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SURFACE FLOW OF ROTATED GRANULAR SYSTEMS AND THE IMPACT OF MACROMECHANICAL STOCHASTICS

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Recent molecular dynamical simulations by Dury and Ristow provide important insights into the dynamics and stochastic aspects of granular flow in rotated drums. We show that their major findings can be explained with a recently proposed deterministic minimal model for granular surface flow that also includes Langevin forces which mimic micromechanically generated stochastics.

1 Introduction

A characteristic property of granular systems such as sand, powder or grains, is that they can flow if they are driven out of their static equilibrium by shear or vibration. The flow that granular systems develop, however, is quite distinct of that of Newtonian fluids such as water. This fact can be exemplified by the avalanching flow along granular piles and the surface flow in rotated drums (cf. the authoritative reviews by Jaeger *et al*[1] and Wolf[2]). In the latter example, a drum that is partly filled with granular material is rotated with a *constant* rotation rate about its horizontally aligned axis of symmetry. Depending on the magnitude of the rotation rate ϖ, two different types of dynamics have been observed experimentally by Rajchenbach[3] (i) for small ϖ, sequences of discrete avalanches that start at the maximum angle of repose φ_s, stop at the minimal angle of repose φ_r, followed by a rigid pile rotation until φ_s has been reached again, and (ii) for larger ϖ, a continuous surface flow with a mean inclination angle of the pile that increases proportional to the square of ϖ for small differences of the angles of repose, $\varphi_s - \varphi_r$. Recently, Dury and Ristow [4] have performed important molecular dynamical simulations on granular flow in rotated drums. Their system differs from the experiment of Rajchenbach in two respects: (i) the difference between maximum and minimum angle of repose for $\varpi \to 0$, $\varphi_s - \varphi_r \simeq 12^o$, is considerably larger than in the experiment,[3] and (ii) the number of grains in the drum is very small in comparison to this experiment. They found that (i) the continuous flow that develops for larger ϖ is not constant in time but fluctuates considerably about a mean value $< \varphi(t) >$, (ii) this mean value $< \varphi(t) >$ does not increase quadratically with ϖ but increases – in the continuous flow range – almost linearly for small ϖ and then crosses over to a weaker increase for larger ϖ, and (iii) the transition from avalanching to continuous flow does not occur at a sharply determined ϖ–value. Based on a stochastic extension of a recently proposed model for surface flow of granular systems, we show that the simulation results [4] can be explained in all major details within this model. Moreover, we discuss the apparent discrepancy of the results[4] by Dury and Ristow and the experiment[3] by Rajchenbach.

2 The Model

In order to discuss the numerical results of Dury and Ristow [4] from a theoretical point of view, we use a stochastic extension of the recently proposed *minimal model*[5,6] for granular surface flow which is able to successfully model a whole variety of granular flow aspects.[5,6] In this model, the surface flow is described by two macromechanical variables, the spatially averaged inclination angle $\varphi(t)$ of the surface of the pile and the characteristic velocity $v(t)$ which is basically given by the square root of the kinetic energy of the flow. The model equations read

$$\dot{v} = [g \sin\varphi - g k_d(v) \cos\varphi + \tilde{\zeta}(t)]\chi(\varphi, v) \tag{1}$$
$$\dot{\varphi} = -av + \overline{\omega} \tag{2}$$

