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THE INFLUENCE OF NON-LINEAR DISSIPATION ON NON-EQUILIBRIUM QUANTUM TUNNELING PROCESSES

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The influence that linear and non-linear coupling to a heat bath has on the dynamics of the tunneling of a particle out of a meta-stable state is examined. The tunneling rate is calculated using the dilute bounce gas method, for finite temperatures. The action corresponding to the bounce is evaluated with the aid of a variational method. It is found that the linear coupling to the heat bath has the effect of decreasing the characteristic cross-over temperature between the regime of thermally activated hopping and the quantum tunneling regime. The linear coupling is found to suppress the decay rate in the quantum tunneling regime, and causes energy to be dissipated. The non-linear coupling has the effect of removing the normal modes of the heat bath to higher energies, and thus reduces the effectiveness of the linear coupling, and thereby increases the tunneling rate.

1 Introduction

The phenomenon of quantal decay of metastable states appears in a myriad of different context in Physics[1-6] and Chemistry[7-9], and has been the subject of continuous investigation since the advent of quantum mechanics. It has long been recognized that the coupling of the tunneling system to the degrees of freedom of its environment does have a significant influence on the decay rates. The early literature on these phenomena have recognized the existence of two temperature regimes in which the decay process has different characteristics[10], namely a low temperature tunneling regime[11] and a high temperature thermally activated hopping regime[12]. The description of quantum mechanical tunneling in terms of path integrals allows both regimes to be given a general and unified treatment. Feynman and Vernon [13] showed that the heat baths described by normal modes which couple linearly to the system can be traced out. The system is then governed by an effective action which contains a term non-local in time; the influence functional. This non-local term represents the influence of the heat bath on the systems motion, and depends upon the spectral properties of the environmental coupling and on the temperature.

The simplest way to obtain an analytical approximation to the path integral is to use the quasi-classical or WKB approximation[14-16], in which it is assumed that the path integral is dominated by the trajectories in the vicinity of those for which the action is stationary. In the quasi-classical approximation, the action associated with each extremal trajectory gives rise to an exponential term in the decay rate, and the neighboring trajectories which deviate from the extremal trajectories only by small amplitude fluctuations give rise to the prefactors. The dominant contribution to the quantum decay rate are due to extremal trajectories which traverse the barrier region with imaginary velocities. These imaginary velocity extremal trajectories, when transcribed in terms of imaginary times are the instanton or bounce trajectories popularized by Coleman[17-18]. Langer[19] has shown that the activated hopping rates can also be expressed in terms of the imaginary part of the free energy of the meta-stable state and has evaluated this in terms of extremal trajectories that traverse the barrier. Caldeira and Leggett[20] have advocated this type of approach to tunneling phenomena, and stressed the importance of the exponential suppression of the quantal tunneling rate by the non-local or dissipative parts of the effective action. This method has been subsumed by the method of periodic orbits[21]. It must be emphasized that this functional integral approach to tunneling reduces the calculation of the decay rate, at all temperatures, to that of the evaluation of a single functional integral in the imaginary time domain.

Caldeira and Legett[20] have indicated how the zero temperature limit of the quantum tunneling rates can be evaluated by this technique. Grabert, Weiss and Hanggi[22-23] have extended this method to low temperatures, and found the leading low temperature corrections to the zero temperature quantum decay rate. However, in this low temperature regime, explicit analytic expressions to the quantum decay rate have been found in a few select exactly soluble cases, such as zero damping[24], weak damping[25-26], an intermediate value of the damping strength[27] and, finally, for an asymptotically large value of the damping strength[20]. Where no exact solution can be found, the scheme can be implemented numerically[28-29], or by using variational methods for both the exponents[30-31] and the prefactor[32], which give excellent agreement with the numerical results. These results continue smoothly from the low temperature quantum tunneling regime into the high temperature thermally activated hopping regime.

In the high temperature regime, the decay rates are of a more universal nature, and don't depend upon the specific details of the bounce solution. For example, Grabert and Weiss[33], and Larkin and Ovchinnikov[34] were able to explicitly calculated the general form of the decay rate in the vicinity where the decay rate process changes from quantum tunneling to thermally activated hopping. Similarly the decay rate in the high temperature region has also been expressed in a general form[35-37].

