift due to the multiplicative noise in eq. (3b). In more technical terms, the Ito as well as the Stratonovitch interpretation of eq. (3) yield the identical result in (6). Note also that the above reasoning immediately allows a generalization to the case with an additional fluctuating parameter $b\to b(t)$.

The master equation in (6) contains the total dynamical information of the nonlinear system. For example, we obtain for the relaxation of the energy $\frac{1}{2}\langle v^2(t)\rangle$ after integration over x and v

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle v^2(t)\rangle = \langle v(t)x(t)\rangle\,\omega_0^2\epsilon - b\langle v(t)x^3(t)\rangle + kT\gamma - \gamma\langle v^2(t)\rangle + \tfrac{1}{2}\lambda S^2\langle x^2(t)\rangle. \tag{7}$$

In accordance with the definition given in ref. [3] we call the nonlinear oscillator stable in the second mean if $(\mathrm{d}/\mathrm{d}t)\langle v^2(t)\rangle$ and $(\mathrm{d}/\mathrm{d}t)\langle x^2(t)\rangle$ tend to zero as t approaches infinity. Clearly, in the linear case ($b=0$) the system of moment equations forms a closed set which has already been studied in ref. [3]. For example, we have with $b=0$ for the stationary moment $\langle v^2\rangle_s$

$$\langle v^2\rangle_s = \omega_0^2\langle x^2\rangle_s = \omega_0^2\,kT\gamma/(\gamma\omega_0^2 - \tfrac{1}{2}\lambda S^2). \tag{8}$$

From eq. (8) we can read off the stability criteria (linear case) $\lambda S^2 < 2\gamma\omega_0^2$.

Next we look at the dynamics in the limit of a large damping constant γ. Then an adiabatic elimination of the velocity variable yields a reduced Langevin equation with multiplicative white Poisson noise

$$\dot{x} = -\gamma^{-1}[\epsilon\omega_0^2 x + bx^3 - f(t) + \epsilon x\zeta(t)]. \tag{9}$$

Following Hasegawa's reasoning with respect to the adiabatic elimination scheme [6] and interpreting the Langevin equation in (9) in the Stratonovitch sense we obtain, in case of fluctuating parameters, the *generalized Smoluchowski master equation*

$$\begin{aligned}\dot{p}(x,t) =& \gamma^{-1}\frac{\partial}{\partial x}\{[\epsilon\omega_0^2 x + bx^3 - \epsilon\lambda Sx]p(x,t)\} + \frac{kT}{\gamma}\frac{\partial^2}{\partial x^2}p(x,t) - \lambda p(x,t)\\ &+ \lambda\exp(\gamma^{-1}\epsilon S)\,p(x\exp[\gamma^{-1}\epsilon S],t),\end{aligned} \tag{10}$$

which for $\lambda = 0$ reduces to the familiar Smoluchowski equation. By expanding up to order $O(S^3)$ we obtain from eq. (10) in terms of the scaling

$$\lambda S^2 = 2kT\gamma a, \quad a > 0 \tag{11}$$

the approximate Fokker–Planck equation

$$\dot{p}(x,t) = \gamma^{-1} \frac{\partial}{\partial x}\{[\epsilon\omega_0^2 x + bx^3 - kTax]\, p(x,t)\} + \frac{kT}{\gamma}\frac{\partial^2}{\partial x^2}[(1+ax^2)p(x,t)], \tag{12}$$

with a stationary (non-equilibrium) probability $p_s(x)$ (N: normalization)

$$p_s(x) = N\frac{1}{1+ax^2}\exp\left[\frac{-(b/a)x^2 + (kT + b/a^2 - \epsilon\omega_0^2/a)\ln|1+ax^2|}{2kT}\right]. \tag{13}$$

For $a = 0$, $p_s(x)$ reduces to the familiar form

$$p_s(x) \propto \exp[-V(x)/kT]. \tag{14}$$

We note that in the diffusion limit $\lambda \to \infty$, $S \to 0$, but λS^2 finite eq. (12) becomes an *exact* Fokker–Planck equation which coincides with the equation obtained after an adiabatic elimination of the corresponding full Fokker–Planck equation in (6). Further, from the expression for the mean first passage time $T(x)$ [7]

$$T(x) = \gamma\int_{-x}^{x} \mathrm{d}y\, \frac{[\int_{-x}^{y} p_s(z)\,\mathrm{d}z]^2}{kT(1+ay^2)p_s(y)} \sim \gamma\int_{-x}^{x}[kT(1+ay^2)p_s(y)]^{-1}\,\mathrm{d}y, \tag{15}$$

we have

$$T(x) \sim \frac{\gamma}{kTN}\int_{-x}^{x}\exp\left[\frac{V(y) - aO(a,y^2)}{kT}\right]\cdot\exp[-\tfrac{1}{2}\ln(1+ay^2)]\,\mathrm{d}y. \tag{16}$$

We see that the expression for Kramers' reaction rate [8] $R(x) \sim 1/T(x)$ is quite different from the one which is obtained in absence of fluctuating parameters ($a = 0$). From a physical point of view, the additional multiplicative noise in eq. (9) gives rise to a decrease for the mean first passage time. In other words, the second temperature-independent (multiplicative) term in eq. (16) corresponds to a "tunneling" contribution to the reaction rate.

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