JOURNAL OF CHEMICAL PHYSICS VOLUME 110, NUMBER 13 1 APRIL 1999

What can be stated by the Glansdorff–Prigogine criterion concerning the stability of mass-action kinetic systems?

Thomas Wilhelm and Peter Hänggi
Departement of Physics, University of Augsburg, Memminger Strasse 6, D-86135 Augsburg, Germany

(Received 17 August 1998; accepted 22 December 1998)

We investigate which general results concerning the local stability of steady states of arbitrary chemical reaction networks can be deduced with the Glansdorff–Prigogine stability criterion. Especially, it is proven that the presence of an autocatalytic reaction is not a necessary condition for a violation of the thermodynamic stability condition. It turns out that every reaction with at least one variable reactant at each side of the reaction equation can potentially destabilize the steady states. An explicit example of a simple reaction system without autocatalytic reactions where the stability of the steady state changes via a supercritical Hopf bifurcation is discussed. Furthermore, in expanding the original concept for proving local stability to global stability analyses, a general way for constructing different Lyapunov functions is given. © *1999 American Institute of Physics.* [S0021-9606(99)51412-8]

I. INTRODUCTION

The theory of thermodynamics allows one to derive in the classical deductive manner according to the ideal prototype of Euklid's ''Elements'' from few general axioms a multitude of prominent results. However, up to now the theory is valid only at equilibrium and near equilibrium. In the general nonequilibrium case the multifarious dynamical phenomena are not covered in such a general way by the thermodynamic theory.

Near equilibrium, where linear relations between the thermodynamic flows and forces are valid, the celebrated Onsager–Casimir theory is fully satisfactory to understand and describe the observed phenomena. The Brussels group developed their theory with the aim to derive in the thermodynamic framework interesting results also for systems far from equilibrium (cf. Ref. 1). An important field of applications has been, from the very beginning, chemical reaction networks.[1–3] Chemical systems have also been used to demonstrate that the Glansdorff–Prigogine stability criterion is not necessary, but only sufficient for the local stability of steady states.[4,5] Nevertheless, it could be that for modeling of chemical systems interesting and relevant results follow from this criterion. For example, one could find a subclass of reaction networks with only thermodynamically stable steady states which directly implies kinetic stability (according to the usual mathematical local stability theory).

In this context it is especially interesting to note that it has often been stated that reaction networks with unstable steady states contain autocatalytic reactions. The Brussels group showed that special autocatalytic reactions yield a destabilizing contribution in the framework of the Glansdorff–Prigogine stability theory.[1–3] Indeed, all known oscillating reaction systems (e.g., model systems such as the Brusselator,[3] the Lotka–Volterra system,[6] the Oregonator,[7] or the smallest chemical reaction system with Hopf bifurcation[8]) contain at least one autocatalytic reaction. On the other hand, e.g., the homogeneous Turing system (cf. Ref. 3) with its cross-catalytic property in the absence of a direct autocatalytic reaction always has a locally stable steady state. Thus, the question arises whether an autocatalytic reaction is really necessary for an unstable steady state (e.g., for a Hopf bifurcation) of the whole network. It should be noted that in all known chemical and biochemical reaction systems which show sustained oscillations the stable limit cycle always arises via a supercritical Hopf bifurcation.[9]

In this work we investigate in a general manner the thermodynamic stability of arbitrary mass-action kinetic reaction networks. In particular, we show that former analyses of single reactions with the result that autocatalytic reactions destabilize the steady state and nonautocatalytic ones do not, has not been carried out in the necessary generality. It is demonstrated that (without knowledge of the steady state concentrations) with the studied criterion the stability of the steady states can be proven only for a very restricted class of reaction networks. Furthermore, we present a simple example of a reaction system with a supercritical Hopf bifurcation, which does not contain an autocatalytic reaction.

In the analysis we confine ourselves to the important case of reaction networks with mass-action kinetics where one reaction can be written as

$$\mathrm{S}+\sum_i \nu_i^+ \mathrm{X}_i \underset{k_-}{\overset{k_+}{\rightleftharpoons}} \mathrm{P}+\sum_i \nu_i^- X_i \quad (\nu_i^+,\nu_i^-,k_+,k_- \geqslant 0). \tag{1}$$

The reaction velocity is assumed to be proportional to the concentrations of the involved reactants (in consideration of their molecularity). Here S and P denote the sum of the constant substances and products and k_+ and k_- the rate coefficients of the forward and backward unidirectional reaction, respectively. X_i is the ith variable reactant and ν_i^+ and ν_i^- its stoichiometric coefficients. In this case a simple unambiguous definition for an autocatalytic reaction can be given. The considered reaction is autocatalytic, if

$$\nu_i^+ \neq 0, \quad \nu_i^- \neq 0, \quad \nu_i^+ \neq \nu_i^- \tag{2}$$

for at least one reactant X_i.

