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Transformation invariance of Lyapunov exponents

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Abstract

Lyapunov exponents represent important quantities to characterize the properties of dynamical systems. We show that the Lyapunov exponents of two different dynamical systems that can be converted to each other by a transformation of variables are identical. Moreover, we derive sufficient conditions on the transformation for this invariance property to hold. In particular, it turns out that the transformation need not necessarily be globally invertible.

Ordinary differential equations constitute a widely used tool to describe the dynamical behavior of physical, biological, chemical and many other systems. Moreover, since Lorenz's [1] discovery of deterministic non-periodic flow, time-continuous dynamical systems play an important role in the exploration of chaotic phenomena [2–5]. In the modern theory of dynamical systems, their properties are mainly analyzed in a qualitative way in terms of their flow in phase space. A first simplification of such analysis might be achieved by transforming the investigated dynamical system to a system with a functional simpler or more convenient form. As a specific example, we mention recent advances [6,7] in the theory of three-dimensional dynamical systems with quadratic non-linearities where coordinate transformations to the so-called jerky dynamics allow for a classification based on functional simplicity of the resulting third-order differential equations. The new dynamical system, however, should have the same dynamical properties as the original one, i.e., the character of the long-time dynamics (fixed point, limit cycle, strange attractor etc.) should not be changed. This leads to the question about the invariance properties of such quantities that can be used to characterize different dynamical long-time behavior, such as dimensions of attractors or Lyapunov exponents.

The concept of the dimension of an attractor is based on its *metric properties*, leading, e.g., to the Hausdorff dimension, or on its *invariant measure*, yielding, e.g., the information dimension [8]. With this concept a simple classification of attractors is possible. For instance, a fixed point has dimension zero, a stable limit cycle has dimension one, a 2-torus has dimension two, while the dimensions of strange attractors being a signature of dissipative, deterministic chaos take on values that are typically non-integer. In Ref. [9], it is shown that the Hausdorff dimension as well as the information dimension are invariant under a wide class of invertible coordinate transformations.

Lyapunov exponents are defined using the *dynamical* long-time properties of the trajectories on an attractor [10]. The type of the attractor is uniquely characterized by its Lyapunov spectrum, i.e., the signs of the corresponding Lyapunov exponents [2]. For the phase space dimension three, e.g., an attracting fixed point possesses the Lyapunov spectrum $\{-,-,-\}$, an attracting limit-cycle $\{0,-,-\}$, an attracting 2-torus

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$\{0, 0, -\}$ and a strange attractor $\{+, 0, -\}$. Besides this, Lyapunov exponents are also of fundamental interest, because the most common definiton of chaos in physics is based on them: a dynamical system is chaotic if its attractor possesses at least one positive Lyapunov exponent.

Moreover, the Lyapunov exponents can also be used to define a dimension-like quantity, the Lyapunov dimension [8,11]. For typical attractors, the Kaplan–Yorke *conjecture* states that this dimension is equal to the information dimension [8,11,12]. This is proven only for the special case of two-dimensional invertible maps [13], though there are heuristic and numerical evidences for its general validity [8,12,14]. Since the information dimension is invariant under coordinate changes [9], one can expect that this is also true for the Lyapunov dimension or even the Lyapunov exponents themselves.

In this paper, we give a detailed and transparent demonstration only based on elementary differential calculus that this assumption is in fact true. The Lyapunov exponents of dynamical systems that are described by ordinary differential equations are invariant under a wide class of transformation of variables. As a consequence, also the Lyapunov dimension is invariant, and the above definition of chaos is independent of the coordinates used. This may not be surprising and, moreover, seems to be a widely accepted fact. Otherwise, the Lyapunov exponents would surely not have become such an important concept in the theory of dynamical systems. Nevertheless, to our knowledge there exists no proof of this fact in the literature. For maps, a corresponding proof is presented by Metzler [15].

