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Vol. 37, No. 2, September 1980
Printed in Belgium

Evaluation of Infinite Series by Use of Continued Fraction Expansions: A Numerical Study

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Received August 29, 1979; revised November 26, 1979

The continued fraction method is applied to the summation of various types of series using a computer-oriented recursive algorithm for the calculation of continued fraction coefficients. For convergent as well as for formally divergent series, the continued fraction representation so obtained shows a marked improvement of the convergence behaviour compared with the corresponding series representation.

1. INTRODUCTION

In recent years there has been some resurgence of interest in the field of application of continued fractions (CF) [1–3]. In particular, the method has proved to be a very useful tool for summing up asymptotic perturbation series as they occur in scattering theory [4, 5], field theory [6] and statistical mechanics [4, 7] as well as applied mathematics [1, 8]. A fundamental advantage of the continued fraction method is the fact that it yields a useful analytic continuation of asymptotic series. An unsolved problem in this context is the derivation of rules which determine conditions under which the CF method provides us with the correct analytic continuation. Further, for the method to be an efficient tool in practical applications one needs a powerful algorithm giving the continued fraction coefficients. Recently, we presented such an algorithm [4], which we expand in this paper. Our aim is to show the usefulness and flexibility of this algorithm for the summation of various types of series, and also shed some light on the difficult theoretical problem of the correct analytical continuation. Further, in the application of the CF method it is also important to have realistic estimates of the truncation error $|F - F_n|$ when, in general, the infinite CF for the result F is approximated by its nth continued fraction approximant F_n.

In Section 2 we review some basic properties of CF expansions with emphasis on methods for obtaining truncation error estimates. In Section 3 we present an algorithm for the summation of general series in terms of continued fractions. The

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0021-9991/80/110242–17$02.00/0

method is then applied in Section 4 to various types of series: absolute convergent series, conditional convergent series and asymptotic series. Finally we discuss the possibility of an analytic continuation of series with the CF method. Our conclusions are stated in Section 5.

2. Basic Properties of Certain Classes of Continued Fractions

Let $K(a_n/b_n)$ denote the (infinite) continued fraction

$$F \equiv K\left(\frac{a_n}{b_n}\right) = \frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots, \qquad a_n, b_n \in \mathbb{C}. \tag{2.1}$$

This continued fraction remains unchanged in value if some partial denominator and some partial numerator, along with the immediately succeeding partial numerator, are multiplied by the same non-zero constant. By use of such an equivalence transformation the continued fraction can be written either in the form

$$K\left(\frac{1}{\beta_n}\right) = \frac{1}{\beta_1+}\frac{1}{\beta_2+}\cdots, \tag{2.2}$$

with the coefficients $\{\beta_n\}$ defined by

$$\begin{aligned} \beta_1 &= \frac{b_1}{a_1}, \\ \beta_{2k} &= \frac{a_1 a_3 \cdots a_{2k-1}}{a_2 a_4 \cdots a_{2k}} b_{2k}, \\ \beta_{2k+1} &= \frac{a_2 a_4 \cdots a_{2k}}{a_1 a_3 \cdots a_{2k+1}} b_{2k+1}, \qquad k \geqslant 1, \end{aligned} \tag{2.3}$$

or in the form

$$K\left(\frac{\alpha_n}{1}\right) = \frac{\alpha_1}{1+}\frac{\alpha_2}{1+}\frac{\alpha_3}{1+}\cdots, \tag{2.4}$$

with

$$\alpha_1 = \frac{a_1}{b_1}, \qquad \alpha_k = \frac{a_k}{b_k b_{k-1}}, \qquad k \geqslant 2. \tag{2.5}$$

We remark that in the special case with $a_n, b_n \in \mathbb{R}$ and $a_n, b_n > 0$ for $n \geqslant 1$ two succeeding approximants F_n form lower and upper bounds, i.e.,

$$F_{2n-1} \leqslant F \leqslant F_{2n}. \tag{2.6}$$

Moreover, we have the stronger relation

$$F_{2n-1} \leqslant F_{2n+1} \leqslant F_{2n} \leqslant F_{2n-2} \leqslant F_{2n-4}. \tag{2.7}$$

