Directed Current Without Dissipation: Reincarnation of a Maxwell–Loschmidt Demon

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Abstract. We investigate whether for initially localized particles a directed current in rocked periodic structures is possible in *absence* of a dissipative mechanism. With a pure Hamiltonian dynamics the breaking of Time-Reversal-Invariance presents a necessary condition to find nonzero current values. Numerical studies are presented for the classical Hamiltonian dynamical case. These support the fact that indeed a finite current does occur when a time-reversal symmetry-breaking signal, such as a harmonic mixing signal, is acting. To gain analytical insight we consider the coherent driven quantum transport in a one-dimensional tight-binding lattice. Here, a finite coherent current is absent for initially localized preparations; it emerges, however, when the initial preparation (with zero initial current) possesses finite coherence. The presence of phase fluctuations will eventually kill any finite current, thereby rendering the nondissipative currents a transient phenomenon.

1 Introduction

Is it possible to get work out of fluctuations? The answer is a definite "yes". This fact is evident from the daily experience with the functioning of mechanical and electrical rectifiers. A typical realization refers to the self-winding wristwatch that works especially well with gesticulating carriers. It must be stressed, however, that all these examples refer to macroscopic fluctuations. The issue becomes more subtle if microscopic fluctuations of classical or quantum Brownian nature are involved. In presence of dissipation, this area of research has been in the limelight over recent years, and it enjoys an ever increasing activity. It is known under the label of *Brownian motors*, *ratchet devices*, and in a biological context it is referred to as *molecular motors*; for recent reviews see the items [1–4]. The issue we want to address with this communication is the problem whether such directed transport in periodic structures can occur if no (!) dissipation is acting on the system. The obvious answer is again an "of course": A ballistic, nondissipative particle with nonzero initial velocity can traverse a periodic structure at no risk. The objective is more tenuous, however, if we define the generation of fluctuation - induced directed current only for the situation that initially the particles are put into the system with zero velocity.

With this prerequisite, a resulting nonvanishing current is indeed counter-intuitive: It reflects the fact that *directed* current is obtained out of fluctua-

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tions in absence of a dissipative mechanism. (Note that persistent ring currents – yielding magnetization – are compatible with thermal equlibrium.) Such a result may be interpreted as a gain of information that in principle can be thought of being put into use for a reduction of thermodynamic entropy. This in turn brings back to our minds the thought construction of a Maxwell's demon. Such a demon was introduced to a public audience by James Clerk Maxwell in his 1871 book *Theory of Heat* [5] with the goal to "pick a hole in the second law". In summary, this demon refers to a hypothetical being or device of arbitrarily small mass that possesses for all times the complete information on all positions and velocities of the molecules in a vessel which is divided into two parts A and B. These two parts are connected by a small hole. He can *see* the individual molecules. Then, without wasting work, he opens and closes this hole so as to allow only the swifter ones to pass from A to B, and only the slower ones to pass from B to A. In doing so, a pressure is build up without having done work, being in violation with the second law of thermodynamics. This idea actually has also been formulated by a colleague of Ludwig Boltzmann, namely Josef Loschmidt in 1869 [6]. We note that in thermodynamics, high-ordered energy can degrade spontaneously into a disordered form, termed *heat*. The everyday experience shows that the time-reversed process seems not to occur naturally. More technically, the second law of thermodynamics states that the total entropy of a closed system cannot decrease. Irreversible transport then necessarily requires an *arrow of time* – causing dissipation –; i.e. the second law (if such a law actually is applicable away from thermal equilibrium) determines the direction of natural processes in a system.

Directed transport is thus generally thought of as being possible only in presence of irreversibility causing some sort of dissipation. Most sucesseful theories such as the Boltzmann transport theory (Stosszahlansatz), the theory of stochastic processes being reflected in the schemes of Fokker-Planck equations or master equations, or also in the approaches that formally start from the Liouville equation and use a closure procedure and/or introduce a time direction via the choice of initial condition, such as the fully uncorrelated many-particle distribution in Bogoliubov's theory, *all* involve an element of Time-Reversal Non-Invariance. It was also the symmetry of Time-Reversal of the autonomous Liouville equation – causing (on a finite time scale) anti-kinetic behavior by reversing all the velocities – that led Loschmidt [7] to his famous objection to Boltzmann's H-theorem, known as the Loschmidt paradox. This paradox should not be confused with Zermélo's paradox [8]; the latter objection to Boltzmann's kinetic theory is formulated in the form of a (Poincaré) recurrence paradox.