with the friction coefficient given by

$$k_d(v) = b_0 + b_2 v^2, \tag{3}$$

and the cutoff function $\chi(\varphi, v) = \Theta(v) + \Theta(\varphi - \varphi_s) - \Theta(v)\Theta(\varphi - \varphi_s)$. Here, $\Theta(y)$ denotes Heaviside's step function $[\Theta(y) = 0\ (1)$ if $y \leq 0\ (y > 0)]$, a, b_0 (being related to φ_d, see below) and b_2 are positive constants, g is the gravitational acceleration, and $\overline{\omega}$ denotes the external (constant) rotation rate of the drum. The stochasticity enters through the Langevin force $\tilde{\zeta}(t)$ being Gaussian white noise with zero mean and correlation strength $\tilde{\Delta}^2$. This mimics the micromechanically generated fluctuations that are present in moving particulate matter. The deterministic part [5,6] of (1) and (2) extends Coulomb's theory of frictional motion to granular surface flow by incorporating viscoplastic facts of the dynamical nature of avalanche/surface motion of granular systems: The latter are (i) a nonlinear dynamical friction coefficient $k_d(v)$ which interpolates between solid friction and the frictious behavior of rapid granular flow ("Bagnold" friction [2]), (ii) the fact that a granular pile is statically stable until the inclination angle φ exceeds the maximum angle of repose φ_s, (iii) the fact that a surface flow $v(t)$ is always directed downward the pile and stops if $v(t)$ reaches zero, and (iv) the fact that a surface flow $v(t) \neq 0$ leads to dynamical changes of the inclination angle φ which counteract the acceleration of the surface flow. The facts (ii) and (iii) are mimicked by the cutoff function χ. Next, it proves useful to non-dimensionalize (1) and (2) by scaling the velocity with $\sqrt{g/a}$ and the time with $1/\sqrt{ga}$. Moreover, we introduce the dimensionless rotation rate $\omega = \overline{\omega}/\sqrt{ga}$ and dimensionless Langevin "force" $\zeta(t) = \tilde{\zeta}(t)/g$. Then, we finally obtain the model equations

$$\dot{v} = \left[\cos\varphi\left(\tan\varphi - tan\varphi_d - \delta\Omega_0^2 v^2\right) + \zeta(t)\right]\chi(\varphi, v) \tag{4}$$
$$\dot{\varphi} = -v + \omega \tag{5}$$

which we want to study in the following. In (4) we have also introduced (i) $\varphi_d = \arctan b_0 < \varphi_s$ being basically the half of the sum of maximum and minimum angle of repose, $\varphi_d = \frac{1}{2}(\varphi_s + \varphi_r)$, (ii) $\Omega_0^2 = 1/\cos\varphi_d$, and (iii) $\delta = (b_2 g/a)\cos\varphi_d$. Note that this model contains only two effective parameters, a and b_2, that describe the deterministic flow and only one stochastic parameter $\Delta = \tilde{\Delta}/g$ which measures

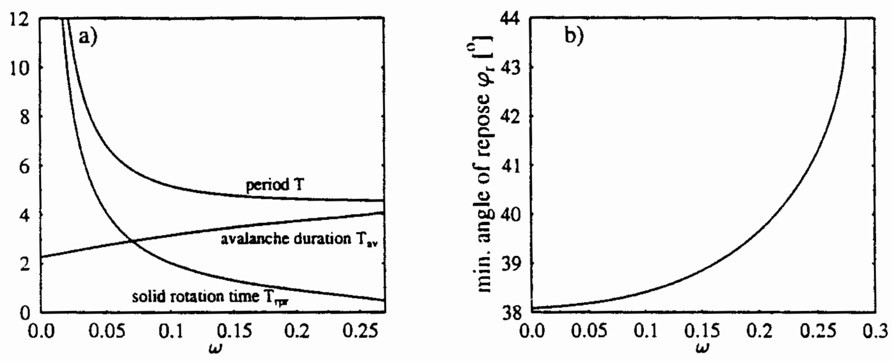

Figure 1: Dependence of (a) the avalanche duration T_{av}, the duration of the rigid pile rotation T_{rpr}, and its overall period $T = T_{av} + T_{rpr}$. (b) The minimum angle of repose φ_r versus the rotation rate ω. The parameters are $\varphi_s = 0.87$ (50^o), $\varphi_d = 0.77$ (44^o), $\Omega_0^2 = 1/\cos\varphi_d = 1.4$ as in the numerical simulation of Dury and Ristow. The nonlinearity parameter δ is 0.1.

the strength of the fluctuating acceleration forces. In experiments, the difference between maximum and minimum angle of repose, $\varphi_s - \varphi_r$, is generically only a few angular degrees. Then, a small angle approximation about $\varphi_d = \arctan b_0$ as used in our previous work[5] leads to further simplifications. To compare with the numerical results of Dury and Ristow[4] where $\varphi_s - \varphi_r \simeq 12^o$, such an approximation is no longer justified.