Next we will give some realistic error estimates for the truncation error $|F - F_n|$, which we believe has not been sufficiently discussed in the literature. A widely applicable result due to Merkes [9], using the information based on the sets $\{a_n\}$, $\{b_n\}$, is: If for all $n \geqslant 2$

$$\left| \frac{a_n}{b_n b_{n-1}} \right| \leqslant r(1-r), \qquad 0 < r < \frac{1}{2}, \tag{2.8}$$

we have the a posteriori bound

$$|F - F_n| \leqslant \frac{r}{1-2r} |F_n - F_{n-1}|. \tag{2.9}$$

These results represent an improvement over the pioneering results of Blanch [1]. Another, very useful result is due to Jones and Thorn [10, 11]: Consider the CF in Eq. (2.2) where

$$|\arg \beta_n| \leqslant \frac{\pi}{2} - \varepsilon, \qquad \varepsilon > 0 \quad \forall n \in \mathbb{N}_0; \tag{2.10}$$

then we obtain the a posteriori bound

$$|F - F_n| \leqslant |F_n - F_{n-1}|. \tag{2.11}$$

These continued fractions have been proved to be convergent if and only if $\sum |\beta_n|$ diverges [12].

3. Outline of the Method for Summation of General Series in Terms of Continued Fractions

In the following we present a convenient method of summing a series

$$S = \sum_{i=1}^{\infty} a_i, \qquad a_i \in \mathbb{C}. \tag{3.1}$$

This series can be recast in the form

$$S(y) = \sum_{i=1}^{\infty} \frac{a_i}{y^{2i-1}} \tag{3.2}$$

with

$$S = S(y = 1). \tag{3.3}$$

Considering $S(y)$ as an asymptotic series, an analytical continuation is obtained by transforming $S(y)$ into the form of a continued fraction

$$S(y) = \frac{d_1}{y+} \frac{d_2}{y+} \frac{d_3}{y+} \cdots. \tag{3.4}$$

The proposed method of summation consists in evaluating the series via the CF in Eq. (3.4) at $y = 1$. The efficiency of this method lies in the fact that the continued fraction coefficients $\{d_i\}$ can be calculated from the set $\{a_i\}$ in a straightforward way by use of a recursive algorithm [4]. For completeness we now sketch this P-algorithm. Starting from the first coefficients

$$\begin{aligned} d_1 &= D_1, & D_1 &= a_1, \\ d_2 &= -D_2/D_1, & D_2 &= a_2, \\ d_3 &= -D_3/D_2, & D_3 &= a_3 + a_2 d_2, \\ d_4 &= -D_4/D_3, & D_4 &= a_4 + a_3(d_2 + d_3), \end{aligned} \tag{3.5}$$

the further coefficients D_n, $n = 4, 5, \ldots$, can be determined recursively. Using the vector X of dimension[1]

$$L = 2 \cdot [(n-1)/2], \tag{3.6}$$

we start from $n = 4$

$$X(1) = d_2, \qquad X(2) = d_2 + d_3, \tag{3.7}$$

interchange

$$X(2) \to X(1), \qquad X(1) \to X(2)$$

and work upwards: setting $X(L-1) = 0$ we have for $n \geqslant 5$

$$X(k) = X(k-1) + d_{n-1} X(k-2), \qquad k = L, L-2, \ldots, 4,$$

and

$$X(2) = X(1) + d_{n-1}. \tag{3.8a}$$

Interchange after each recursion step the odd and even components

$$\begin{aligned} X(2) &\to X(1), & X(4) &\to X(3); \\ X(1) &\to X(2), & X(3) &\to X(4); \quad \text{etc.} \end{aligned} \tag{3.8b}$$

Then the coefficient d_n is given by

$$d_n = \frac{D_n}{D_{n-1}}, \tag{3.9}$$

[1] Note that the notation [ ] implies the integer part.

where

$$D_n = a_n + \sum_{i=1}^{L/2} a_{n-i} X(2i-1). \tag{3.10}$$

In practice we generally have to break off the infinite CF in Eq. (3.4) at an arbitrary order without knowing the exact truncation error. An approximate error estimate for the truncated CF can be obtained in the following way. In Eq. (3.4) setting

$$\Sigma_n(y) = \frac{d_n}{y + \Sigma_{n+1}(y)} \tag{3.11}$$

we obtain with $\Sigma_n(y) \simeq \Sigma_{n+1}(y)$:

$$\Sigma_n(y) \simeq -\frac{1}{2}(y \pm (y^2 + 4d_n)^{1/2}). \tag{3.12}$$

In order to obtain the correct asymptotic limit ($\Sigma \sim y^{-1}$ for $y \gg 1$) select the minus sign in Eq. (3.12).

4. Application of the Continued Fraction Method

The numerical evaluation of the series in this section is done by using double-precision arithmetic with 72 bytes on a UNIVAC 1110.