Our prime challenge here is as follows: Is it possible that a deterministic Hamiltonian dynamics alone – with no reference to the concept of dissipation – is able to support a directed current? If the answer is yes, we in essence (on the level of few-degrees Hamiltonian chaos) deal with a re-incarnation of the brainchild created by Maxwell and Loschmidt, namely finite directed transport (carrying information) starting out from zero occurs without dissipation. The fact that this is – in principle allowed – lies hidden in the Loschmidt paradox: What happens when we break Time-Reversal-Symmetry in a purely deterministic Hamiltonian dynamics? Before addressing this situation we first recall the deterministic dissipative case.

2 Periodically Rocked Deterministic Ratchets

2.1 Rocking Ratchets in Presence of Dissipation

By now it is well known that directed transport can occur on a pure dissipative, deterministic level. The deterministically rocked overdamped dynamics in a ratchet potential, i.e. a periodic potential profile that breaks the spatial reflection-symmetry, has been studied in Refs. [9–11]. There exist many macroscopic devices that allow directed transport by use of a periodic rocking force. The situation is typified by the deterministic, overdamped ratchet dynamics

$$\dot{x} = -\frac{d}{dx}U_R(x) + A\cos(\omega t) \tag{1}$$

with the ratchet potential from Ref. [11] given explicitly by

$$U_R(x) = -\frac{1}{2\pi}[\sin(2\pi x) + \frac{1}{4}\sin(4\pi x)] \tag{2}$$

Given the simple one-dimensional non-autonomous first order differential equation the resulting current behavior is rather rich, exhibiting co-existing stable periodic solutions $x(t)$, current quantization phenomena [11] and a devil staircase behavior for the current itself [10, 11]. In particular, in this case of overdamped motion no deterministic current reversal occurs. The latter happens only in presence of noise within the non-adiabatic driving regime [11], note also the contribution by Reimann in this volume. In Figure 1 we depict this complex phase-diagram for the current behavior as a function of driving frequency ω and driving amplitude A.

The influence of finite inertia with a $\ddot{x}$ - contribution occurring in Eq.(1) has been studied in Ref. [13], and recently in Ref. [14]. Now the current behavior is even richer, exhibiting multiple deterministic current reversals, current carrying solutions in the regular regime and chaotic regime as well, an universal Gaussian scaling regime [13], intermittent chaotic behavior and anomalous deterministic diffusion [13, 14]. Note, however, that the dissipative term proportional to $\dot{x}$ breaks Time-Reversal Symmetry. Finite dissipation thus generically yields a *stationary* current in an extended parameter regime of driving strength, driving frequency, and mass value. As noted already in Ref. [13], see footnote [16] therein, this very form of a driven inertial dynamics *in absence of dissipation* is now invariant under Time Reversal Symmetry $t \longrightarrow -t$: as a consequence, the absence of the $\dot{x}$ – *term* allows for zero current throughout the whole parameter regime.

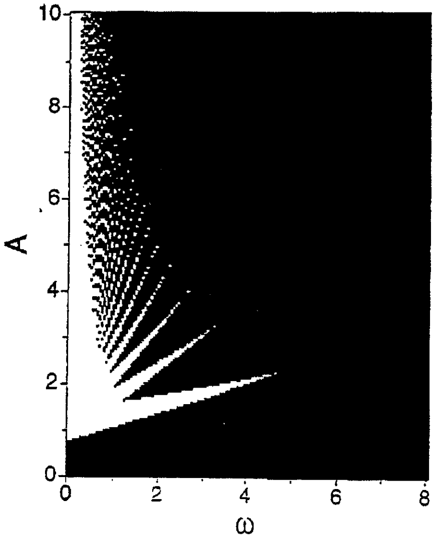

Fig. 1. Phase diagram for finite directed current (white areas) and zero current (black areas) for the overdamped periodically rocked ratchet dynamics in (1), in the potential (2), as a function of dimensionless amplitude strength A and dimensionless driving frequency ω. Note that the onset of current with increasing driving frequency requires an increased amplitude strength over the adiabatic limit $A = 0.75$. This figure has originally been provided by Peter Talkner *et al.* in (1995) [12].