As starting point we consider the n-dimensional autonomous dynamical system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{1}$$

where $\mathbf{x}$ denotes a point in an n-dimensional phase space $\Gamma^x \subseteq \mathbb{R}^n$, $\mathbf{F}(\mathbf{x})$ is an n-dimensional continuously differentiable vector field and the overdot denotes the derivative with respect to time t. The solution of this dynamical system for a fixed initial value $\mathbf{x}_0 \in \Gamma^x$ is given by the trajectory $\varphi(t; \mathbf{x}_0)$. Then, $\varphi(t = 0; \mathbf{x}_0) = \mathbf{x}_0$ holds and $\varphi(t; \mathbf{x}_0)$ fulfills the equation

$$\dot{\varphi}(t; \mathbf{x}_0) = \mathbf{F}[\varphi(t; \mathbf{x}_0)] \tag{2}$$

for all $t \geqslant 0$. To define the Lyapunov exponents for the dynamical system (1), we consider a reference trajectory $\varphi_\mathrm{r} = \varphi(t; \mathbf{x}_\mathrm{r})$ with initial value $\mathbf{x}_\mathrm{r}$ and a nearby trajectory $\varphi(t; \mathbf{x}_\mathrm{r} + \widetilde{\mathbf{x}}_0)$ which belongs to a small initial deviation $\widetilde{\mathbf{x}}_0$ from $\mathbf{x}_\mathrm{r}$. The dynamical behavior of such nearby trajectories can be described approximately by the linearization of Eq. (1) with respect to the reference trajectory φ_r, i.e., by the linear system of differential equations

$$\dot{\widetilde{\mathbf{x}}} = \mathrm{D}_\mathbf{x}\mathbf{F}(\varphi_\mathrm{r})\widetilde{\mathbf{x}} \tag{3}$$

with a time-dependent $n \times n$ coefficient matrix $(\mathrm{D}_\mathbf{x}\mathbf{F})_{ij}(\varphi_\mathrm{r}) = \partial_{x_j} F_i(\varphi_\mathrm{r}) = (\partial F_i/\partial x_j)(\varphi_\mathrm{r})$ $(i, j = 1, 2, \ldots, n)$. We denote the solution of Eq. (3) for the initial value $\widetilde{\mathbf{x}}_0 \in \mathbb{R}^n$ by $\widetilde{\varphi}_{\mathbf{x}_\mathrm{r}}(t; \widetilde{\mathbf{x}}_0)$. The subscript $\mathbf{x}_\mathrm{r}$ shall elucidate the dependence of this solution on the reference trajectory φ_r which is determined by its initial value $\mathbf{x}_\mathrm{r}$.

The Lyapunov exponents with respect to a reference trajectory φ_r that is supposed to be *bounded* for all $t \geqslant 0$ are defined as [10]

$$\lambda_{\mathbf{x}_\mathrm{r}}(\widetilde{\mathbf{x}}_0) = \lim_{t \to \infty} \frac{1}{t} \ln \|\widetilde{\varphi}_{\mathbf{x}_\mathrm{r}}(t; \widetilde{\mathbf{x}}_0)\|, \tag{4}$$

where we assume that the limit exists. Therefore, they constitute a measure for the mean exponential divergence or convergence of nearby trajectories. With the above definition, the following statements originally derived by Oseledec hold [2,10,16]:

(i) There are n Lyapunov exponents $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, i.e., there are s $(1 \leqslant s \leqslant n)$ different Lyapunov exponents $\lambda^{(k)}$ with multiplicity n_k and with $\lambda^{(1)} > \lambda^{(2)} > \cdots > \lambda^{(s)}$. For any $\lambda^{(k)}$, there is a linear vector space $U^{(k)}_{\mathbf{x}_\mathrm{r}} = \{\widetilde{\mathbf{x}}_0 \in \mathbb{R}^n : \lambda_{\mathbf{x}_\mathrm{r}}(\widetilde{\mathbf{x}}_0) \leqslant \lambda^{(k)}\}$ with $\mathbb{R}^n = U^{(1)}_{\mathbf{x}_\mathrm{r}} \supset U^{(2)}_{\mathbf{x}_\mathrm{r}} \supset \cdots \supset U^{(s)}_{\mathbf{x}_\mathrm{r}}$.