This technique and many of its applications to systems with linear couplings to heat baths, have been reviewed and compared with the results of other methods in the book by Weiss[38] and also in the review article by Hanggi, Borovec and Talkner[39]. The assumption of linear coupling to the degrees of freedom of the environment is an idealization, made to make the elimination of the heat bath normal coordinates tractable. It is based on the assumption of weak coupling to each individual normal mode of the heat bath, in which case only the first term in a Taylor series expansion in powers of the normal coordinates need be retained. However, when the effect of the coupling to the heat bath is large, the effects of higher order coupling may become important in some cases[40]. Also the non-linear terms may provide the leading effects of the dissipation, in cases where the linear terms are forbidden by symmetry[41]. Therefore, in this article we shall investigate the effects that the non-linear couplings to a heat bath have on the decay rate.

2 Model

The microscopic system and its environment is modeled by a Hamiltonian containing three terms,

$$H = H_s + H_r + H_{r-s} \qquad (1),$$

where H_s describes the one dimensional motion of a particle in a potential $V(q)$, and H_r describes the dynamics of the thermal reservoir and H_{r-s} describes the coupling between the system and its environment. The Hamiltonian for the system is given by

$$H_s = \frac{p^2}{2M} + V(q) \qquad (1a)$$

where p and q are, respectively canonically conjugates operators representing the particle's momenta and coordinates. The Hamiltonian describing the thermal reservoir, H_r, can be written as

$$H_r = \sum_\alpha \left[-\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x_\alpha^2}) + \frac{m}{2}\omega_\alpha^2 x_\alpha^2 \right] \qquad (1b),$$

where the sum over α ranges over all the normal modes of the oscillators and x_α and ω_α are, respectively, the normal mode coordinates and frequency. Thus, each normal mode describes an Einstein oscillator with frequency ω_α. The interaction term that couples the particle with the thermal reservoir, H_{r-s}, is written as

$$H_{r-s} = \sum_{\alpha} \left[A_{\alpha} q x_{\alpha} + B_{\alpha} q^2 x_{\alpha}^2 \right] \qquad (1c),$$

The first term represents the usual bilinear term coupling of the particle's corrdinate to the coordinate x_{α} of the α-th normal mode. The second term is the non-linear coupling term. The latter is quadratic in q, as opposed to being linear, so that in the absence of asymetry in the potential $V(q)$ the total system of particle plus heat bath are invariant under spatial inversions. This ensures that any parity violation is solely due to $V(q)$ and not due to the coupling with the heat bath. The second term may be regarded as producing a position dependent change in the oscillators frequency from ω_{α} to $\omega_{\alpha}(q)$, where

$$\omega_{\alpha}(q)^2 = \omega_{\alpha}^2 + \frac{2}{m} B_{\alpha} q^2 \qquad (2),$$

where the coefficient B_{α} must be positive for the heat bath to be stable for all values of q. The linear term, then has the effect of displacing the equilibrium value of the normal mode coordinates from 0 to a_{α} where

$$a_{\alpha} = A_{\alpha} q / [m \omega_{\alpha}^2 + 2 B_{\alpha} q^2] \qquad (3).$$

With few exceptions[40−43], the heat bath coupling is assumed to be bi-linear, where $B_{\alpha} = 0$. This corresponds to the assumption that any single degree of freedom of the environment is only weakly perturbed by the particle's motion. This, of course, does not imply that the accummulative influence of all the degrees of freedom of the environment on the particle's motion is weak.

Since we are interested in the effects of the dissipation on quantum tunneling, we shall separate out the dynamic effects from the static effects of the coupling to the environment by adding a counter-term $\Delta V(q)$ to the potential, which takes the zero temperature limiting form of

$$\Delta V(q) = \frac{1}{2} \sum_{\alpha} \left[\frac{A_{\alpha}^2 q^2}{[m \omega_{\alpha}^2 + 2 B_{\alpha} q^2]} - \hbar \omega_{\alpha}(q) + \hbar \omega_{\alpha} \right] \qquad (4),$$

This addition has the effect that the total potential experienced by the particle is temperature independent.