4.1. *Summation of Absolute Convergent Series*

We start the numerical investigation of the CF method with the absolute convergent series

$$G(x) = x + \frac{x^3}{1 \cdot 3} + \frac{x^5}{1 \cdot 3 \cdot 5} + \cdots, \qquad x \in \mathbb{R}, \tag{4.1}$$

which is connected with the complementary error function erfc(x) by [11]:

$$\begin{aligned} G(x) &= e^{x^2/2} \int_0^x e^{-y^2/2}\, dy \\ &= \left(\frac{\pi}{2}\right)^{1/2} e^{x^2/2} \left(1 - \operatorname{erfc}\left(\frac{x}{2^{1/2}}\right)\right). \end{aligned} \tag{4.2}$$

Expanding the integrand in Eq. (4.2) we obtain for $G(x)$ the alternating series representation

$$G^{\mathrm{alt}}(x) = e^{x^2/2} \left(x - \frac{x^3}{1! \cdot 2 \cdot 3} + \frac{x^5}{2! \cdot 2^2 \cdot 5} - \frac{x^7}{3! \cdot 2^3 \cdot 7} + \cdots\right). \tag{4.3}$$

Furthermore, the complementary error function can be written for $x > 0$ in the form of the explicit CF [13]

$$F\left(\frac{x}{2^{1/2}}\right) = \pi^{1/2} e^{x^2/2} \operatorname{erfc}\left(\frac{x}{2^{1/2}}\right) = \frac{1}{t+}\,\frac{1/2}{t+}\,\frac{1}{t+}\,\frac{3/2}{t+}\cdots \qquad \left(t = \frac{x}{2^{1/2}}\right)$$

$$= \frac{1}{b_1 t+}\,\frac{1}{b_2 t+}\,\frac{1}{b_3 t+}\cdots, \tag{4.4}$$

with

$$b_1 = 1, \qquad b_2 = 2;$$

$$b_{2k+2} = b_{2k}\frac{2k}{2k+1}, \tag{4.5}$$

$$b_{2k+1} = b_{2k-1}\frac{2k-1}{2k}, \qquad k \geqslant 1.$$

TABLE I

Evaluation of the Series $G(x)$ Given by Eq. (4.1) for $x = 1$ and $x = 5$, Respectively[a]

x	n	A	B	C	D	E
1	2	1.3333333	1.5000000	1.3739344	1.4131897	1.5663656
	5	1.4105820	1.4106667	1.4107226	1.4106888	1.3740580
	8	1.4106861	1.4106861	1.4106861	1.4106861	1.4223866
	11	1.4106861	1.4106861	1.4106861	1.4106861	1.4061236
	⋮	⋮	⋮	⋮	⋮	⋮
	82					1.4106861
	⋮					⋮
5	2	4.6666667 (1)	−6.818 (−1)	−4.202 (6)	2.591 (5)	3.3631071 (5)
	5	3.0658465 (3)	8.451 (0)	1.061 (8)	4.093 (5)	3.3631072 (5)
	8	3.1851683 (4)	1.093 (1)	−5.042 (8)	3.271 (5)	⋮
	11	1.1780459 (5)	−2.901 (2)	8.532 (8)	3.359 (5)	
	⋮	⋮	⋮	⋮	⋮	
	20	3.2853151 (5)	2.854 (5)	−7.361 (7)	3.3631067 (5)	
	29	3.3630674 (5)	3.3631075 (5)	2.884 (5)	3.3631072 (5)	
	38	3.3631072 (5)	3.3631072 (5)	3.3636 (5)	3.3631072 (5)	
	47	⋮	⋮	3.3631072 (5)	⋮	
	⋮			⋮		

[a] Column A: direct summation of Eq. (4.1); Column B: summation via the CF representation (3.4); Column C: direct summation of the alternating series in Eq. (4.3); Column D: summation via the corresponding CF (3.4); Column E: summation using Eq. (4.7) with the analytic CF representation (4.4). The correct values are given by the last values of each column. For the case $x = 5$ the entry is given by a number and the power of 10 by which it should be multiplied.

From Eq. (4.4) with Eq. (2.11) it follows that the truncation error is bounded by

$$|G(x) - G_n(x)| \leqslant \frac{1}{2^{1/2}} \left| F_n\left(\frac{x}{2^{1/2}}\right) - F_{n-1}\left(\frac{x}{2^{1/2}}\right) \right|. \tag{4.6}$$

Here, $G_n(x)$ denotes the nth continued fraction approximant of

$$G(x) = \left(\frac{\pi}{2}\right)^{1/2} e^{x^2/2} - \frac{1}{2} F\left(\frac{x}{2^{1/2}}\right). \tag{4.7}$$

The convergence behaviour of the different evaluation schemes of the series for $G(x)$ is illustrated in Table I for the values $x = 1$ and $x = 5$. We deduce from this table that the continued fraction evaluation (Eq. (3.4)) shows only a slight improvement of convergence compared with the direct summation method. For small x-values we note that the continued fraction representation in terms of Eq. (3.4) converges more rapidly than the analytic continued fraction representation given in Eq. (4.4).