(ii) Which Lyapunov exponent $\lambda^{(k)}$ will result from Eq. (4) depends on the choice of the initial perturbation $\widetilde{\mathbf{x}}_0$ in the following manner: $\lambda_{\mathbf{x}_\mathrm{r}}(\widetilde{\mathbf{x}}_0) = \lambda^{(k)}$ if $\widetilde{\mathbf{x}}_0 \in U^{(k)}_{\mathbf{x}_\mathrm{r}} \setminus U^{(k+1)}_{\mathbf{x}_\mathrm{r}}$ (with $U^{(s+1)}_{\mathbf{x}_\mathrm{r}} = \emptyset$). In particular, for all vectors $\widetilde{\mathbf{x}}_0$ that are not in the subspace $U^{(2)}_{\mathbf{x}_\mathrm{r}}$ limit (4) yields the largest Lyapunov exponent $\lambda^{(1)}$.

(iii) In general, the Lyapunov exponents are identical for almost all reference trajectories that belong to the basin of attraction of a certain attractor.

Next, we specify the transformation of variables whose effect on the Lyapunov exponents of the dynamical system (1) we want to study. Such a transformation $\mathbf{T} : \mathbf{x} \mapsto \mathbf{y}$ with

$$\mathbf{y} = \mathbf{T}(\mathbf{x}) \tag{5}$$

maps the phase space $\Gamma^x \subseteq \mathbb{R}^n$ to a phase space $\Gamma^y \subseteq \mathbb{R}^n$ that is again n-dimensional. We suppose that $\mathbf{T}$ is invertible and possesses an inverse $\mathbf{T}^{-1} : \mathbf{y} \mapsto \mathbf{x}$ with

$$\mathbf{x} = \mathbf{T}^{-1}(\mathbf{y}) \tag{6}$$

and that $\mathbf{T}$ and $\mathbf{T}^{-1}$ are at least twice continuously differentiable with respect to $\mathbf{x}$ and $\mathbf{y}$, respectively. Applying the transformation $\mathbf{T}$ to Eq. (1) leads to the new dynamical system

$$\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y}), \tag{7}$$

where the transformed vector field $\mathbf{G}(\mathbf{y})$ is given by

$$\mathbf{G}(\mathbf{y}) = \mathrm{D}_{\mathbf{x}}\mathbf{T}[\mathbf{T}^{-1}(\mathbf{y})]\mathbf{F}[\mathbf{T}^{-1}(\mathbf{y})]. \tag{8}$$

This follows immediately from the differentiation of Eq. (5) with respect to time t. For a solution of system (7) that belongs to an initial value $\mathbf{y}_0$, we write $\psi(t;\mathbf{y}_0)$. The relation between the trajectories of the original and the transformed dynamical systems (1) and (7) is given by

$$\varphi(t;\mathbf{x}_0) = \mathbf{T}^{-1}\{\psi[t;\mathbf{T}(\mathbf{x}_0)]\} \tag{9}$$

or, equivalently,

$$\mathbf{T}[\varphi(t;\mathbf{x}_0)] = \psi[t;\mathbf{T}(\mathbf{x}_0)]. \tag{10}$$

This can be verified in the following way. Differentiating Eq. (10) with respect to time **t** and using (2) and (9), one obtains an equation for $\dot{\psi}[t;\mathbf{T}(\mathbf{x}_0)]$ that is consistent with $\mathbf{G}\{\psi[t;\mathbf{T}(\mathbf{x}_0)]\}$ from (8).

For the definition of its Lyapunov exponents, the dynamical system (7) has to be linearized with respect to a reference trajectory $\psi_\mathrm{r} = \psi(t;\mathbf{y}_\mathrm{r})$. This results in the linear system of differential equations

$$\dot{\tilde{\mathbf{y}}} = \mathrm{D}_{\mathbf{y}}\mathbf{G}(\psi_\mathrm{r})\tilde{\mathbf{y}}, \tag{11}$$

where $(\mathrm{D}_{\mathbf{y}}\mathbf{G})_{ij}(\psi_\mathrm{r}) = \partial_{y_j}G_i(\psi_\mathrm{r}) = (\partial G_i/\partial y_j)(\psi_\mathrm{r})$ $(i,j = 1,2,\ldots,n)$. With the solution $\tilde{\psi}_{\mathbf{y}_\mathrm{r}}(t;\tilde{\mathbf{y}}_0)$ of this linear differential equation (for an initial value $\tilde{\mathbf{y}}_0$), the limit