2.2 Directed Transport in Absence of Dissipation

We next take a closer look at an inertial Hamiltonian dynamics in a periodic potential $U(x) = U(x + a)$ which is subject to a time-periodic, on average unbiased forcing $F(t+T) = F(t)$. The time-dependent Hamiltonian dynamics for a particle of mass m in scaled units thus reads

$$m\ddot{x} = -\frac{d}{dx}U_R(x) + F(t) \,. \tag{3}$$

We shall define the mean velocity v of this driven Hamiltonian dynamics by considering (with $\dot{x}(0) = 0$) the limiting procedure

$$v = \lim_{t\to\infty} v(t) = \frac{x(t)}{t} \,. \tag{4}$$

For Eq.(4) to make sense, we implicitly assume here a self-averaging behavior for the current. By merely glancing at Figure 2, which exhibits the tilted potential profile at a maximal forward and backward tilt, one naively conceives that the condition of breaking the reflection symmetry (a ratchet profile) should suffice to induce a directed current for symmetric rocking

$F(t) = E_1 \cos(\omega t)$. As already noted above, however, this inertial dynamics satisfies the Time - Reversal Symmetry, $t \longrightarrow -t + t_0$, with t_0 in general being a nonvanishing constant. In our case of a pure cos - drive this constant is zero. Recently, Flach and collaborators [15] realized that the introduction of a broken time-space symmetry can indeed yield nonzero directed current. In particular, they considered the case of a harmonic mixing drive, see [16], consisting of two harmonics with commensurate frequencies at ω and 2ω, i.e.

$$F(t) = E_1 \cos[\omega t + \psi(t)] + E_2 \cos[2\omega t + 2\psi(t) + \phi], \tag{5}$$

where ϕ denotes a fixed *relative* phase between the harmonics. Flach *et al.* [15] considered the case with $\psi(t)$ set identically to zero.

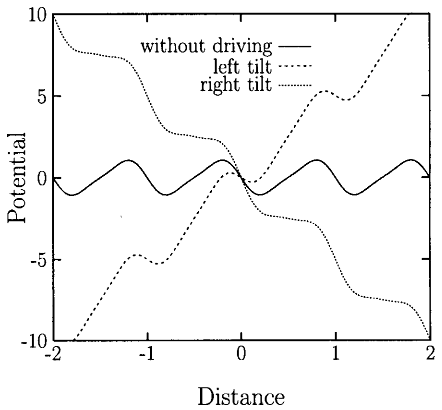

Fig. 2. Sketch of the untilted and (left and right) maximally tilted ratchet potential configurations for the scaled Hamiltonian dynamics $\ddot{x} = -\frac{d}{dx}U_R(x) + 5\cos(t)$ with $U_R(x) = -\sin(2\pi x) - 0.2\sin(4\pi x)$.

For later purposes, we allow in general, however, for realistic *locked fluctuations* of the absolute phases of the two harmonics. These nonzero phase fluctuations do reflect the fact that under realistic conditions the quality of coherent sources is never perfect, so that the case $\psi(t) = 0$ must be considered as a (mathematical) idealization. Nevertheless, we shall defer the impact of such finite phase fluctuations to section 3.2 below and consider first the ideal situation. As noted by Flach and collaborators, this harmonic mixing signal $F(t)$ yields Time-Reversal-Non-Invariance for the Hamiltonian dynamics in (3). Our numerical findings for the time evolution of the position $x(t)$ and the mean velocity, or current, are depicted with Figures 3(a) and 3(b). Care must be involved in integrating the Hamiltonian dynamics so as not to introduce a spurious dissipation induced by the numerical scheme. In doing so, we have employed a *symplectic* integrator [17], which guarantees the conservation of phase volume.