3 No Fluctuations

To start, we discuss the dynamics of Eqs. 4 and 5 without the Langevin force, i.e. $\zeta(t) = 0$. Similarly as in our previous work,[5] the model (4) and (5) possesses two types of dynamics.

(i) For sufficiently small ω, the dynamics consists of alternations of avalanches and rigid pile rotations. The avalanches start with zero initial velocity at the maximum angle of repose φ_s and stop at the minimum angle of repose φ_r after the avalanche duration T_{av}. Then, the whole pile rotates as rigid body without surface flow until φ_s has been reached again and this lasts $T_{rpr} = (\varphi_s - \varphi_r)/\omega$. The sum of these two processes defines the period T of the successive avalanching process. In Fig. 1 we show the dependence of T_{av}, T_{rpr}, T, and φ_r on the rotation rate ω.

(ii) For larger ω, the surface flow $v(t)$ once started at φ_s stays positive for all times and saturates in a continuous flow with constant velocity and surface inclination. The long time evolution of the continuous flow solution is given by the fixed points of Eqs. 4 and 5,

$$v_{CF} = \omega \quad \text{and} \quad \varphi_{CF} = \varphi_d + \arctan\left(\frac{\delta\omega^2}{\Omega_0^2 + \delta\omega^2 \tan\varphi_d}\right). \tag{6}$$

From (6), it follows that the inclination angle of the surface for the continuous

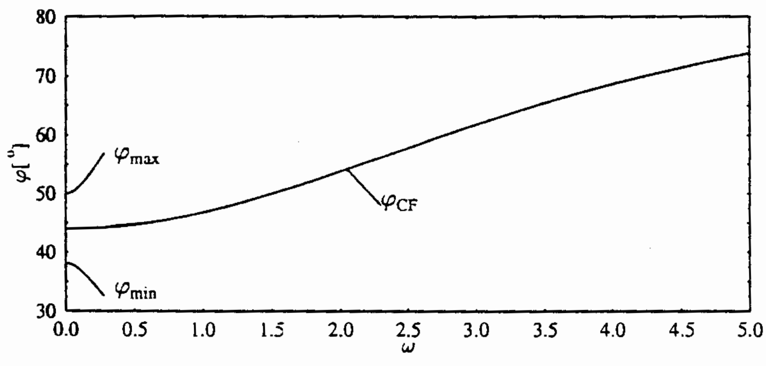

Figure 2: Dependence of φ_{max}, φ_{min} and φ_{CF} on the rotation rate ω. All parameters have the same values as in Fig 1. The transition to a continuous flow occurs at $\omega_T \simeq 0.35$.

flow, φ_{CF}, increases quadratically with ω only if ω is sufficiently small. Since δ is typically of the order 10^{-1} and Ω_0^2 of order 1, the curvature δ/Ω_0^2 is comparably small. For larger ω, φ_{CF} crosses over to a much weaker increase with ω (cf. also Fig 2). In order to distinct between the ω-ranges where successive avalanching or continuous flow are present, one can take advantage of the following fact: Avalanches can be characterized by the minimum and maximum dynamical angles φ_{max} and φ_{min} during the avalanching process. For non-zero ω, $\varphi_{max} > \varphi_s$ and $\varphi_{min} < \varphi_r$ holds. In Fig. 2, we show the dependence of φ_{max}, φ_{min} and φ_{CF} on the external rotation rate ω. For small ω, there is successive avalanching and φ_{max} (φ_{min}) increases (decreases) quadratically with ω. The point where φ_{max} and φ_{min} cease to exist, determines the transition point ω_T to continuous flow. Above ω_T, there is a continuous flow and the increase of φ_{CF} is close to a linear dependence on ω first. Therefore, we conclude that Rajchenbach's results[3] and Dury and Ristow's results[4] are both compatible with our model depending on the ω-range that is studied.