$$\mu_{\mathbf{y}_\mathrm{r}}(\tilde{\mathbf{y}}_0) = \lim_{t\to\infty} \frac{1}{t} \ln \|\tilde{\psi}_{\mathbf{y}_\mathrm{r}}(t;\tilde{\mathbf{y}}_0)\| \tag{12}$$

defines the Lyapunov exponents of the transformed dynamical system (7). Clearly, from this definition analogous consequences result as from Eq. (4). In particular, which of the different Lyapunov exponents are obtained from limit (12) depends on the choice of the initial perturbation $\tilde{\mathbf{y}}_0$.

After describing the considered situation in detail, giving the relevant definitions and fixing the notation, we are now in the situation to prove that the Lyapunov exponents of the original dynamical system (1) and of its transformed counterpart (7) agree. Clearly, this equality can be valid only for Lyapunov exponents of such reference trajectories of (1) and (7) that can be mapped to each other. We consider an arbitrary bounded reference solution $\varphi_\mathrm{r} = \varphi(t;\mathbf{x}_\mathrm{r})$ of system (1) and, moreover, suppose that $\mathbf{y}_\mathrm{r} = \mathbf{T}(\mathbf{x}_\mathrm{r})$ holds. It follows that $\psi_\mathrm{r} = \psi(t;\mathbf{y}_\mathrm{r}) = \mathbf{T}[\varphi(t;\mathbf{x}_\mathrm{r})]$ or, inverted and in shortened form

$$\varphi_\mathrm{r} = \mathbf{T}^{-1}(\psi_\mathrm{r}) \tag{13}$$

are valid. As discussed above, the initial deviations $\tilde{\mathbf{x}}_0$ or $\tilde{\mathbf{y}}_0$, respectively, determine which Lyapunov exponent of the sets of all different Lyapunov exponents results from Eqs. (4) or (12). Therefore, the invariance of *all* of them is proven, if one can show that $\lambda_{\mathbf{x}_\mathrm{r}}(\tilde{\mathbf{x}}_0) = \mu_{\mathbf{T}(\mathbf{x}_\mathrm{r})}(\tilde{\mathbf{y}}_0)$ is valid for an appropriate transformed initial perturbation $\tilde{\mathbf{y}}_0$. To perform this proof, we subdivide it into four steps:

(i) It must be guaranteed that the boundedness of φ_r implies that also $\psi_r = \mathbf{T}(\varphi_r)$ is bounded for all $t \geqslant 0$. This, however, is an immediate consequence of the invertibility and differentiability of the transformation $\mathbf{T}$.

(ii) Since the definitions (4) and (12) are based on the solutions $\widetilde{\varphi}_{\mathbf{x}_r}(t;\widetilde{\mathbf{x}}_0)$ and $\widetilde{\psi}_{\mathbf{y}_r}(t;\widetilde{\mathbf{y}}_0)$ of the linearized systems (3) and (11), respectively, we have to investigate the relations between these two solutions. Therefore, we search for a transformation $\mathbf{L}:\widetilde{\mathbf{x}} \mapsto \widetilde{\mathbf{y}}$ with $\widetilde{\mathbf{y}} = \mathbf{L}(\widetilde{\mathbf{x}})$ that converts the solutions of system (3) to the solutions of (11), i.e.,

$$\mathbf{L}[\widetilde{\varphi}_{\mathbf{x}_r}(t;\widetilde{\mathbf{x}}_0)] = \widetilde{\psi}_{\mathbf{T}(\mathbf{x}_r)}[t;\mathbf{L}(\widetilde{\mathbf{x}}_0)]. \tag{14}$$

To determine the transformation $\mathbf{L}$, we first derive an equation for $\mathbf{L}$ and then use this equation to find $\mathbf{L}$.