3 Formulation

The path integral for the reduced density matrix can be written in terms of an effective action $S[q(\tau)]$, where τ is an imaginary time. the path integral is to be evaluated over all trajectories $q(\tau)$, which satisfy the periodic boundary conditions $q(\tau+\Theta) = q(\tau)$ where Θ is an inverse temperature $\Theta = \frac{\hbar}{k_B T}$. The effective action is given, to leading order in the non-linear dissipation, as

$$S[q(\tau)] = \int_{-\Theta/2}^{\Theta/2} d\tau \Big[\frac{m\dot{q}(\tau)^2}{2} + V(q(\tau))\Big]$$

$$+\frac{1}{2}\int_{-\Theta/2}^{\Theta/2} d\tau \int_{-\Theta/2}^{\Theta/2} d\tau' K(\tau-\tau')\Big[q(\tau) - q(\tau')\Big]^2$$

$$+\frac{1}{2}\int_{-\Theta/2}^{\Theta/2} d\tau \int_{-\Theta/2}^{\Theta/2} d\tau' \int_{-\Theta/2}^{\Theta/2} d\tau" L_2(\tau'-\tau,\tau"-\tau) q(\tau)^2 \Big[q(\tau')q(\tau") - q(\tau)^2\Big]^2$$

$$+\int_{-\Theta/2}^{\Theta/2} d\tau \int_{-\Theta/2}^{\Theta/2} d\tau' M(\tau-\tau')\Big[q(\tau)^2 - q(\tau')^2\Big]^2$$

$$+\cdots\cdots \qquad (5),$$

where $K(\tau)$ represents the kernel for the usual linear damping mechanism and is given by

$$K(\tau) = \frac{1}{2\pi}\int_0^{\infty} d\omega J(\omega)\Big[\big[1+N(\omega)\big]exp(-\omega|\tau|) + N(\omega)exp(+\omega|\tau|)\Big] \qquad (6.a),$$

in which $N(\omega)$ is the Bose-Einstein distribution function and $J(\omega)$ is the spectrum of the environmental coupling,

$$J(\omega) = \frac{\pi}{2}\sum_{\alpha}\Big[\delta(\omega-\omega_\alpha)\frac{A_\alpha^2}{m\omega_\alpha}\Big] \qquad (6.b).$$

The non-linear dissipative terms have the kernels $L_2(\tau',\tau")$ and $M(\tau)$. The double time kernel $L_2(\tau',\tau")$ is given as

$$L_2(\tau',\tau") = \frac{1}{2\pi}\int_0^{\infty} d\omega I_1(\omega)\Big[\big[1+N(\omega)\big]exp(-\omega|\tau'|) + N(\omega)exp(+\omega|\tau'|)\Big]$$

$$\times\Big[[1+N(\omega)]exp(-\omega|\tau''|)+N(\omega)exp(+\omega|\tau''|)\Big] \qquad (7.a),$$

and the coupling spectrum $I_n(\omega)$ is given by

$$I_n(\omega)=\frac{\pi}{2n!}\sum_\alpha\Big[\delta(\omega-\omega_\alpha)\frac{B_\alpha^n A_\alpha^2}{(m\omega_\alpha)^{n+1}}\Big] \qquad (7.b).$$

Likewise, the single time kernel $M(\tau)$ is given by

$$M(\tau)=\frac{1}{2\pi}\int_0^\infty d\omega H(\omega)\Big[[1+N(\omega)]exp(-\omega|\tau|)+N(\omega)exp(+\omega|\tau|)\Big]^2 \qquad (8.a),$$

in which the spectral coupling is

$$H(\omega)=\frac{\pi}{2}\sum_\alpha\Big[\delta(\omega-\omega_\alpha)\frac{B_\alpha^2}{(m\omega_\alpha)^2}\Big] \qquad (8.b).$$

It can be clearly seen that $M(\tau)$ is of higher in powers of $\hbar$ than $L_2(\tau',\tau'')$, and can be neglected within the WKB approximation. However, the multi-time kernels, containing $I_n(\omega)$, are of the same order in $\hbar$ as those explicitly dispayed here. In the following we shall assume that the effective potential $V(q)$ can be approximated by the general form

$$V(q)=\frac{M}{2}\omega_0^2q^2\Big[1-(\frac{q}{\Delta q})^n\Big] \qquad (9),$$

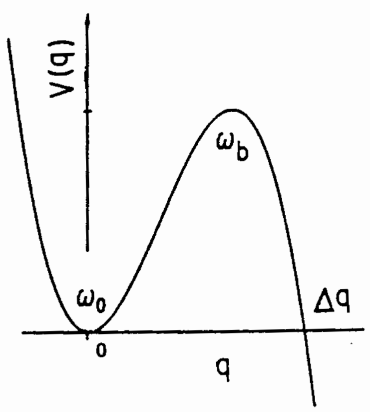

Figure 1: The form of the potential $V(q)$, with a metastable minimum at $q=0$, a curvature at the minimum of $M\omega_0^2$. The barrier regime extends between $q=0$ and $q=\Delta q$.