Of necessity, numerical approximations are always finite processes. For the alternating series in Eq. (4.3) the $(n+1)$th term $|a_{n+1}(x)|$ gives an upper bound for the truncation error of the actual truncation error of the power series. In Table II we present the actual truncation errors with a comparison of the a posteriori truncation error estimates of Eq. (4.6). As indicated in Table II, the a posteriori error bounds

TABLE II

Actual Truncation Errors and Truncation Error Bounds for the Series $G(x)$ for $x = 1$ and $x = 5$, Respectively[a]

x	n	A	B	C	D
1	2	2.50 (−3)	4.91 (−3)	1.56 (−1)	5.00 (−1)
	5	2.66 (−6)	3.90 (−5)	3.66 (−2)	9.23 (−2)
	8	3.44 (−10)	7.92 (−8)	1.17 (−2)	2.84 (−2)
	11	1.08 (−15)	8.77 (−14)	4.56 (−3)	1.07 (−2)
	⋮	⋮	⋮	⋮	⋮
	100			8.04 (−9)	1.69 (−8)
	⋮			⋮	⋮
5	2	7.66 (4)	2.10 (7)	1.76 (−2)	5.32 (−1
	5	7.34 (4)	3.10 (8)	1.25 (−5)	1.19 (−4)
	8	8.93 (3)	1.17 (9)	5.17 (−8)	3.37 (−7)
	11	5.89 (2)	1.70 (9)	2.76 (−10)	6.28 (−10)
	⋮	⋮	⋮	⋮	⋮
	20	4.96 (−2)	1.17 (8)		
	30	1.44 (−9)	6.70 (4)		
	⋮	⋮	⋮		

[a] Column A: exact CF truncation error $|G^{\text{alt}}(x) - G_n^{\text{alt}}(x)|$; Column B: $|a_{n+1}|$; Column C: $|G(x) - G_n(x)|$; Column D: $(1/2^{1/2})\,|F_n(x/2^{1/2}) - F_{n-1}(x/2^{1/2})|$. The entry is given by a number and the power of 10 by which it should be multiplied.

give very good estimates for the exact truncation errors and compare, for large x-values, favourably with the upper bounds $|a_{n+1}(x)|$. The summation method in Eq. (3.4) has been tested for various types of absolute convergent series. As a rule, the continued fraction summation method shows for alternating absolute series an efficient improvement of convergence, whereas for absolute convergent series with terms of equal signs the improvement is in general not dramatic. This is demonstrated for the typical case of the zeta function, defined by [13]

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \qquad \operatorname{Re}(z) > 1, \tag{4.8}$$

or, alternatively by

$$\zeta(z) = (1 - 2^{(1-z)}) \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n^z}, \qquad \operatorname{Re}(z) > 0, \quad z \neq 1. \tag{4.9}$$

In Table III the results for the direct summation and for the continued fraction method are shown for $z = 2$ for the two representations given by Eqs. (4.8) and (4.9).

It is worthwhile to emphasize that the series summation technique presented may not be applicable in certain cases. As as counterexample we study the power series (cf. Ref. [14])

$$P(x) = \frac{1 + 6x + (1 - 24x^3)^{1/2}}{2(1+x)} = \sum_{n=0}^{\infty} c_n x^n. \tag{4.10}$$

This series converges absolutely for $|x| \leqslant 1/(24)^{1/3}$ and the coefficients $\{c_n\}$ are calculated to be

$$c_0 = 1,$$
$$c_n = \frac{(-)^n}{n!} \left\{ \sum_{i=0}^{[n/3]} \binom{\frac{1}{2}}{i} 24^i - 5 \right\}, \qquad n \geqslant 1. \tag{4.11}$$

TABLE III

Evaluation of the Zeta Function for $z = 2$[a]

n	A	B	C	D
5	1.464	1.55161	1.67722	1.6451804035
10	1.550	1.61517	1.63592	1.6449340351
15	1.580	1.63037	1.64908	1.6449340669
20	1.596	1.63635	1.64256	1.6449340668
⋮	⋮	⋮	⋮	⋮
100	1.634	1.64345	1.64483	1.6449340668
⋮	⋮	⋮	⋮	⋮

[a] Column A: direct summation of Eq. (4.8); Column B: summation via the corresponding CF (3.4); Column C: direct summation of Eq. (4.9); Column D: summation via the corresponding CF (3.4). The correct value is given by the last value of Column D.