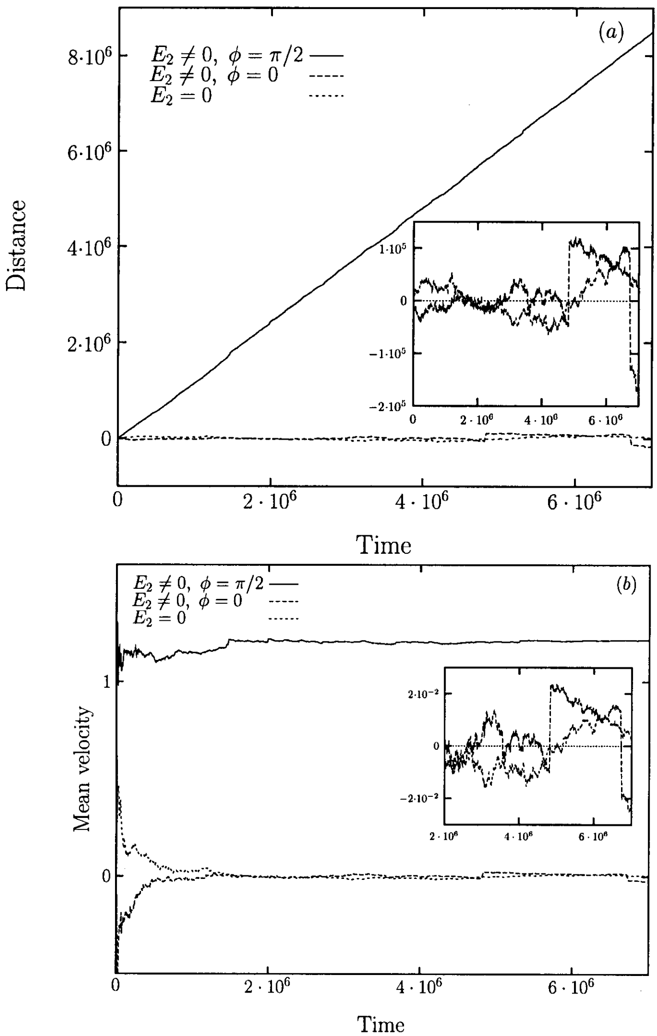

Fig. 3. Position (a) and mean velocity (b) of a particle in the ratchet potential $U_R(x) = -\sin(2\pi x) - 0.2\sin(4\pi x)$ driven by the harmonic mixing force (5) with unit frequency $\omega = 1$. The particle is initially localized at the bottom of the potential well near $x = 0$, and starts out with zero velocity. The strength of the first harmonic in (5) was fixed, $E_1 = 5$, which just corresponds to the situation depicted in Fig. 2. The strength of the second harmonic is chosen either zero, or $E_2 = 2$. Moreover, we use $\psi(t) := 0$, and the relative phase ϕ is chosen either zero, or $\pi/2$. The numerics were performed according to the leapfrog/ Verlet algorithm, cf. Eq. (6), with the time step $\Delta t = 2\pi/6 \times 10^{-3}$, or smaller. When the time reversal symmetry is unbroken, i.e. $E_2 = 0$ or $E_2 \neq 0$, but $\phi = 0$, no mean velocity emerges. However, the particle experiences large, Levy-flight like excursions (cf. inset in Fig. 3a). Moreover, the mean velocity itself undergoes zero-mean random fluctuations (cf. inset in Fig. 3b). The Time-Reversal-Non-Invariance of the dynamics yields nonzero current.

In practice, we have used the symplectic integrator given by the leap-frog/Verlet scheme:

$$
\begin{aligned}
x_{n+1} &= x_n + \Delta t \frac{p_n}{m} - \frac{1}{2}(\Delta t)^2 \Big[\nabla U_R(x_n) - F(t_n) \Big] \\
p_{n+1} &= p_n - \frac{1}{2}\Delta t \Big[\nabla U_R(x_n) + \nabla U_R(x_{n+1}) - F(t_n) - F(t_{n+1}) \Big] .
\end{aligned} \tag{6}
$$

We thus find that the *breaking* of Time-Reversal Invariance presents the key to obtain possible finite, rocking induced directed current. The breaking of reflection symmetry is not important for this result; a finite current appears as well for a reflection - symmetric potential when driven by a harmonic mixing signal. We further note that the trajectories $x(t)$, as well as the current $v(t)$, exhibit some intermittent, Levy-flight [18] like diffusive behavior. Therefore, we suspect that the x-motion and the velocity, eq. (4), are not strictly self-averaging, but presumably (no strong super-diffusion) exhibit long-time fluctuations that decay for the velocity variable as $t \to \infty$. Moreover, the condition of Time-Reversal-Non-invariance is only necessary (but *not* sufficient; see Eq. (12) below) to induce a finite current. With the relative phase set at $\phi = 0$ one regains Time-Reversal Symmetry, yielding again a zero current as depicted in Fig. 3 (a) and 3 (b), note the dashed line behavior therein. We conjecture here, that the slightest amount of absolute phase fluctuations $\psi(t) \neq 0$ will kill any nonvanishing asymptotic, stationary current. In practice, this then implies that the current induced by the Time-Reversal Non-Invariance will survive at most only as a transient.

To gain deeper analytical insight into this intriguing question we next study the ratchet transport in a one-dimensional tight-binding lattice for which exact analytical results can be derived. This requires the consideration of a quantum transport scheme.