where the meta-stable minimum is located at $q=0$, and in the absence of dissipation the classical forbidden region extends from $q=0$ to $q=\Delta q$. This potential is shown in figure 1. We shall also assume that the linear dissipation spectral density is of the ohmic form

$$J(\omega) = M\eta\omega \qquad (10).$$

In evaluating the effects of the non-linear dissipation we shall assume that for small ω_α , B_α scales with $(\frac{m\omega_\alpha^2}{2\Delta q^2})$, as suggested by the form of (2). Hence, we shall write

$$I_1(\omega) = \lambda M\eta\Big(\frac{\omega}{\Delta q}\Big)^2 \qquad (11),$$

where λ is a dimensionless measure of the relative strength of the non-linear dissipation. With this particular form of scaling, the terms originating from $I_n(\omega)$ are of the order of λ^n.

4 Tunneling Rate

The probability for decay from a meta-stable state can be characterized by a decay rate, at times sufficiently long so the transients have died away and yet sufficiently short so that the decay can be approximated by an exponential[44]. In the dilute bounce gas approximation, the reduced density matrix of the meta-stable system is given by the weighted sum over a set of extremal paths called bounce trajectories, together with the quantum fluctuations around these trajectories. A typical bounce trajectory is shown in figure 2. The multi-bounce trajectory start from the meta-stable minimum, near $q = 0$ at time $\tau = -\Theta/2$, and cross the barrier region any even number of times, returning to the starting point at a later time $\tau = +\Theta/2$. In the dilute bounce gas approximation the sum of the contributions from all the multi-bounce trajectories exponentiates, and gives rise to an imaginary part to the free energy. This approximation neglects any interactions between the bounces and any possible interference from paths surrounding different extremal trajectories [33–34].

Under the above conditions, the decay rate can be expressed as

$$\Gamma = \Big(\frac{S_K}{2\pi\hbar}\Big)^{\frac{1}{2}}\Big(\frac{D_0}{D'_B}\Big)^{\frac{1}{2}} exp\big[(S_0 - S_B)/\hbar\big] \qquad (12),$$

in which S_0 and S_B are, respectively, the values of the action representing the equilibrium solution $q_0(\tau) = 0$ and the single bounce trajectory $q_B(\tau)$. The factors D_0 and D_B represent the products of eigenvalues of $\frac{\delta S^2}{\delta q^2}$ evaluated about these trajectories. The prime in the contribution from the single bounce trajectory indicates that the zero eigenvalue has to be omitted and this is replaced by the zero mode normalization factor proportional to S_k. For a derivation of this process see refs.[27,45]. The factor S_k is given by

$$S_k = \int_{-\Theta/2}^{+\Theta/2} d\tau \frac{M}{2} \dot{q}_B(\tau)^2 \qquad (13).$$

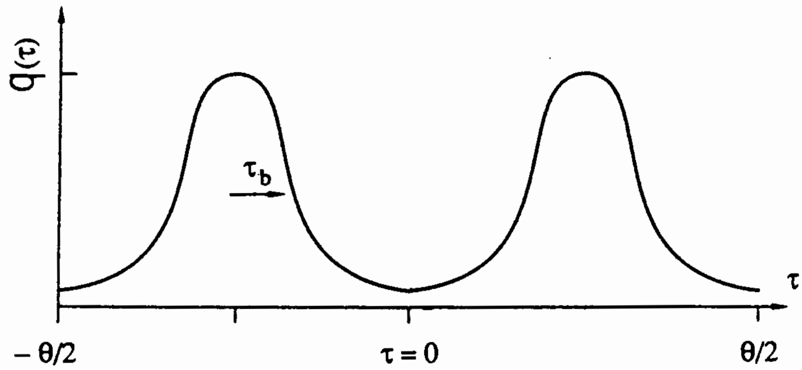

Figure 2: A typical multi-bounce trajectory $q_{nB}(\tau)$. The bounce time τ_B is the time for one traversal of the barrier region.