II. THERMODYNAMIC STABILITY ANALYSIS OF ARBITRARY MASS-ACTION KINETIC REACTION NETWORKS

According to the general theory developed by the Brussels group (cf. Ref. 1), the thermodynamic equilibrium state is locally stable if the excess entropy $\delta^2 S$, taken at equilibrium, is negative for arbitrary small deviations, i.e.,

$$(\delta^2 S)_{\mathrm{eq}} < 0. \tag{3}$$

If the system is thermally and mechanically stable, this condition reduces to the condition for stability with respect to diffusion, i.e.,

$$\sum_i \delta\mu_i \delta n_i > 0, \tag{4}$$

where μ_i, n_i denote the chemical potential and the mole number of reactant X_i, respectively.

The thermodynamic theory for local stability of nonequilibrium steady states (ss) is based on the assumption of stable local equilibrium (cf. Ref. 1). In purely mathematical terms, this can be expressed in terms of a negative definite function, which for physical reasons may be termed *excess entropy density*:

$$(\delta^2 s)_{\mathrm{ss}} < 0. \tag{5}$$

In the domain of phase space where this is a negative definite function it serves as an appropriate Lyapunov function which can prove the stability of the steady states.

The analyzed steady state is globally stable in a domain D of the phase space if the time derivative of the excess entropy density, i.e., the *excess entropy production density* is positive for all deviations contained in D:

$$\frac{d(\delta^2 s)_{\mathrm{ss}}}{dt} > 0. \tag{6}$$

To prove the local stability it is sufficient to analyze the linearized form at the steady state (cf. Appendix B, Ref. 4). It is clear that this criterion is only sufficient for the usual mathematical local stability, because the latter does not require any Lyapunov function. For homogeneous chemical reaction systems (6) simplifies to

$$\sum_j \delta v_j \delta A_j > 0, \tag{7}$$

where v_j, A_j denote the reaction velocity and the affinity of the jth reaction, respectively. This expression has the interesting property that it consists of a sum of different terms, each belonging exactly to one reaction. Of course the thermodynamic stability condition (7) can only be violated if at least one term is negative.

We now successively prove the following (1) For systems under constant temperature and pressure the excess entropy density is a negative definite function in the whole phase space $[\delta^2 s = f(c_i)$, where c_i denotes the concentration of reactant $X_i]$. (2) Already the simple reaction $\mathrm{X} \rightleftharpoons \mathrm{Y}$ (this is the simplest nonautocatalytic reaction with variable substrate and product) always yields a negative term in the excess entropy production density. (3) Reaction networks consisting exclusively of reactions where one side of the reaction equation contains only constant (''outer'') reactants always have locally stable steady states.

A. The excess entropy density is a negative definite function in the whole phase space

For systems under constant temperature and pressure condition Eq. (5) reduces to

$$L = \sum_{i,k} (\partial\mu_i/\partial c_k)_{\mathrm{ss}} \delta c_i \delta c_k > 0, \tag{8}$$

where L is proportional to the negative excess entropy density. This quadratic form is positive definite if its form matrix $F = (\partial\mu_i/\partial c_k)_{\mathrm{ss}}/(RT)$ is positive definite (R and T denote the gas constant and temperature, respectively). With the chemical potential of the ith variable reactant given by

$$\mu_i = \mu_i^0 + RT \ln c_i, \tag{9}$$

F reduces to

$$F = \mathrm{diag}\left(\frac{1}{\bar{c}_1}, \ldots, \frac{1}{\bar{c}_n}\right). \tag{10}$$

Because for positive-valued steady states $\{\bar{c}_i\}$ all eigenvalues of F are positive, the function L is positive definite (and thus the excess entropy density negative definite) and therefore well suited to serve as a Lyapunov function.

B. The simplest example

The tendency of an autocatalytic reaction to destabilize the steady state of the whole system has been explained with the help of the Glansdorff–Prigogine stability criterion (e.g., Refs. 1–3). In order to analyze a special reaction for its ability to destabilize the stationary state one must take into account the variation of all involved variable substances. In doing so, one recognizes that already the simplest monomolecular reaction $\mathrm{X} \underset{k_-}{\overset{k_+}{\rightleftharpoons}} \mathrm{Y}$ with variable substrate and product always yields a negative term in the excess entropy production density.