It is easy to verify that the fifth term d_5 of the corresponding continued fraction, Eq. (3.4), equals zero. The continued fraction so obtained clearly does not equal the correct value in (4.10). This difficulty can be circumvented by presumming the first term in the original series, Eq. (4.10), and then applying the continues fraction expansion to the remaining series. The resulting continued fraction coincides with the periodic fraction first discussed by Perron [14]:

$$P(x) = 1 + \frac{2x}{1+}\,\frac{x}{1+}\,\frac{-3x}{1+}\,\frac{2x}{1+}\,\frac{x}{1+}\,\frac{-3x}{1+}\cdots. \tag{4.12}$$

This fraction shows normal convergence behaviour for $|x| < 1/(24)^{1/3}$ from a numerical viewpoint except at the value $x = \frac{1}{3}$, where the continued fraction in (4.12) is seen to be mathematically not convergent.

4.2 *Conditional Convergent Series*

As a first example of conditional convergent series we study the series of the logarithm

$$\log(1+z) = \sum_{n=1}^{\infty} (-)^{n+1}\frac{z^n}{n}, \qquad |z| \leqslant 1, \quad z \neq -1, \tag{4.13}$$

for the case $\mathrm{Re}(z) = 1$.

Our method of summation for the logarithm coincides up to an equivalence transformation with the following explicit CF representation [13] for:

$$F(z) \equiv \log(1+z) = \frac{z}{1+}\,\frac{1^2z}{2+}\,\frac{1^2z}{3+}\,\frac{2^2z}{4+}\,\frac{2^2z}{5+}\,\frac{3^2z}{6+}\cdots, \qquad z \notin (-\infty, -1]. \tag{4.14}$$

By use of the results of Jones and Thorn [10], the a posteriori truncation error estimates for the CF in Eq. (4.14) read

$$\begin{aligned} |\log(1+z) - F_n(z)| &\leqslant |F_n(z) - F_{n-1}(z)| \\ &\quad \text{if} \quad |\arg z| \leqslant \pi/2 \\ &\leqslant \sec(|\arg z| - \pi/2)\,|F_n(z) - F_{n-1}(z)| \\ &\quad \text{if} \quad \pi/2 < |\arg z| < \pi. \end{aligned} \tag{4.15}$$

In Table IV we give the a posteriori truncation error bounds for the calculation of $\log 2$ and $\log(2 + 10i)$. These bounds compare well with the exact truncation error bounds. Further, compared with the a priori upper bounds obtainable for real z-values from the $(n+1)$th term of the alternating series, Eq. (4.13), we find a remarkable improvement.

Conditional convergent series tend to any given value if they are suitably

TABLE IV

Actual Truncation Errors and Truncation Error Bounds for $\log(1+z)$ for $z=1$ and $z=1+10i$, Respectively[a]

z	n	A	B	C
1	3	2.50 (−1)	6.85 (−3)	3.33 (−2)
	6	1.43 (−1)	2.55 (−5)	2.12 (−4)
	9	1.00 (−1)	1.52 (−7)	9.15 (−7)
	12	7.69 (−2)	6.73 (−10)	5.08 (−9)
	15	6.25 (−2)	3.75 (−12)	2.37 (−11)
	⋮	⋮	⋮	⋮
$1+10i$	3	2.55 (3)	1.62 (0)	2.36 (0)
	6	1.48 (6)	2.07 (−1)	6.97 (−1)
	9	1.05 (9)	6.49 (−2)	1.40 (−1)
	12	8.21 (11)	1.27 (−2)	3.71 (−2)
	15	8.77 (14)	3.59 (−3)	8.23 (−3)
	⋮	⋮	⋮	⋮
	20	5.29 (19)	2.92 (−4)	8.12 (−4)
	30	3.76 (29)	2.59 (−6)	7.00 (−6)
	⋮	⋮	⋮	⋮
	50	2.52 (49)	2.01 (−10)	5.31 (−10)
	⋮	⋮	⋮	⋮

[a] Column A: $|a_{n+1}|$; Column B: exact CF truncation error $|\log z - \log_n z|$; Column C: $|\log_n z - \log_{n-1} z|$. The entry is given by a number and the power of 10 by which it should be multiplied.

rearranged. So we obtain from the series for $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ by adding $\frac{1}{2}\log 2 = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots$ the new rearranged series

$$\tfrac{3}{2}\log 2 = 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \cdots. \qquad (416)$$

To test our method we have applied the CF method to this series. Although the convergence is less rapid than in the series discussed above for log 2, the CF expansion converges to the exact result: after 40 terms its relative error is less than 10^{-6}.