Since both systems are linear, $\mathbf{L}$ is also linear and, therefore, can be written in the form

$$\widetilde{\mathbf{y}} = \mathsf{L}\widetilde{\mathbf{x}} \tag{15}$$

with a non-trivial $n \times n$ matrix L. Taking the derivative of this relation with respect to time and using Eq. (3), one obtains $\dot{\widetilde{\mathbf{y}}} = \dot{\mathsf{L}}\widetilde{\mathbf{x}} + \mathsf{L}\mathrm{D}_{\mathbf{x}}\mathbf{F}(\varphi_r)\widetilde{\mathbf{x}}$. However, $\dot{\widetilde{\mathbf{y}}}$ is determined by differential equation (11), where, again, $\widetilde{\mathbf{y}}$ is given by Eq. (15). Collecting all this and taking into account that the resulting equation is valid for arbitrary $\widetilde{\mathbf{x}}$, one finally obtains

$$\dot{\mathsf{L}} = \mathrm{D}_{\mathbf{y}}\mathbf{G}(\psi_r)\mathsf{L} - \mathsf{L}\mathrm{D}_{\mathbf{x}}\mathbf{F}(\varphi_r). \tag{16}$$

This differential equation constitutes an equation that determines the matrix L. However, we will not evaluate L by integrating (16), since we do not know the initial value $\mathsf{L}(t=0)$ at all. We will, however, derive a solution of (16) by calculating $\mathrm{D}_{\mathbf{y}}\mathbf{G}(\psi_r)$ using Eq. (8). So, we consider

$$\mathrm{D}_{\mathbf{y}}\mathbf{G}(\mathbf{y})\Big|_{\psi_r} = \mathrm{D}_{\mathbf{y}}\{\mathrm{D}_{\mathbf{x}}\mathbf{T}[\mathbf{T}^{-1}(\mathbf{y})]\mathbf{F}[\mathbf{T}^{-1}(\mathbf{y})]\}\big|_{\psi_r}, \tag{17}$$

where the notation $|_{\psi_r}$ explicitly elucidates that the above expressions have to be evaluated at $\mathbf{y} = \psi_r$ *after* taking the derivative. Since $\mathbf{G}$ depends on $\mathbf{y}$ only via $\mathbf{T}^{-1}(\mathbf{y})$, we can write $\mathbf{G}(\mathbf{y}) = \overline{\mathbf{G}}[\mathbf{T}^{-1}(\mathbf{y})]$ with

$$\overline{\mathbf{G}}(\mathbf{x}) = \mathbf{D}_{\mathbf{x}}\mathbf{T}(\mathbf{x})\mathbf{F}(\mathbf{x}). \tag{18}$$

Therefore, for the derivative of $\mathbf{G}$, we obtain

$$\mathrm{D}_{\mathbf{y}}\mathbf{G}(\mathbf{y})\big|_{\psi_r} = \mathrm{D}_{\mathbf{y}}\overline{\mathbf{G}}[\mathbf{T}^{-1}(\mathbf{y})]\Big|_{\psi_r} = \mathbf{D}_{\mathbf{x}}\overline{\mathbf{G}}[\mathbf{T}^{-1}(\mathbf{y})]\mathbf{D}_{\mathbf{y}}\mathbf{T}^{-1}(\mathbf{y})\big|_{\psi_r} = \mathbf{D}_{\mathbf{x}}\overline{\mathbf{G}}[\mathbf{T}^{-1}(\mathbf{y})](\mathbf{D}_{\mathbf{x}}\mathbf{T})^{-1}[\mathbf{T}^{-1}(\mathbf{y})]\Big|_{\psi_r}. \tag{19}$$