From the affinity $A = RT \ln(qx/y)$, where q is the equilibrium constant, and the corresponding reaction velocity $v = k_+ x - k_- y$, where lower case characters denote the concentration of the reactants, one obtains for the first variation

$$\delta A = RT\left(\frac{\delta x}{\bar{x}} - \frac{\delta y}{\bar{y}}\right), \quad \delta v = k_+ \delta x - k_- \delta y. \tag{11}$$

Upon combining one finds the quadratic form

$$\delta v\, \delta A = RT\left(\frac{k_+}{\bar{x}}(\delta x)^2 + \frac{k_-}{y}(\delta y)^2 - \left(\frac{k_-}{\bar{x}} + \frac{k_+}{\bar{y}}\right)\delta x\, \delta y\right). \tag{12}$$

Note that the excess entropy production density contains the negative $\delta x\, \delta y$ term. The whole expression is positive semidefinite if and only if $2k_+k_-/(\overline{xy})\geqslant(k_-/\overline{x})^2+(k_+/\overline{y})^2$, which for $(x,y)\neq(\overline{x,y})$ never holds. The equality sign is obtained in the equilibrium case.

Generally one studies the definiteness of quadratic forms with n variables via its form matrices F. A quadratic form is positive definite if all main minors of its F are positive definite. For the studied reaction, F reads

$$F=\begin{pmatrix} \dfrac{k_+}{\overline{\overline{x}}} & -\dfrac{1}{2}\left(\dfrac{k_+}{\overline{\overline{y}}}+\dfrac{k_-}{\overline{\overline{x}}}\right) \\ -\dfrac{1}{2}\left(\dfrac{k_+}{\overline{\overline{y}}}+\dfrac{k_-}{\overline{\overline{x}}}\right) & \dfrac{k_-}{\overline{\overline{y}}} \end{pmatrix}. \tag{13}$$

Both 1×1 main minors are always positive, because the unidirectional reaction rates as well as the steady state concentrations are positive, but the determinant $\det(F)$ is negative semidefinite:

$$\det(F)=-\frac{1}{4}\left(\frac{k_+}{\overline{\overline{y}}}-\frac{k_-}{\overline{\overline{x}}}\right)^2. \tag{14}$$

Note that in the equilibrium case the determinant is zero and the whole quadratic form positive semidefinite $(\overline{y}/(RTk_-)\,\delta v\,\delta A=(q\,\delta x-\delta y)^2)$.

Herewith, we have shown that for this simplest case the excess entropy production density [cf. Eqs. (6) and (7)] is not positive semidefinite. Therefore, already the simplest nonautocatalytic reaction can destabilize the steady state.

C. The general case

In order to decide whether an arbitrary reaction of the form (1) yields a positive or a negative contribution in the excess entropy production density, i.e., in the sum (7), one generally has to analyze an $n\times n$ form matrix F. The considered reaction always stabilizes the steady state if F is positive semidefinite, i.e., if F has only non-negative main minors. It will be shown that for all drawn conclusions it is sufficient to analyze a general reaction with two variable substances of the form

$$a\mathrm{X}+b\mathrm{Y}\underset{k_-}{\overset{k_+}{\rightleftharpoons}}c\mathrm{X}+d\mathrm{Y}, \tag{15}$$

where possible constant reactants are incorporated into the rate constants (which are therefore sometimes termed ''apparent'' rate constants). The velocity (mass-action kinetics) and affinity (ideal solute systems) of this reaction can be written as

$$v=k_+x^ay^b-k_-x^cy^d,\quad A=RT\ln(qx^{a-c}y^{b-d}), \tag{16}$$

yielding

$$\delta v=(k_+a\overline{x}^{a-1}\overline{y}^b-k_-c\overline{x}^{c-1}\overline{y}^d)\,\delta x +(k_+b\overline{x}^a\overline{y}^{b-1}-k_-d\overline{x}^c\overline{y}^{d-1})\,\delta y, \tag{17}$$

$$\delta A=RT\left(\frac{a-c}{\overline{x}}\,\delta x+\frac{b-d}{\overline{y}}\,\delta y\right). \tag{18}$$

Therefore, $\delta v\,\delta A/(RT)=F\,\delta x\,\delta y$, with the form-matrix explicitly reading

$$F=\begin{pmatrix} \dfrac{a-c}{\overline{x}}m_{11} & \dfrac{1}{2}\left(\dfrac{a-c}{\overline{x}}m_{22}+\dfrac{b-d}{\overline{y}}m_{11}\right) \\ \dfrac{1}{2}\left(\dfrac{a-c}{\overline{x}}m_{22}+\dfrac{b-d}{\overline{y}}m_{11}\right) & \dfrac{b-d}{\overline{y}}m_{22} \end{pmatrix}, \tag{19}$$

where $m_{11}=k_+a\overline{x}^{a-1}\overline{y}^b-k_-c\overline{x}^{c-1}\overline{y}^d$, $m_{22}=k_+b\overline{x}^a\overline{y}^{b-1}-k_-d\overline{x}^c\overline{y}^{d-1}$.