Next, as a typical example of a conditional convergent series, we discuss the eta function [13]

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n^z}, \qquad \mathrm{Re}(z) > 0, \qquad (4.17)$$

which is connected with Riemann's zeta function (see Eq. (4.9)) by

$$\eta(z) = (1 - 2^{(1-z)})\,\zeta(z). \qquad (4.18)$$

TABLE V

Evaluation of the Eta Function for $z = 0.5$[a]

n	A	B
5	0.817	0.605043537436285
10	0.451	0.604898627352368
15	0.732	0.604898643424325
20	0.494	0.604898643421630
⋮	⋮	⋮

[a] Column A: direct summation of Eq. (4.17); Column B: summation via the corresponding CF (3.9). The correct value is given by the last value of Column B.

This function can be easily evaluated by its integral representation

$$\eta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t + 1}\, dt, \qquad \mathrm{Re}(z) > 0. \tag{4.19}$$

The series Eq. (4.17) is conditional convergent for $0 < |z| \leqslant 1$ and obviously we have $\eta(1) = \log 2$. The dramatic improvement of convergence with the CF method for the series (4.17) is shown in Table V for the value $z = 0.5$. An accuracy of 15 significant digits is obtained after only 20 terms using the CF method, while the direct summation does not even give the first significant digit. The same behaviour is obtained for all other values of $0 < |z| \leqslant 1$.

We have investigated numerous other types of conditional convergent series, all showing the same dramatic improvement of convergence as in the case of the eta function.

For example, let us discuss the series

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{(-)^n}{n^p \cdot \log n} &= 0.924\,299\,897\,222\,939 \qquad \text{for} \quad p = 0 \\ &= 0.526\,412\,246\,533\,310 \qquad \text{for} \quad p = 1, \end{aligned} \tag{4.20}$$

where the numerical values can be obtained by the contour integration methods presented in Appendix A. Again, as shown in Table VI, for the two cases $p = 0$ and $p = 1$, the CF method gives 15 correct digits after only 20 terms. Compare this convergence behaviour with that obtained by direct summation. By checking the truncation error of this alternating series, one can see that for the case of $p = 0$ up to $e^{10^{15}}$ terms are necessary to obtain an absolute accuracy of 10^{-15}!

As a last example of a conditional convergent series we mention the Fourier series

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\frac{1}{2} \log(2(1 - \cos x)). \tag{4.21}$$

TABLE VI

Evaluation of the Series Given by Eq. (4.20) for $p = 0$ and $p = 1$, Respectively[a]

p	n	A	B
0	5	1.191	0.924575821884318
	10	0.720	0.924299865188788
	15	1.103	0.924299897228137
	20	0.761	0.924299897222939
	⋮	⋮	⋮
1	5	0.567	0.526539505225282
	10	0.509	0.526412230841441
	15	0.537	0.526412246535892
	20	0.519	0.526412246533310
	⋮	⋮	⋮

[a] Column A: direct summation of Eq. (4.20); Column B: summation via the corresponding CF (3.4).

The CF method reproduces 15 correct digits after only a few terms. The behaviour is quite analoguous to the cases discussed above and so we have not shown it in a separate table.

4.3 *Summation of Asymptotic Series*

A very important class of series frequently encountered in applied mathematics is asymptotic series. We study our method by applying it to the asymptotic series of the incomplete gamma function [13]:

$$\Gamma(a, z) \sim z^{a-1}e^{-z}\left\{1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} + \cdots\right\},$$

$$z \to \infty \quad \text{in} \quad |\arg z| < \frac{3\pi}{2}. \tag{4.22}$$

In Table VII we have represented the improvement of convergence behaviour for the typical case $a = 0$ and $z = 5$ and 10, respectively. (In this case the incomplete gamma function is identical to the exponential integral $E_1(z)$.)

It can be seen that the CF method converges rapidly to 11 correct digits after about 20 steps. On the other hand, only 2 digits for $z = 5$ and 4 digits for $z = 10$, respectively, can be obtained by direct summation. We may compare our CF expansion with the well-known analytic continued fraction for the incomplete gamma function [13], valid for real values of $x > 0$:

$$\Gamma(a, x) = e^{-x}x^a\left(\frac{1}{x+}\,\frac{1-a}{1+}\,\frac{1}{x+}\,\frac{2-a}{1+}\,\frac{2}{x+}\cdots\right). \tag{4.23}$$

TABLE VII

Evaluation of the Incomplete Gamma Function for the Case $a = 0$ and $z = 5$ and $z = 10$, Respectively[a]

z	n	A	B
5	3	1.185 (−3)	1.1550766284 (−3)
	9	1.235 (−3)	1.1483018337 (−3)
	15	1.540 (−2)	1.1482956273 (−3)
	21	2.756 (1)	1.1482955918 (−3)
	27	3.062 (5)	1.1482955913 (−3)
	⋮	⋮	⋮
10	3	4.1768 (−6)	4.1616602282 (−6)
	9	4.1578 (−6)	4.1569693101 (−6)
	15	4.1593 (−6)	4.1569689300 (−6)
	21	4.1792 (−6)	4.1569689297 (−6)
	⋮	⋮	⋮

[a] Column A: direct summation of Eq. (4.22); Column B: summation via the corresponding CF (3.4). The correct values are given by the last values of Column B. The entry is given by a number and the power of 10 by which it should be multiplied.