For the last implication, we have used the identity $\mathrm{D}_{\mathbf{y}}\mathbf{T}^{-1}(\mathbf{y}) = (\mathrm{D}_{\mathbf{x}}\mathbf{T})^{-1}[\mathbf{T}^{-1}(\mathbf{y})]$. This follows from differentiating $\mathbf{T}[\mathbf{T}^{-1}(\mathbf{y})] = \mathbf{y}$ with respect to $\mathbf{y}$, provided that $\det[\mathrm{D}_{\mathbf{x}}\mathbf{T}(\mathbf{x})] \neq 0$ holds for all $\mathbf{x} \in \Gamma^x$ and, therefore, $\mathrm{D}_{\mathbf{x}}\mathbf{T}(\mathbf{x})$ is invertible. Eq. (19) can also be written as

$$\mathrm{D}_{\mathbf{y}}\mathbf{G}(\mathbf{y})\big|_{\psi_r} = \mathrm{D}_{\mathbf{x}}\overline{\mathbf{G}}(\mathbf{x})(\mathbf{D}_{\mathbf{x}}\mathbf{T})^{-1}(\mathbf{x})\Big|_{\varphi_r}, \tag{20}$$

if one takes into account relation (13), $\varphi_r = \mathbf{T}^{-1}(\psi_r)$. The derivative of $\overline{\mathbf{G}}(\mathbf{x})$ is obtained from Eq. (18),

$$\mathrm{D}_{\mathbf{x}}\overline{\mathbf{G}}(\mathbf{x}) = \mathbf{D}_{\mathbf{x}}[\overset{\downarrow}{\mathbf{D}}_{\mathbf{x}}\mathbf{T}(\mathbf{x})\mathbf{F}(\mathbf{x})] + \mathbf{D}_{\mathbf{x}}\mathbf{T}(\mathbf{x})\mathbf{D}_{\mathbf{x}}\mathbf{F}(\mathbf{x}). \tag{21}$$

Here, the arrow in the first summand indicates that the outer derivative is only applied to this labeled term and not to $\mathbf{F}$. To simplify this first summand, we study it component-wise by writing

$$(\mathrm{D}_{\mathbf{x}}[\overset{\downarrow}{\mathrm{D}}_{\mathbf{x}}\mathbf{T}(\mathbf{x})\mathbf{F}(\mathbf{x})])_{ij} = \partial_{x_j}[\overset{\downarrow}{\mathrm{D}}_{\mathbf{x}}\mathbf{T}(\mathbf{x})\mathbf{F}(\mathbf{x})]_i = [\partial_{x_j}(\mathrm{D}_{\mathbf{x}}\mathbf{T})_{ik}(\mathbf{x})]F_k(\mathbf{x}) = F_k(\mathbf{x})\partial_{x_j}\partial_{x_k}T_i(\mathbf{x}), \tag{22}$$

where the summation convention has been used. Since $\mathbf{T}$ is supposed to be twice continuously differentiable, the order of the derivatives can be changed. Then, $\partial_{x_j}T_i(\mathbf{x})$ can be rewritten as $(\mathrm{D}_{\mathbf{x}}\mathbf{T})_{ij}(\mathbf{x})$ and, using $F_k(\mathbf{x}) = \dot{x}_k$, the sum $F_k(\mathbf{x})\partial_{x_k}$ corresponds to a total time derivative. Altogether, we have $F_k(\mathbf{x})\partial_{x_j}\partial_{x_k}T_i(\mathbf{x}) = (\mathrm{d}/\mathrm{d}t)(\mathrm{D}_{\mathbf{x}}\mathbf{T})_{ij}(\mathbf{x})$ and, therefore,

$$D_{\mathbf{x}}[D_{\mathbf{x}}^{\downarrow}\mathbf{T}(\mathbf{x})\mathbf{F}(\mathbf{x})] = \frac{d}{dt}D_{\mathbf{x}}\mathbf{T}(\mathbf{x}). \tag{23}$$

Now, combining Eqs. (20), (21) and (23) and, then, multiplying the result by $D_{\mathbf{x}}\mathbf{T}(\varphi_r)$, one obtains

$$D_{\mathbf{y}}\mathbf{G}(\psi_r)D_{\mathbf{x}}\mathbf{T}(\varphi_r) = \frac{d}{dt}D_{\mathbf{x}}\mathbf{T}(\varphi_r) + D_{\mathbf{x}}\mathbf{T}(\varphi_r)D_{\mathbf{x}}\mathbf{F}(\varphi_r). \tag{24}$$