1. One-dimensional case

For example, for $b=d$, $a\neq c$ the problem is reduced to the one-dimensional case:

$$\mathrm{sgn}(F_{1\mathrm{D}})=\mathrm{sgn}((a-c)(aq-c\overline{x}^{c-a})). \tag{20}$$

If one side of the corresponding reaction equation in Eq. (15) is constant, i.e., for $a=0$ or $c=0$, one recognizes $F_{1\mathrm{D}}>0$. In all other cases $\mathrm{sgn}(F_{1\mathrm{D}})$ generally depends on the steady state concentration $\overline{x}$. In the special case where the steady state equals the thermodynamic equilibrium $\overline{x}^{c-a}=q$, $F_{1\mathrm{D}}$ is positive. At sufficient distance from equilibrium, however, one cannot estimate the sign of $F_{1\mathrm{D}}$ without knowledge of $\overline{x}$. Both 1×1 main minors of F in Eq. (19) are positive if and only if

$$(a=0\ \text{ or }\ c=0)\ \text{ and }\ (b=0\ \text{ or }\ d=0). \tag{21}$$

2. Two-dimensional case

The 2×2 main minor of F (19) is its determinant

$$\det(F)=-\frac{1}{4}\left(\frac{a-c}{\overline{x}}m_{22}-\frac{b-d}{\overline{y}}m_{11}\right)^2\leqslant0. \tag{22}$$

We analyze under what conditions this expression vanishes. In the two-variable case ($a\neq c$ and $b\neq d$) $\det(F)=0$ can be rewritten as

$$(bc-ad)(q\overline{x}^{a-c}\overline{y}^{b-d}-1)=0. \tag{23}$$

This expression vanishes at equilibrium $q=\overline{x}^{c-a}\overline{y}^{d-b}$, or if

$$ad=bc. \tag{24}$$

In the nonequilibrium case it follows from Eqs. (21) and (24) that the only possibility for positive semidefinite F reads

$$(a=0, \quad b=0)$$

or

$$(c=0, \quad d=0). \tag{25}$$

As can be seen already from Eqs. (17) and (18) arbitrary reactions with *n* variables and only constant reactants at one side of the reaction equation always stabilize the steady state, thus completing our analysis. Therefore, we have shown that only reactions with constant reactants at one side of the reaction equation tend to stabilize the steady state(s) of the reaction network.

III. A SIMPLE MASS-ACTION KINETIC SYSTEM WITH HOPF BIFURCATION CONSISTING ONLY OF NONAUTOCATALYTIC REACTIONS

We demonstrate that the sample system:

$$\begin{aligned}
&\mathrm{A} \xrightarrow{k_0} \mathrm{X},\\
&2\mathrm{Y} \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{Z},\\
&\mathrm{X}+\mathrm{Z} \xrightarrow{k_2} \mathrm{Y}+\mathrm{Z},\\
&\mathrm{Y} \xrightarrow{k_3} \mathrm{B},
\end{aligned} \tag{26}$$

(A and B denote constant reactants) which does not contain an autocatalytic reaction according to definition (1) and (2), nevertheless possesses for special parameter values an unstable steady state. In particular, it exhibits a supercritical Hopf bifurcation yielding stable limit cycles near the bifurcation point.

Assuming simple mass-action kinetics and upon introducing the dimensionless quantities $(x,y,z)/(k_0 a)k_3 \to (x,y,z)$, $k_3 t \to t$ the corresponding differential equation system is written in the form:

$$\begin{aligned}
&\dot{x} = 1 - k_2 x z,\\
&\dot{y} = -y - 2k_1 y^2 + 2k_{-1} z + k_2 x z,\\
&\dot{z} = k_1 y^2 - k_{-1} z.
\end{aligned} \tag{27}$$

This system has one steady state

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{k_{-1}}{k_1 k_2}, 1, \frac{k_1}{k_{-1}} \right). \tag{28}$$