Regarding the correspondent coefficients d_n in the CF (3.4) one again sees that both continued fractions coincide up to a simple equivalence transformation. However, our method is valid for all values of $z \in \mathbb{C}$ and, more important, it can be used in all the cases where no explicit continued fraction is known, e.g., for the case of Bessel and Coulomb functions. In all these cases we obtain results analogous to the example just discussed.

4.4 *Analytic Continuation with the Continued Fraction Expansion*

As a last application of the CF method we discuss the analytic continuation of series. For this we may use any function which is defined by its series expansion only in a restricted domain. So, from the examples given above we choose the series expansion in (4.17) for the eta function, which is convergent only for $\mathrm{Re}(z) > 0$. On the other hand, the eta function can be continued analytically into the left complex half plane. Using the integral representation (4.19) one obtains [13]

$$\eta(z) = \frac{2^z - 2}{1 - 2^z} \pi^{z-1} \sin \frac{\pi z}{2} \Gamma(1-z)\, \eta(1-z). \tag{4.24}$$

Especially simple is the case $z = 0, -1, -2, \ldots$, where we have

$$\eta(0) = \frac{1}{2}, \quad \eta(-2n) = 0 \quad \text{and} \quad \eta(1-2n) = (2^{2n} - 1)\frac{B_{2n}}{2n}, \quad n \in \mathbb{N}. \tag{4.25}$$

Here the Bernoulli numbers are denoted by B_n. Applying the CF method to the corresponding diverging series

$$\eta(-p) = \sum_{n=1}^{\infty} (-)^{n+1} n^p, \qquad p \in \mathbb{N}_0, \tag{4.26}$$

one can show that the coefficient d_{2n+1} of the CF expansion equals zero. Therefore, it terminates after this term, giving the result (4.25). This behaviour can be understood by relating series (4.26) with the binomial series

$$(1+z)^q = \sum_{n=0}^{\infty} \binom{q}{n} z^n = 1 + qz + \frac{q(q-1)}{2!} z^2 + \cdots. \tag{4.27}$$

This series is absolutely convergent for $|z| < 1$ and divergent for $|z| > 1$ if $q \notin \mathbb{N}_0$.

If we apply the CF method to (4.27) for the divergent case, $|z| > 1$ and $q = -n$ ($n \in \mathbb{N}_0$), again the coefficient d_{2n} equals zero and the terminated continued fraction yields the correct result. In this case the binomial series is a rational function and consequently the continued fraction has to be finite by construction. Now it can be shown that $\eta(-p)$ can be formally written as a finite sum over binomial series:

$$\left.\frac{1}{(1+z)^{M+1}}\right|_{z=1} = \frac{1}{M} \sum_{m=0}^{M-1} a_m^M \eta(-(m+1)), \qquad M \in \mathbb{N}, \tag{4.28}$$

with

$$a_m^M = \sum_{l_1 < l_2 < \cdots < l_m = 1}^{M-1} \frac{1}{l_1 \cdot l_2 \cdots l_m}. \tag{4.29}$$

By inverting Eq. (4.28) it follows that the continued fraction for $\eta(-p)$ with $p \in \mathbb{N}_0$ is also finite.

The CF method can also be used for the analytical continuation of the other non-

TABLE VIII

Evaluation of the Eta Function for $z = -1.5$[a]

n	A	B
5	6.55	0.1208726817
10	−16.88	0.1186808255
15	30.62	0.1186808707
20	−46.28	0.1186808707
⋮	⋮	⋮

[a] Column A: direct summation of Eq. (4.17); Column B: summation via the corresponding CF (3.4). The correct value is given by the last value of Column B.

TABLE IX

Evaluation of the Binomial Series for the Case $t = \frac{1}{2}$ and $q = \pi$[a]

n	A	B
5	31.70	19.6624787596506
10	31.47	31.5420409084769
15	31.83	31.5442810282513
20	29.24	31.5442806996167
25	57.91	31.5442807001975
⋮	⋮	⋮

[a] Column A: direct summation of Eq. (4.30); Column B: summation via the corresponding CF (3.4). The correct value is given by the last value of Column B.

trivial cases. In Table VIII, for example, we have chosen the case $z = -1.5$. The result is that the CF method leads to the correct result after a few terms.