3 Directed Versus Nondirected Quantum Current in Absence of Dissipation

3.1 Quantum Rectifiers Working Without Dissipation

To start, let us consider a charged particle such as an electron $e < 0$ moving on a periodic lattice under the influence of a harmonic mixing electric field signal $F(t)/e := \mathcal{E}(t)$ of the form in (5) with the force amplitudes E_1, E_2 having now the meaning of scaled electric field amplitudes. We set again the phase fluctuations $\psi(t) = 0$. The undriven energy levels for the electron in the periodic potential possess a band structure. For the sake of simplicity and clarity, we restrict our analysis to the motion of the electron in the lowest energy band, neglecting thereby interband transitions. Then, in the representation of localized Wannier states, $|n\rangle$, the Hamiltonian of the driven quantum transport problem acquires the form [19, 20]

$$H_{\mathrm{TB}}(t) = -\frac{\hbar\Delta}{2}\sum_{n=-N}^{N}(|n\rangle\langle n+1| + |n+1\rangle\langle n|) - ea\mathcal{E}(t)\sum_{n=-N}^{N} n|n\rangle\langle n| \ , \quad (7)$$

where $\hbar\Delta$ is the tunneling matrix element between neighboring states and $2N+1$ denotes the number of sites. We shall describe the dynamics in terms of the density matrix $\rho_{n,m}$. By doing so, we go beyond the standard picture of pure Bloch states. The density matrix approach allows one to consider electrons prepared in mixed quantum-mechanical states as well. In the limit of an infinite ($N \to \infty$) number of states this single-band tight-binding model is *integrable* and can be solved exactly for *arbitrary* external driving fields $\mathcal{E}(t)$ [19, 21, 22]. By the term "exactly" we mean that one can obtain an *explicit analytical* expression for the characteristic function, $F(k,t) = \sum_n e^{ikn} p_n(t)$ (with $-\pi \le k < \pi$), of the probability distribution $p_n(t) = \rho_{n,n}(t)$ to find the electron on the site n [22]. Following the reasoning in Ref. [19], we find (in the limit $N \to \infty$) the explicit result

$$F(k,t) = \frac{1}{2\pi}\sum_{nm}\rho_{n,n+m}(0)\int_{-\pi}^{\pi} dk' e^{i\{[k'-\eta(t,0)]m+kn-\Sigma(k,k',t)\}} \ , \quad (8)$$

where

$$\Sigma(k,k',t) = \Delta\int_0^t \{\cos[k'-\eta(t,\tau)] - \cos[k'-k-\eta(t,\tau)]\}d\tau, \quad (9)$$

and

$$\eta(t,\tau) = \frac{ea}{\hbar}\int_\tau^t \mathcal{E}(t')dt'. \quad (10)$$

With Eq. (8) at hand, any moment of the probability distribution $p_n(t)$ can be found from Eqs. (8), (9), (10) by taking the respective number of derivatives. The first moment, $\langle x(t)\rangle = -iaF_k'(0,t)$, describes the mean position of the electron on the lattice, $\langle x(t)\rangle = a\sum_n np_n(t)$. It reads [21, 22]

$$\langle x(t)\rangle = \langle x(0)\rangle + a|K|\Delta\int_0^t d\tau \sin[\eta(\tau,0)+\varphi] \ , \quad (11)$$

with $K = \sum_n \rho_{n,n-1}(0)$ being the coherence parameter and $\tan\varphi = \frac{\mathrm{Im}K}{\mathrm{Re}K}$. Next we introduce the quasi-momentum $p(t)$ obeying the so-called acceleration theorem, i.e., $\dot{p}(t) = e\mathcal{E}(t)$, with the pseudo-Hamiltonian given by [20]

$$H(\langle x\rangle, p, t) = |K|\epsilon(p) - e\langle x\rangle\mathcal{E}(t),$$

where $\epsilon(p) = -\hbar\Delta\cos(pa/\hbar)$ is the undriven energy spectrum. One can readily demonstrate that Eq. (11) provides the explicit solution of the driven *nonlinear classical dynamics* described with the pseudo-Hamiltonian $H(\langle x\rangle, p)$

for the initial quasi-momentum $p(0) = \hbar\varphi/a$ and the initial position $\langle x(0)\rangle$. These latter two quantities are *defined* by the initial density matrix $\rho_{n,m}(0)$.