Carrying out a linear stability analysis, the characteristic polynomial of the Jacobian may be written in the form

$$\lambda^3 - T\lambda^2 - K\lambda - D = 0, \tag{29}$$

where *T*, *D* indicate the trace and determinant, respectively. It follows from the Hurwitz criterion that the steady state is locally stable if the coefficients at this point fulfill the condition $T, K, D, -(TK+D) < 0$. A Hopf bifurcation occurs [with the special property that the real eigenvalue is negative (cf. Ref. 8)] if $T, K, D < 0$ and $TK + D = 0$. The calculation shows that the coefficients at the steady state (28) read:

$$\begin{aligned}
&T = -(k_1 k_2 + 5k_{-1} + k_{-1}^2)/k_{-1},\\
&K = -(k_1 k_{-1}^2 + 5k_1^2 k_2 + k_1^2 k_2 k_{-1} - 2k_{-1}^2)/(k_1 k_{-1}),\\
&D = -k_1 k_2.
\end{aligned} \tag{30}$$

For the choice that $k_1 = 1$, $k_{-1} = 1$ one obtains

$$\begin{aligned}
&T = -(6 + k_2),\\
&K = 1 - 6k_2,\\
&D = -k_2,\\
&TK + D = k_2^2 + 17/3 k_2 - 1,
\end{aligned} \tag{31}$$

i.e., the Hopf bifurcation occurs at $k_2^H := k_2 = (\sqrt{325} - 17)/6 \approx 0.1713$ (from $6k_2^H > 1$ it follows that $K < 0$). Numerical integration shows that it is supercritical bifurcation, i.e., a stable limit cycle arises, which becomes unstable if k_2 is lowered to $k_2^* \approx 0.1576$.

IV. DISCUSSION

In this work we systematically have studied the capability of the Glansdorff–Prigogine stability citerion for general statements about the stability of steady states of mass-action kinetic systems. Let us summarize the most important results.

Generally, i.e., without the restriction to mass-action kinetic systems, three points need to be emphasized

(1) Originally the concept of local stability proof by use of the *excess entropy density* has been based on the assumption of local equilibrium. This very concept can be used in a strict mathematical formulation without reference to local equilibrium. An arbitrary function may be used as a Lyapunov function to prove the global stability of the steady state contained in a domain *D* of the phase space if only this function is (positive or negative) definite in this domain. One should note that the definiteness of the underlying quadratic form at the *equilibrium* point does not mean in general the definiteness of the corresponding quadratic form at the *nonequilibrium* steady state point. Without giving here an explicit example, one can imagine systems where the equilibrium quadratic form is negative definite, but the corresponding nonequilibrium form *not*. By contrast, the function *excess entropy density* can be ideally suited to serve as a Lyapunov function for steady states, because it is (positive or negative) definite in a surrounding of this point without fulfilling this condition at the equilibrium point. (An example for nondefinite excess entropy density at the equilibrium point is the regular solutions as in the special sense, the simplest generalization of ideal solutions (cf., e.g., Ref. 1).

(2) The Glansdorff–Prigogine stability citerion in the formulation of Eqs. (5) and (6) is, although based on the idea of Lyapunov functions, a local criterion. It is therefore interesting to extend this concept for a global stability analysis. This has been discussed in Appendix B, where we have outlined a general scheme for the construction of Lyapunov functions.

(3) The Glansdorff–Prigogine stability citerion is only a sufficient, but not a necessary condition for local stability of the corresponding steady state. Nevertheless, two decades ago there was an intensive discussion about this point in the

literature (cf. Ref. 5, and citations therein). For example, Eigen obviously overestimated the validity of this criterion, when, in his famous article,[10] he stated: ''A steady state is unstable whenever a negative fluctuation $\delta_x \sigma$ occurs.'' This is equivalent to the statement that condition (7) is violated (cf. Ref. 10).

In Ref. 4 we have discussed the last two points in detail using the simple example system of Ref. 8.

At this point it is worth to refer to other efforts to extend the thermodynamic nonequilibrium stability theory to the whole nonequilibrium range. A remarkable approach has been given by Keizer in a series of papers (cf. Ref. 11, and citations therein), summarized in his monograph.[12] He proposed the ''σ function,'' a kind of generalized entropy, whose second variation is proportional to the inverse of the covariance matrix. This function serves as a Lyapunov function and is indeed for certain situations of the Gauss–Markov limit a necessary *and* sufficient condition for stability of the considered steady state. Keizer and Chang were also able to give an experimental verification for this theory.[13] Note also related work for the so termed ''stochastic potential'' (or generalized thermodynamic potential), originally introduced by Graham[14] for Fokker–Planck processes, generalized by Hänggi[15] for Markovian master equations and recently studied in related form e.g., by Ross and co-workers (for a recent publication see Ref. 16).

Discussing the possibility of a violation of condition (7), in Ref. 10 Eigen also argued that ''autocatalytic reaction systems... are the candidates for such instabilities.'' In this work we show that autocatalytic reactions are not a necessary condition for this violation.