The bounce trajectory $q_B(\tau)$ extremalizes the action, and so is given by the solution of the Euler-Lagrange equation vbox

$$-M\ddot{q}(\tau) + \left.\frac{\partial V}{\partial q}\right|_{q_B(\tau)} + 2 \int_{-\Theta/2}^{+\Theta/2} d\tau' K(\tau - \tau')\left[q(\tau) - q(\tau')\right]$$

$$+2 \int_{-\Theta/2}^{+\Theta/2} d\tau' \int_{-\Theta/2}^{+\Theta/2} d\tau" L_2(\tau - \tau', \tau - \tau")q(\tau)\left[q(\tau')q(\tau") - 2q(\tau)^2\right]$$

$$+2 \int_{-\Theta/2}^{+\Theta/2} d\tau' \int_{-\Theta/2}^{+\Theta/2} d\tau" L_2(\tau' - \tau, \tau' - \tau")\left[q(\tau')q(\tau")^2\right]$$

$$+2 \int_{-\Theta/2}^{+\Theta/2} d\tau' M(\tau - \tau')q(\tau)\left[q(\tau)^2 - q(\tau')^2\right] = 0 \qquad (14),$$

subject to the periodic boundary conditions.

5 The Bounce Tajectory

We solve the Euler-Lagrange equation in a variational approximation[29-30] using the ansatz,

$$q(\tau) = \frac{a}{\left[1 - b\cos(\frac{2\pi\tau}{\Theta})\right]} \tag{15}$$

in which a and b are variational parameters. The acceptable physical range for these parameters is $a > 0$ and $1 > b > -1$. The choice of ansatz for the bounce trajectory, represented by (15), is motivated by the following considerations; (i) it is periodic in Θ, (ii) it reduces to the exact trajectory in the vicnity of the cross-over temperature $T_0 > T$ [29,33-34], (iii) it reduces to the asymptotically exact bounce trajectory in the limit $\alpha \to \infty$ [20,46].

On substituting the trial bounce trajectory into the expression for the action, we find the expression

$$S_B = \pi M\omega_0 \frac{a^2}{(1-b^2)}\left(\frac{\Theta\omega_0}{2\pi}\right)\left[\frac{1}{2}\left(\frac{2\pi}{\Theta\omega_0}\right)^2 \frac{b^2}{(1-b^2)^{\frac{3}{2}}}\right.$$

$$+\frac{1}{(1-b^2)^{\frac{1}{2}}} - \left[\frac{a}{\Delta q}\frac{1}{\sqrt{(1-b^2)}}\right]^n P_{n+1}\left(\frac{1}{\sqrt{(1-b^2)}}\right)$$

$$+\alpha\left(\frac{2\pi}{\Theta\omega_0}\right)\frac{b^2}{(1-b^2)}$$

$$\left.-4\lambda\alpha\left(\frac{2\pi}{\Theta\omega_0}\right)\left[\frac{a}{\Delta q}\frac{1}{\sqrt{(1-b^2)}}\right]^2 F\left(\frac{1}{\sqrt{(1-b^2)}}\right)\right] \tag{16}$$

In this expression $P_n(x)$ is the n-th order Legendre Polynomial and $F(x)$ is given by the expression,

$$F(x) = \sum_{n,n'}\left[\frac{(x+1)}{(x-1)}\right]^{(|n|+|n'|+|n+n'|)/2}\left[|n+n'|+x\right]\left[|n|+|n'| - \frac{|nn'|}{(|n|+|n'|)}\right] \tag{17}$$

where the sum runs over all positive and negative values of n and n'. The first term in eqn.(16) represents the contribution from the action due to the kinetic energy, the second and third term represent the contributions from the potential energy $V(q)$, and the last two terms, respectively, represent the action of the linear and the non-linear dissipation. We have neglected terms of order $\hbar$, originating from $M(\tau - \tau')$.

Clearly, the form of the action simplifies when expressed in terms of the parameterization, where

$$y = \frac{(\frac{a}{\Delta q})}{\sqrt{(1-b^2)}}$$

and

$$x = \frac{1}{\sqrt{(1-b^2)}} \qquad (18).$$

On minimizing the action with respect to a and b we obtain the set of simultaneous equations,

$$\Bigg[(\frac{2\pi}{\Theta\omega_0})^2\frac{x(3x^2-1)}{2} + \alpha(\frac{2\pi}{\Theta\omega_0})2x - 4\lambda\alpha(\frac{2\pi}{\Theta\omega_0})y^2F'(x)$$