The result in Eq. (11) has some truly remarkable consequences: for arbitrary external fields we find the prominent result that for a particle being prepared in a mixed state characterized by the *diagonal* density matrix $\rho_{n,m}(0) = p_n(0)\delta_{nm}$, implying a zero coherence parameter $K = 0$, the mean particle position $\langle x(t)\rangle = \langle x(0)\rangle$ is not affected by the *arbitrary driving fields.* As a consequence we find that the current is identically zero [22], i.e.,

$$j = 0. \tag{12}$$

This initial diagonal preparation mimics the classical situation of an initially localized particle. In clear contrast to the motion of a classical particle in a periodic potential, the quantum mechanical motion of an initially localized particle – *being restricted to the single band dynamics* – does not support a net current. This result is counter-intuitive, – even for a fixed bias there results no finite current. On the other hand it is also intuitive because a finite current generically would cause dissipation. The counterintuitive classical result in section 2.2 can therefore be approached from a quantum transport scheme only if we allow as well for *interband* transitions.

Is a directed quantum current in absence of dissipation possible nevertheless? There is still the possibility that with a finite coherence parameter $K \neq 0$ a finite current emerges, e.g., due to broken Time-Reversal-Invariance. In doing so, we prepare pure initial states given by the Bloch waves $|\Psi(0)\rangle = \sum_n c_n|n\rangle$, where $c_n = \frac{1}{\sqrt{2N+1}}e^{ip(0)na/\hbar}$. For this case, $\rho_{nm}(0) = c_n c_m^*$ and the coherence parameter is maximal, $|K| = 1$ $(N \to \infty)$. Note that, in the absence of the external driving, Eq. (11) indicates that any Bloch state carries a current $j = e\langle\dot{x}(t)\rangle = ea\Delta\sin(p(0)a/\hbar)$. In view of our stated prerequisite in the introduction, we consider here the case that the *initial* current is zero, yielding $p(0) = 0$. Then, the tight-binding dynamics driven by a harmonic mixing signal with *fixed* relative phase, Eq. (5), yields a finite limit in Eq. (4). The current in absence of dissipation emerges in this case as

$$\begin{aligned} j_{hm} &= e\Delta a \sum_{k=-\infty}^{\infty} J_{2k}(\xi_1) J_k(\xi_2/2) \\ &\times \sin\left[k\phi - \xi_1\sin(\psi) - \frac{1}{2}\xi_2\sin(2\psi+\phi)\right], \end{aligned} \tag{13}$$

where $\xi_{1,2} = eaE_{1,2}/(\hbar\omega)$. This result holds for the fixed phase $\psi(t) := \psi$, cf. Eq. (5). In the lowest order of the electric field amplitudes and for $\psi(t) = 0$ we find

$$j_{hm} = \left[\frac{1}{16}e\Delta a\Big(\frac{eaE_1}{\hbar\omega}\Big)^2\frac{eaE_2}{\hbar\omega} - \frac{1}{2}ea\Delta\frac{eaE_2}{\hbar\omega}\right]\sin(\phi). \tag{14}$$

We notice that the current is identically zero when $\phi = 0$, cf. Eq. (14). However, this current is finite for $\phi \neq 0$. This fact can – misleadingly – be interpreted as a current that originates due to broken Time-Reversal-Invariance.

Note, however, if we put $E_1 = 0$ in Eq. (14) the current still exists for $\phi \neq 0$, although the time-reversal symmetry is restored. This means that the finite current emerges already in the presence of a *single* harmonic driving due to the initial phase shift, $\phi \neq 0$. The latter one generates an effective initial momentum of the particle, and thus results in a finite current. In absence of any phase fluctuations this (coherent) current carries no dissipation and it fails to decohere. Nevertheless, if we assume that the initial field phase ψ is randomly distributed in the interval $[0, 2\pi]$ with the probability density $P(\psi) = 1/2\pi$, one can show that after the corresponding *static* averaging over ψ the current j_{hm} in Eq. (13) is zero *independently* of the relative phase shift ϕ. Moreover, we will show now that the *dynamical* phase fluctuations $\psi(t)$ also yield decaying current toward zero.

3.2 Role of Phase Fluctuations

Let us consider a simple model of dichotomous Markovian phase fluctuations $\psi(t)$. This model allows for an exact analytic treatment, i.e. we set

$$\psi(t) = \psi_0 \alpha(t), \tag{15}$$

where ψ_0 denotes the amplitude of phase fluctuations and $\alpha(t) = \pm 1$ is the dichotomous Markov process (DMP) with the stationary autocorrelation function $\langle \alpha(t)\alpha(t')\rangle = \exp(-\nu|t-t'|)$ [24]. The parameter ν is the mean frequency of random phase jumps and defines the dephasing time $\tau_0 = 1/\nu$. Using the relations, $\cos[\psi(t)] = \cos(\psi_0)$ and $\sin[\psi(t)] = \sin(\psi_0)\alpha(t)$, the external driving can be recast into the form