Comparing this result with Eq. (16), we finally find the central relation

$$\mathsf{L} = D_{\mathbf{x}}\mathbf{T}(\varphi_r). \tag{25}$$

Therefore, the transformation $\mathbf{L}$ between the linearized systems (3) and (11) is given by the linearized transformation $\mathbf{T}$ taken at the reference trajectory $\varphi_r = \varphi(t;\mathbf{x}_0)$. In particular, the relation between the initial perturbations $\tilde{\mathbf{x}}_0$ and $\tilde{\mathbf{y}}_0$ reads $\tilde{\mathbf{y}}_0 = \mathsf{L}(t=0)\tilde{\mathbf{x}}_0 = D_{\mathbf{x}}\mathbf{T}(\mathbf{x}_r)\tilde{\mathbf{x}}_0$.

(iii) Based on Eq. (25) we now derive properties of $\mathbf{L}$ which are needed to show the invariance of the Lyapunov exponents. Since the reference trajectory φ_r is supposed to be bounded and the derivative $D_{\mathbf{x}}\mathbf{T}(\mathbf{x})$ is finite for finite $\mathbf{x}$, a first consequence of Eq. (25) is the boundedness of L for all $t \geqslant 0$. Therefore, there exists a *time-independent* constant $L^+ < \infty$ such that

$$\|\mathbf{L}(\tilde{\mathbf{x}})\| = \|\mathsf{L}\tilde{\mathbf{x}}\| \leqslant L^+\|\tilde{\mathbf{x}}\| \tag{26}$$

holds for all $\tilde{\mathbf{x}} \in \mathbb{R}^n$. Similarly, taking into account that $\det\mathsf{L} = \det D_{\mathbf{x}}\mathbf{T}(\varphi_r) \neq 0$ is valid for all $t \geqslant 0$, a corresponding estimation can also be derived for the inverse L^{-1}. Rewritten as an estimation for L, it reads

$$\|\mathbf{L}(\tilde{\mathbf{x}})\| = \|\mathsf{L}\tilde{\mathbf{x}}\| \geqslant L^-\|\tilde{\mathbf{x}}\| \tag{27}$$

with a time-independent constant $L^- > 0$. Eq. (27) is valid for all $\tilde{\mathbf{x}} \in \mathbb{R}^n$.

(iv) With Eqs. (26) and (27), the proof that $\lambda_{\mathbf{x}_r}(\tilde{\mathbf{x}}_0)$ and $\mu_{\mathbf{T}(\mathbf{x}_r)}[\mathbf{L}(\tilde{\mathbf{x}}_0)]$ are identical is now straightforward. From the definitions of the Lyapunov exponents (4) and (12) follows:

$$\mu_{\mathbf{T}(\mathbf{x}_r)}[\mathbf{L}(\tilde{\mathbf{x}}_0)] - \lambda_{\mathbf{x}_r}(\tilde{\mathbf{x}}_0) = \lim_{t\to\infty}\frac{1}{t}\ln\frac{\|\mathbf{L}[\tilde{\varphi}_{\mathbf{x}_r}(t;\tilde{\mathbf{x}}_0)]\|}{\|\tilde{\varphi}_{\mathbf{x}_r}(t;\tilde{\mathbf{x}}_0)\|}, \tag{28}$$

where also the relation $\tilde{\psi}_{\mathbf{T}(\mathbf{x}_r)}[t;\mathbf{L}(\tilde{\mathbf{x}}_0)] = \mathbf{L}[\tilde{\varphi}_{\mathbf{x}_r}(t;\tilde{\mathbf{x}}_0)]$ [cf. Eq. (14)] has been used and the two appearing logarithms have been combined. With Eq. (26) one further obtains $\mu_{\mathbf{T}(\mathbf{x}_r)}[\mathbf{L}(\tilde{\mathbf{x}}_0)] - \lambda_{\mathbf{x}_r}(\tilde{\mathbf{x}}_0) \leqslant \lim_{t\to\infty}(1/t)\ln L^+$ and, therefore

$$\mu_{\mathbf{T}(\mathbf{x}_r)}[\mathbf{L}(\tilde{\mathbf{x}}_0)] - \lambda_{\mathbf{x}_r}(\tilde{\mathbf{x}}_0) \leqslant 0. \tag{29}$$