The general result that reaction networks consisting exclusively of reactions where one side containes only constant substances are always locally stable is already known with the work of Clarke (cf. Ref. 17). He instead used for his analyses a graph theoretic approach. However, because the analyses in this work are based on Lyapunov functions, we in addition can study global stability.

In Sec. II A we have shown that for mass-action kinetic systems the function *excess entropy density* (L) always fulfills the necessary condition for a Lyapunov function to be (positive or negative) definite. In Appendix A we have proved the same result for the more general function L_{mf}. From a mathematical point of view, this result is closely related to a corresponding result given in Ref. 18, wherein exclusively equilibrium situations were addressed.

The sample system studied in Sec. III has been known explicitly at least since the paper of Cook *et al.*[19] It is constructed as an extension of the well-known trimolecular model,[20] which has been criticized, like the ''Brusselator,'' because of the assumption of its trimolecular reaction. System (27) is one of three different schemes which have been demonstrated in Ref. 19 to yield as asymptotic limit cases the original trimolecular model, whereby one of these systems still contains an autocatalytic reaction. (The authors stated that ''These three-variable schemes are of great interest in themselves as the simplest oscillatory 'real chemical' schemes involving only first- and second-order steps.'' A comparison with the system of Ref. 8 shows that this is obviously not the case.) The decomposition of the trimolecular reaction according to system (26) is only bimolecular in the forward and backward direction of all reactions. However, the stability analysis, especially the determination of the Hopf bifurcation hyperplane, has not been done in our general way.

Simple systems such as Eq. (27) are well suited for the application of local bifurcation theory (cf., e.g., Ref. 21) to analytically prove that the occurring Hopf bifurcation is indeed supercritical.

One further point is especially interesting with regard to actual studies. The stable limit cycle of the original trimolecular two-variable model becomes unstable (and the system diverges) if the velocity constant of the autocatalytic reaction is sufficiently small. If one simply adds a further intermediate variable as a storage substance, the stable limit cycle exhibits a period doubling cascade into chaos when lowering the autocatalytic reaction rate.[20(c),22] If the rate is lowered even further, the stable chaotic attractor becomes unstable and the system diverges again. In contrast, in system (27) we observed the same (two-dimensional) behavior than in the original two-variable model.[20] Of course the period doubling and the chaotic region in the parameter space could be so narrow that we did not find it numerically, but it seems that the trajectories of the whole three-dimensional model are in fact confined to a two-dimensional surface in the long time limit. Thus, system (27) is a good candidate for applying ''no-chaos theorems'' in similar form as recently presented in Ref. 23.

ACKNOWLEDGMENTS

This work has been supported by the *Graduiertenkolleg* ''Nichtlineare Probleme in Analysis, Geometrie und Physik'' (GRK 283) financed by the DFG and the state of Bavaria.

APPENDIX A: PROOF OF THE POSITIVE DEFINITENESS OF A FUNCTION CONSTRUCTED USING MOLE NUMBERS INSTEAD OF CONCENTRATIONS

If one uses the chemical potential of reactant X_i as a function of the mole fraction x_i (note that in this appendix x_i is used for the mole fraction of reactant X_i and *not* for its concentration) instead of the concentration c_i [like in Eq. (9)], one obtains another function for the excess entropy density. This can be used as a Lyapunov function if it fulfills the condition of definiteness. We here prove that this condition is always fulfilled, because the form matrix of the according quadratic form is positive definite. With

$$\mu_i = \mu_i^0 + RT \ln x_i . \tag{A1}$$

(with the mole fraction $x_i = c_i / (\Sigma_i c_i + c_c)$) one obtains the function

$$L_{\mathrm{mf}} := \sum_{i,k} (\partial \mu_i / \partial c_k)_{\mathrm{ss}} \delta c_i \delta c_k > 0. \tag{A2}$$

Here, c_c denotes the concentration of the constant substances. If c_c is very large, L_{mf} becomes identical to L of Eq. (8). The form matrix F of this quadratic form can be written as

$$F=\frac{1}{(\Sigma_i c_i+c_c)\Pi_i c_i}\tilde{F} \tag{A3}$$

with

$$\overline{F}=\begin{pmatrix} \Pi_{i=2}^{n}c_i(\Sigma_{i=2}^{n}c_i+c_c) & -\Pi_i c_i & \cdots & -\Pi_i c_i \\ -\Pi_i c_i & c_1\Pi_{i=3}^{n}c_i(c_1+\Sigma_{i=3}^{n}c_i+c_c) & \cdots & -\Pi_i c_i \\ \vdots & \vdots & \ddots & \vdots \\ -\Pi_i c_i & -\Pi_i c_i & \cdots & \Pi_{i=1}^{n-1}c_i(\Sigma_{i=1}^{n-1}c_i+c_c) \end{pmatrix}. \tag{A4}$$