$$\mathcal{E}(t) = \tilde{\mathcal{E}}(t) - \frac{\hbar}{ea} g(t)\alpha(t), \tag{16}$$

where $\tilde{\mathcal{E}}(t)$ is given by Eq. (5) for $\psi(t) = 0$, but with the renormalized amplitudes $E_1 \to \tilde{E}_1 = E_1 \cos(\psi_0)$ and $E_2 \to \tilde{E}_2 = E_2 \cos(2\psi_0)$. Moreover, the function $g(t)$ in Eq. (16) reads

$$g(t) = g_1 \sin(\omega t) + g_2 \sin(2\omega t + \phi), \tag{17}$$

where the amplitudes $g_1 = eaE_1 \sin(\psi_0)/\hbar$ and $g_2 = eaE_2 \sin(2\psi_0)/\hbar$ have the dimension of a frequency.

In the presence of random phase fluctuations the stochastically averaged time-dependent current is given by [22]

$$j(t) = e\frac{d\langle\langle x(t)\rangle\rangle_\psi}{dt}, \tag{18}$$

where the outer average $\langle ...\rangle_\psi$ denotes the average over the phase fluctuations. The stationary current then is given by $j = \lim_{t\to\infty} j(t)$. For the most interesting case with a coherence parameter of $|K| = 1$, it follows from Eq. (11) that

$$j(t) = ea\Delta \times \mathrm{Im}\Big\{ \exp\Big(i[\tilde{\eta}(t,0) + p(0)a/\hbar]\Big)\langle U(t)\rangle_\psi \Big\}, \tag{19}$$

where $\tilde{\eta}(t,0)$ is defined in Eq. (10), but with $\tilde{\mathcal{E}}(t)$ taken from Eq. (16). The function $\langle U(t)\rangle_\psi := \Big\langle \exp\left[-i\int_0^t \alpha(\tau)g(\tau)d\tau\right]\Big\rangle_\psi$ in Eq. (19) is the averaged solution of the auxiliary stochastic differential equation

$$\dot{U}(t) = -ig(t)\alpha(t)U(t), \tag{20}$$

which describes a generalized Kubo oscillator [24] with the stochastic frequency $\Omega(t) = g(t)\alpha(t)$. The averaged solution of Eq. (20) can be written by virtue of the Floquet theorem in the form

$$\langle U(t)\rangle_\psi = e^{-\Gamma_1 t}u_1(t) + e^{-\Gamma_2 t}u_2(t), \tag{21}$$

with positive-valued Floquet values $\Gamma_{1,2} > 0$, and time-periodic Floquet modes $u_{1,2}(t+2\pi/\omega) = u_{1,2}(t)$. It then follows from Eq. (19) that the smallest of the two Floquet exponents, $\Gamma = \min\{\Gamma_1, \Gamma_2\}$ in Eq. (21) characterizes the time scale on which the (transient) current due to the broken Time Reversal Symmetry does exist. Even if the current exists in the absence of phase fluctuations $\psi(t)$, it will relax in real life situations for times $t \gg \Gamma^{-1}$. This result that in the presence of a random driving a stationary ($t \to \infty$) current cannot be realized in absence of dissipation has been shown by us previously in Ref. [22]. Next, we shall evaluate this time scale explicitly.

The averaged solution of Eq. (20) with $g(t) = const$ has been given by Kubo [25]. It can also be looked up in the book by Van Kampen [24]. In our case, the problem is more intricate. To solve the task we apply the formalism of the so-termed "formulae of differentiation" [26] to yield the coupled set of differential equations

$$\begin{aligned} \frac{d}{dt}\langle U(t)\rangle_\psi &= -ig(t)\langle\alpha(t)U(t)\rangle_\psi, \\ \frac{d}{dt}\langle\alpha(t)U(t)\rangle_\psi &= -\nu\langle\alpha(t)U(t)\rangle_\psi - ig(t)\langle U(t)\rangle_\psi \end{aligned} \tag{22}$$

for the average $\langle U(t)\rangle_\psi$ and the correlation $\langle\alpha(t)U(t)\rangle_\psi$. The initial conditions in Eq. (22) follow as $\langle U(0)\rangle_\psi = 1$ and $\langle\alpha(0)U(0)\rangle_\psi = 0$. From here on, sailing becomes smooth by observing that the set (22) is indeed equivalent to a Hill equation for $\langle U(t)\rangle_\psi$. To solve (22) we use the transformation $\langle U(t)\rangle_\psi = r(t)\cos[\Phi(t)/2]$, $\langle\alpha(t)U(t)\rangle_\psi = ir(t)\sin[\Phi(t)/2]$, and end up with