Correspondingly, using (27) instead of (26), we find

$$\mu_{\mathbf{T}(\mathbf{x}_r)}[\mathbf{L}(\tilde{\mathbf{x}}_0)] - \lambda_{\mathbf{x}_r}(\tilde{\mathbf{x}}_0) \geqslant 0. \tag{30}$$

Taken together, Eqs. (29) and (30) yield the desired result

$$\mu_{\mathbf{T}(\mathbf{x}_r)}[\mathbf{L}(\tilde{\mathbf{x}}_0)] = \lambda_{\mathbf{x}_r}(\tilde{\mathbf{x}}_0). \tag{31}$$

To obtain this invariance of the Lyapunov exponents under coordinate changes, we had to make several assumptions on the transformation $\mathbf{T}$ that describes the change of coordinates: $\mathbf{T}$ has to be invertible, at least twice continuously differentiable, and the determinant of its Jacobian $D_{\mathbf{x}}\mathbf{T}$ must not vanish anywhere in the phase space.

These requirements on $\mathbf{T}$, however, can be weakened by an alternative derivation of Eq. (25) that is based on a theorem about the solutions of linear systems like (3) or (11). These systems are obtained by linearizing the original dynamical system along a reference solution. Now, the theorem states [17] that the solution of such a linear system for an initial value that is the normalized vector in the *i*th direction of the underlying (euclidean) space, is given by the derivative of the reference solution of the original dynamical system with respect to the ith component of its initial value. For the dynamical system (1) and its linearized counterpart (3), e.g., this means

$$\frac{\partial\varphi(t;\mathbf{x}_r)}{\partial x_{r,i}} = \widetilde{\varphi}_{\mathbf{x}_r}(t;\widetilde{\mathbf{e}}^i), \tag{32}$$

where $\widetilde{\mathbf{e}}^i$ is a normalized vector in the ith direction of the $\widetilde{\mathbf{x}}$-space. Clearly, for systems (7) and (11) an analogous relation holds.

From this theorem follows the relation

$$\mathrm{D}_{\mathbf{x}}\mathbf{T}(\varphi_r)\mathrm{D}_{\mathbf{x}_r}\varphi(t;\mathbf{x}_r) = \mathrm{D}_{\mathbf{y}_r}\psi[t;\mathbf{T}(\mathbf{x}_r)]\mathrm{D}_{\mathbf{x}}\mathbf{T}(\mathbf{x}_r), \tag{33}$$

which is obtained by setting $\mathbf{x}_0 = \mathbf{x}_r$ in Eq. (10) and differentiating it with respect to $\mathbf{x}_r$, is equivalent to Eq. (25). Therefore, we have derived the explicit form of the linear transformation $\mathbf{L}$ only by using the property (10) of the transformation $\mathbf{T}$. In other words, the Lyapunov exponents of two different dynamical systems (1) and (7), whose solutions are mapped to each other by a transformation $\mathbf{T}$ according to (10), are identical even if $\mathbf{T}$ is *not* invertible (and, moreover, $\mathbf{T}$ needs to be differentiable only once). However, for the validity of the relations (26) and (27) it is necessary that $\|\mathrm{D}_{\mathbf{x}}\mathbf{T}(\mathbf{x})\| < \infty$ and $\det\mathrm{D}_{\mathbf{x}}\mathbf{T}(\mathbf{x}) \neq 0$ hold. The first condition also guarantees that a bounded reference solution φ_r of (1) is transformed to a bounded trajectory ψ_r of Eq. (7). Therefore, for non-invertible transformations $\mathbf{T}$, the invariance of the Lyapunov exponents is only valid for such trajectories that for all $t \geqslant 0$ are confined to those regions in phase space, where these two conditions are fulfilled. Typically, this will be the whole phase space except a final number of isolated points.

Summarizing our results, we have shown that Lyapunov exponents are invariant under invertible transformations of variables. Moreover, the invariance property is not restricted to such coordinate changes, but is also valid for non-invertible transformations that map the trajectories of two different dynamical systems to each other according to Eq. (10). Then, the Lyapunov exponents of these systems are identical for almost all of their trajectories.

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