This real symmetric matrix is positive definite if—after the triangle decomposition $\tilde{F}=CB$, where without loss of generality we can set $c_{ii}=1$ for all i—all diagonal elements b_{ii} are larger than zero (cf. Ref. 24). One immediately recognizes $b_{11}>0$. Applying the Gaussian algorithm to $\tilde{F}$ to generate the upper triangular matrix B, after elimination of the first row and the first column the new matrix $\tilde{F}_1$ reads

$$\tilde{F}_1=\frac{(\Sigma_i c_i+c_c)c_1}{\Sigma_{i=2}^{n}c_i}\tilde{F}_1 \tag{A5}$$

with

$$\tilde{\tilde{F}}_1=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \Pi_{i=3}^{n}c_i(\Sigma_{i=3}^{n}c_i+c_c) & \cdots & -\Pi_{i=2}^{n}c_i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -\Pi_{i=2}^{n}c_i & \cdots & \Pi_{i=2}^{n-1}c_i(\Sigma_{i=2}^{n-1}c_i+c_c) \end{pmatrix}. \tag{A6}$$

One now recognizes that $b_{22}>0$ and that this likewise works for all remaining $n-2$ steps. In each step, one c_i is eliminated and the new appearing b_{ii} is larger than zero which finishes the proof.

It follows that the function L_{mf} is, like the function L (8), for the ideal solute systems under consideration always a suited Lyapunov function, because it fulfills the necessary condition of definiteness.

In Ref. 4 we have shown that in the considered sample system the region of the parameter space for which the (local) stability of the steady state can be proven with the help of the Lyapunov function L_{mf} is generally smaller than the region for which the (local) stability can be proven with L.

APPENDIX B: CONSTRUCTION OF DIFFERENT LYAPUNOV FUNCTIONS ON THE BASIS OF THE GLANSDORFF–PRIGOGINE CRITERION

We demonstrate a simple procedure for the construction of Lyapunov functions for proving the global stability of steady states in a domain of the parameter space containing the equilibrium point. For the sake of simplicity we confine ourselves to the simple case of thermal and mechanical stable homogeneous systems.

According to the Glansdorff–Prigogine theory the equilibrium of these systems is locally stable if condition (4) is fulfilled. With the assumption of stable local equilibrium these authors derived the inequality (7) being a condition for local stability of the considered steady state. Because this derivation is based on the idea of using the excess entropy density as a Lyapunov function, it is natural to study with this function global stability as well. As we have shown in Sec. II A (and in Appendix A for the more general function L_{mf}), the function

$$L_{\mathrm{eed}}:=\sum_i \frac{(c_i-\overline{c_i})^2}{\overline{c_i}} \tag{B1}$$

is positive definite, and therefore well suited to serve as a Lyapunov function. L_{eed} is the function L (8) with finite deviations Δc_i instead of infinitesimal deviations δc_i.

Starting from

$$(\delta^2 s)_{\mathrm{ss}}\propto\sum_i (\delta\mu_i\delta c_i)_{\mathrm{ss}}, \tag{B2}$$

we show that related Lyapunov functions can be obtained. The function L_{eed} follows directly from Eq. (B2) if one uses for the variation of the concentrations the finite deviation from the steady state Δc_i. If, in addition, one uses for the chemical potentials the finite deviation $\Delta\mu_i$, one obtains

$$L_{\mathrm{fd}}=\sum_i (c_i-\overline{c_i})\ln\frac{c_i}{\overline{c_i}}. \tag{B3}$$

This function is positive definite (for $c_i\neq\overline{c_i}$) and therefore suited to serve as a Lyapunov function.

A third related function can be obtained from the relation

$$\frac{1}{2}\frac{\partial}{\partial t}\delta^2 s=-\sum_i \delta\left(\frac{\mu_i}{T}\right)\frac{\partial}{\partial t}\delta c_i\geqslant 0, \tag{B4}$$

which one obtains in deriving Eq. (7) (cf. Ref. 1). If one uses instead of the first variation for both, the chemical potential and the concentration, the finite deviation Δ, the time derivative of this third function reads: $dL_{\text{Shear}}/dt=\sum_i \ln(c_i/\overline{c_i})\dot{c}_i$, yielding

$$L_{\text{Shear}}=\sum_i c_i \ln\frac{c_i}{\overline{c_i}}-c_i+\overline{c_i}, \tag{B5}$$

which is positive definite. This is the well-known Lyapunov function for chemical reaction systems used first by Shear, who deduced the expression from the Boltzmann H-Theorem.