4 Impact of Fluctuations

Now we turn to the impact of a non-zero Langevin force in the model equations (4) and (5). Clearly, the inclusion of non-zero fluctuations $\zeta(t)$ leads to permanent fluctuations of the inclination angle $\varphi(t)$ and the velocity $v(t)$ while flowing. These fluctuations superpose the deterministic dynamics discussed in the previous section. For small enough $\Delta = \tilde{\Delta}/g$, however, the basic deterministic dynamics will survive in an averaged sense. There is still successive avalanching present for small enough ω and for larger ω, and there is still a continuous flow with a mean inclination angle $< \varphi(t) > \simeq \varphi_{CF}$ and a mean velocity $< v(t) >= \omega$. In Fig. 3, we show numerical calculations for the dependence of the averaged inclination angle $< \varphi(t) >$ on the rotation rate ω for the whole range of ω-values and the averages of the maximum and minimal dynamical angles, $< \varphi_{max} >$ and $< \varphi_{min} >$. In comparison with the

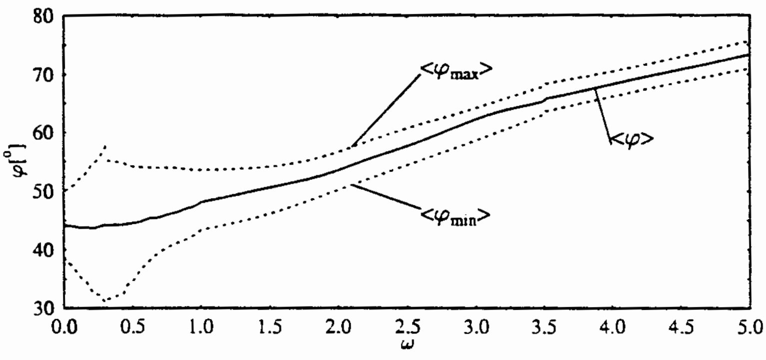

Figure 3: Dependence of $<\varphi_{max}>$, $<\varphi_{min}>$ and $<\varphi>$ on the rotation rate ω for $\Delta = 0.05$. All other parameters have the same values as in Fig 1.

deterministic limit in Fig. 2, several facts are remarkable: (i) For small ω, $<\varphi_{max}>$ and $<\varphi_{min}>$ increase – as in the deterministic case – quadratically with ω. For the chosen fluctuation strength $\Delta = 0.05$, $<\varphi_{max}>$ and $<\varphi_{min}>$ assume values that are basically the same as their deterministic limits. (ii) Close to the deterministic transition point ω_T, there is a drastic change of the behavior of $<\varphi_{max}>$ and $<\varphi_{min}>$ as function of ω. This characterizes the transition from successive avalanching to continuous flow and is slightly larger than ω_T in the deterministic limit. Our numerical calculations show that this transition is not truly sharp. In a very narrow vicinity about this transition, the dynamics alternates intermittently between avalanching and continuous flow. Therefore, the location of the transition slightly depends on the criterion used to distinguish between avalanching and continuous flow. In our calculations, avalanching means that the dynamics shows at least 200 successive avalanches. (iii) Above this discontinuous transition, $<\varphi_{max}>$ ($<\varphi_{min}>$) decreases (increases) towards the averaged inclination angle $<\varphi(t)>$ first, and for larger ω, the difference $<\varphi_{max}> - <\varphi_{min}>$ of the distribution of the inclination angle $\varphi(t)$ is almost constant. These results agree very well with the finding of Dury and Ristow [4] (cf. their Fig. 4). As an aside, we have also investigated whether the transition between avalanching and continuous flow is hysteretic or not. For the parameters used in Fig. 3, we have adiabatically increased and subsequently decreased the rotation rate ω and have found that – as an effect of the Langevin force – there is no hysteresis observable.

5 Conclusions

Our main result is as follows. An extension of the deterministic minimal model [5] that includes Langevin forces shows – as function of the external rotation rate ω – a transition scenario from avalanching to continuous flow which has a striking similarity with the recent numerical simulations of Dury and Ristow.[4] Although most of averaged quantities considered here are close to their deterministic limits, the inclusion of stochastics seems to be an important aspect of the granular surface dynamics even on a macromechanical level. In subsequent publications,[7,8] we will elaborate in greater detail on the role of macromechanical fluctuations for the granular dynamics.

Acknowledgments

We thank R. Bartussek and P. Reimann for helpful discussions on the stochastic aspects of the problem.

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