$$\begin{aligned} \dot{r}(t) &= -\nu\sin^2(\Phi(t)/2)r(t), \\ \dot{\Phi}(t) &= -\nu\sin[\Phi(t)] - g(t) \end{aligned} \tag{23}$$

for the new variables $r(t)$ and $\Phi(t)$. The initial conditions transform into $r(0) = 1$, and $\Phi(0) = 0$. In terms of the unknown solution $\Phi(t)$, Eq. (23) yields for the averaged solution of the stochastic differential equation (20) the result,

$$\langle U(t)\rangle_\psi = \exp\{-\nu \int_0^t \sin^2[\Phi(\tau)/2]d\tau\} \cos[\Phi(t)/2]. \tag{24}$$

This formal expression holds for an *arbitrary* function $g(t)$ and is not restricted to the class of periodic functions only. Because the second equation in (23) cannot be integrated in closed form for $g(t)$ given in Eq. (17) the solution $\langle U(t)\rangle_\psi$ remains implicit. However, one can deduce the corresponding decay rate Γ in an analytical form for relevant limiting cases. The most interesting one is the case of a highly coherent field with small field strengths such that $\nu \ll \omega$ and $g_{1,2} \ll \omega$. In this case, the amplitude of oscillations of $\Phi(t)$ is small, $\Phi(t) \ll 1$, and one can expand $\sin[\Phi(t)] \approx \Phi(t)$ in (23). After some straightforward calculations we obtain in the lowest order in the driving field strengths for the rate

$$\Gamma = \frac{1}{8}\nu\Big\{ \sin^2(\psi_0)\Big(\frac{eaE_1}{\hbar\omega}\Big)^2 + \sin^2(2\psi_0)\Big(\frac{eaE_2}{2\hbar\omega}\Big)^2\Big\} \tag{25}$$

implying for the transient current (for $t \gg \nu^{-1}$) the central result

$$j(t) = j_{hm} \exp(-\Gamma t), \tag{26}$$

with j_{hm} is given in Eq. (14).

This result inherits the following consequences. (i) A nonzero stationary current does not exist. (ii) The decay rate Γ for the transient current is determined by the mean rate of phase fluctuations ν, the amplitude of the phase fluctuations ψ_0 and further also by the intensities of the field components $eaE_{1,2}/\hbar\omega$. (iii) With the increasing field strength, not only does the field-induced current j_{hm} increase, cf. Eq. (14), but at the same time the current decay rate is also enhanced, cf. Eq. (25). This latter fact may render considerably more difficult the experimental observation of the decaying transient current.

To decide whether the *coherent* field induced current is stationary or not one can put forward the following criterion: (1) introduce small phase fluctuations $\psi(t)$ in the otherwise strictly periodic field $\mathcal{E}(t)$, cf. Eq. (5), and (2) evaluate the limits in the sequence

$$j = e \lim_{\nu \text{ or } \psi_0 \to 0} \lim_{t\to\infty} \frac{d\langle\langle x(t)\rangle\rangle_\psi}{dt}. \tag{27}$$

The order of limits in Eq. (27) is very important and cannot be interchanged. The application of this criterion to the current in the absence of dissipation yields $j = 0$, meaning that no finite stationary contribution survives.

3.3 Stationary Quantum Current in the Presence of Dissipation

In the previous section we demonstrated that the field-induced current is at most a transient phenomenon under realistic conditions. The role of finite dissipation is thus crucial to produce possibly nonvanishing, *stationary* directed currents. Directed net current emerging in a tight-binding model due to the combined action of dissipation and external driving has been studied in the recent literature in Refs. [16, 22, 23]. The main result of these works is as follows: The directed current appears in form of a nonlinear response to the external driving field if any odd moment of the unbiased driving field, $\langle \mathcal{E}^{2n+1}(t)\rangle_\omega$ is different from zero. Here, $\langle ...\rangle_\omega$ denotes averaging over the period of driving. In the lowest third order of the harmonic mixing driving strengths one finds that the dissipative result j_{st} [16, 23] is proportional to

$$j_{st} \propto \langle \mathcal{E}^3(t)\rangle_\omega \propto E_1^2 E_2 \cos(\phi). \tag{28}$$

The nontrivial prefactor involves the dissipation strength and temperature, see in Ref. [16, 23]. We state (without proof) that the application of the criterion in (27) does not influence the result in Eq. (28); the current is stationary and does not decay in time.

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