Both functions L_{fd} and L_{Shear} prove the local stability of the considered steady state exactly for the same region in the parameter space as the function L_{eed}, which follows by expansion of the logarithm of these functions. However, in Ref. 4 we have shown that L_{Shear} is better suited for proving the global stability. Indeed, in Ref. 25 it has been shown that with the help of L_{Shear} the global stability of the thermodynamical equilibrium point can be proven in the whole phase space for arbitrary generalized mass-action kinetic systems.

Further related Lyapunov functions follow from L_{fd} and L_{Shear} if one again takes the chemical potential as a function of the mole fraction as outlined for L_{mf} in Appendix A.

This idea for the construction of different Lyapunov functions is interesting also with respect to more general systems than only thermal and mechanical stable homogeneous ones, for which more general expressions of the excess entropy density, e.g., $1/2\,\delta^2 s=\delta(1/T)\,\delta u+\delta(p/T)\,\delta v-\delta(\mu_i/T)\,\delta c_i$, exist.

[1] P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structure, Stability, and Fluctuations* (Wiley, London, 1971).

[2] G. Nicolis and I. Prigogine, *Self-organization in Nonequilibrium Systems* (Wiley, New York, 1977).

[3] I. Prigogine and R. Lefever, J. Chem. Phys. **48**, 1695 (1968).

[4] T. Wilhelm, S. Schuster, and R. Heinrich, Nonlinear World **4**, 295 (1997).

[5] R. F. Fox, Proc. Natl. Acad. Sci. USA **77**, 3763 (1980).

[6] A. J. Lotka, J. Am. Chem. Soc. **42**, 1595 (1920). V. Volterra, Nature (London) **118**, 558 (1926).

[7] R. J. Field and R. M. Noyes, J. Chem. Phys. **60**, 1877 (1974).

[8] T. Wilhelm and R. Heinrich, J. Math. Chem. **17**, 1 (1995).

[9] M. Eiswirth, A. Freund, and J. Ross, Adv. Chem. Phys. **80**, 127 (1991).

[10] M. Eigen, Naturwissenschaften **58**, 465 (1971).

[11] J. Keizer, J. Chem. Phys. **65**, 4431 (1976).

[12] J. Keizer, *Statistical Thermodynamics of Nonequilibrium Processes* (Springer, New York, 1987).

[13] J. Keizer and O.-K. Chang, J. Chem. Phys. **87**, 4064 (1987).

[14] R. Graham, in *Stochastic Nonlinear Systems*, edited by L. Arnold and R. Lefever (Springer, New York, 1981), p. 202; R. Graham, in *Stationary Nonequilibrium Systems with Applications to Lasers and Nonlinear Optics*, Springer Tracts in Modern Physics Vol. 66 (Springer, Berlin, 1973).

[15] P. Hänggi, Phys. Rev. A **25**, 1130 (1982).

[16] W. Vance and J. Ross, J. Chem. Phys. **108**, 2088 (1998).

[17] B. L. Clarke, Adv. Chem. Phys. **43**, 1 (1980).

[18] W. R. Smith and R. W. Missen, *Chemical Reaction Equilibrium Analysis: Theory and Algorithms* (Krieger, Malabar, 1991).

[19] G. B. Cook, P. Gray, D. G. Knapp, and S. K. Scott, J. Phys. Chem. **93**, 2749 (1989).

[20] This model, being more simple than the ''Brusselator,'' constitutes the ''simplest'' two-component reaction system with limit cycle oscillations and is known as (a) Selkov system [E. E. Selkov, Eur. J. Biochem. **4**, 70 (1968)], (b) Schnakenberg system [J. Schnakenberg, J. Theor. Biol. **81**, 389 (1979)], or (c) Gray–Scott system (cf. K. Alhumaizi and R. Aris, *Surveying a Dynamical System: A study of the Gray-Scott Reaction in a Two-phase Reactor* (Longmans, New York, 1995)].

[21] T. Wilhelm and R. Heinrich, J. Math.Chem. **19**, 111 (1996).

[22] J. Wolf and R. Heinrich, BioSystems **43**, 1 (1997).

[23] R. Eichhorn, S. J. Linz, and P. Hänggi, Phys. Rev. E **58**, 7151 (1998).

[24] J. H. Wilkinson, *The Algebraic Eigenvalue Problem* (Clarendon, Oxford, 1978).

[25] S. Schuster and R. Schuster, J. Math. Chem. **3**, 25 (1989).