Furthermore, we mention that in many cases the analytic continuation with the CF method is equivalent to the summation of an asymptotic series. Let us, for example, use the binomial series Eq. (4.27). Setting $t = 1/z$, we obtain the series

$$\left(1 + \frac{1}{t}\right)^q = 1 + \frac{q}{t} + \frac{q(q-1)}{2!\,t^2} + \cdots, \tag{4.30}$$

which is convergent for $|t| > 1$. For $|t| < 1$ this series can be interpreted as an asymptotic series. Therefore, as shown above, the CF method yields the correct result, even if we choose, e.g., $q = \pi$ and $t = \frac{1}{2}$ (corresponding to $z = 2$). This is illustrated in Table IX.

As a rule the CF method can be suitably used for the analytic continuation of any series. Even if these series are not Borel summable, they can be summed in terms of a continued fraction. For example the series

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1} \tag{4.31}$$

is not Borel summable for $|z| > 1$ (cf. Appendix B). However, the terminating continued fraction yields for $|z| > 1$ 15 correct digits after four terms.

5. Conclusions

We have presented a suitable summation method for various types of series using a continued fraction representation. The terms occurring in the series can be looked upon as coefficients of an asymptotic expansion $S(y)$ which represents the starting point for the construction of the continued fraction representation. Its evaluation at

$y = 1$ yields the desired result for the summation of the series under consideration. Henceforth, our method of conversion of the original series furnishes new properties in the sense that an nth-order truncated approximation of the continued fraction does not coincide with the nth partial sum but rather provides an approximately correct analytical continuation of the series. The method of evaluation is based on a computer-oriented recursive algorithm for the calculation of the continued fraction coefficients. Its practical application is thus limited only by a possible numerical instability of this algorihtm due to round-off errors.

Appendix A

Many infinite series can be summed by use of complex integration techniques. Using Cauchy's formula we can write (see, e.g., [15])

$$\sum_{n=N_0}^{\infty} (-)^n a_n = \frac{1}{2i} \oint_C \frac{a(z)}{\sin \pi z}\, dz. \tag{A1}$$

The contour C is given by an infinite line parallel to the imaginary axis through the point ε, with $N_0 - 1 < \varepsilon < N_0$, and is closed in counterclockwise direction in the right half plane by an infinite half circle. Hereby we have assumed that $a(z)$ has no poles inside this contour.

Because in the cases discussed above the contribution from the infinite half circle vanishes, we obtain:

$$\sum_{n=N_0}^{\infty} (-)^n a_n = \int_0^{\infty} \frac{dy}{\cosh \pi y} \operatorname{Re}\{a(\varepsilon + iy)\}. \tag{A2}$$

For the series (4.20) we choose $\varepsilon = \frac{3}{2}$ yielding

$$\sum_{n=2}^{\infty} \frac{(-)^n}{n^p \cdot \log n} = \int_0^{\infty} \frac{dy}{\cosh \pi y} \operatorname{Re}\left\{\frac{1}{(\frac{3}{2} + iy)^p \cdot \log(\frac{3}{2} + iy)}\right\}, \qquad \operatorname{Re}(p) \geqslant 0. \tag{A3}$$

This integral converges rapidly and can be calculated with standard integration techniques giving the values in Eq. (4.20).

Note however, that for many physically interesting series the analytic continuation of the coefficients a_n is not known and therefore the series cannot be evaluated in this way.

Appendix B

Considering the series

$$h(z) = \sum_{n=1}^{\infty} n z^{n-1}, \tag{B1}$$

we introduce its first Borel transform

$$H(z) = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!}. \tag{B2}$$

Thus, we obtain for $h(z)$ the representation

$$\begin{aligned} h(z) &= \int_0^{\infty} xe^{-x} H(xz)\, dx \\ &= \int_0^{\infty} xe^{-x} \left\{ 1 + \frac{2xz}{2!} + \frac{3(xz)^2}{3!} + \cdots \right\} dx \\ &= \int_0^{\infty} xe^{x(z-1)}\, dx = \frac{1}{(1-z)^2} \int_0^{\infty} ye^{y}\, dy \qquad \text{with} \quad y = x(z-1). \end{aligned} \tag{B3}$$

Equation (B3) clearly demonstrates that for $z > 1$, $h(z)$ is not Borel summable.

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Printed by the St. Catherine Press Ltd., Tempelhof 37, Bruges, Belgium