$$+1 - y^n P'_{n+1}(x)\Bigg] = 0 \qquad (19.a),$$

and

$$\Bigg[(\frac{2\pi}{\Theta\omega_0})^2\frac{x(x^2-1)}{2} + \alpha(\frac{2\pi}{\Theta\omega_0})(x^2-1) - 8\lambda\alpha(\frac{2\pi}{\Theta\omega_0})y^2F(x)$$

$$+x - y^n\frac{n+2}{2}P_{n+1}(x)\Bigg] = 0 \qquad (19.b),$$

which determines a and b. For temperatures less than the cross-over temperature, T_0, these equations posses three solutions. They are the solution representing the meta-stable equilibrium $q_0(\tau) = 0$ which corresponds to $y = 0$. The solution corresponding to the unstable equilibrium is $q_T(\tau) = (\frac{2}{2+n})^{(\frac{1}{n})}\Delta q$, ie., $y = (\frac{2}{2+n})^{(\frac{1}{n})}$, $x = 1$. The bounce trajectory $q_B(\tau)$ is the non-trivial solution where $y \neq 0$ and $x \neq 1$.

At the cross-over temperature, T_0, the bounce solution coalesces with the unstable equilibrium solution. Thus, at this temperature, the bounce action becomes equal to the Arrhenius factor from the top of the potential barrier[27,33-34],

$$S_B = \frac{n}{2^{1+\frac{2}{n}}(n+2)^{1-\frac{2}{n}}}\Theta_0 M\omega_0^2\Delta q^2 \qquad (20).$$

The condition for the two solutions to become degenerate determines the cross-over temperature, which is found as

$$(\frac{2\pi k_B T}{\hbar\omega_0}) = (\frac{2\pi}{\Theta_0\omega_0})$$

$$= \left[\alpha^2\left(1 - 11\lambda\left(\frac{2}{(n+2)}\right)^{\frac{2}{n}}\right)^2 + n\right]^{\frac{1}{2}} - \alpha\left[1 - 11\lambda\left(\frac{2}{(n+2)}\right)^{\frac{2}{n}}\right] \qquad (21).$$

Without dissipation, the cross-over temperature is simply given in terms of the curvature of the potential at the barrier top[13]. The effect of the linear dissipation is such as to decrease the cross-over temperature from the non-dissipative value[46]. On the other hand, the non-linear dissipation has the opposite effect of the linear term.

Below the cross-over temperature, the bounce action decreases and thus quantum tunneling gives rise to the dominant contribution to the decay rate. In the limit of zero temperature, $T = 0$ or $\Theta \to \infty$, the solution for the bounce trajectory takes on the limiting form $y \to 0$ and $x \to \infty$, where the product xy is held constant. Under these limiting conditions, the solution becomes more apparent when the trial trajectory is re-parameterized in terms of

$$c = \frac{\left(\frac{a}{\Delta q}\right)}{(1-b)}$$

and

$$\tau_B = \left(\frac{\Theta}{\pi}\right)\left[\frac{1-b}{2b}\right]^{\frac{1}{2}} \qquad (22),$$

as c is a dimensionless measure of the amplitude of the bounce and τ_B is the bounce time. The bounce time is the characteristic imaginary time scale that the extremal trajectory spends in the classically forbidden regime. The bounce time should not be identified with the traversal time for a wave-packet to tunnel across the barrier region[47], as direct studies have shown that at the instant the maximum of the wave packet reaches the barrier, the transmitted packet emerges [48].

In this low temperature limit, the simultaneous equations can be reduced to,

$$\frac{1}{2^4} = \left(\frac{\omega_0\tau_B}{2}\right)^2\left[\frac{1}{2} - \frac{(2n+2)!}{2^{2(n+1)}(n+1)!^2}c^n\right] \qquad (23.a),$$

and

$$\left(\frac{\omega_0\tau_B}{2}\right)\left[\frac{n+4}{2}\frac{(2n+2)!}{2^{2(n+1)}(n+1)!^2}c^n - 1\right] = \frac{\alpha}{4}\left[1 - 2\lambda\left(4\ln 2 + \frac{7}{24}\right)c^2\right] \qquad (23.b).$$

In this limit, the bounce action reduces to the form

$$S_B = \pi M\omega\Delta q^2 c^2\left[\frac{1}{2^4}\left(\frac{\omega_0\tau_B}{2}\right)^{-1} + \left(\frac{\omega_0\tau_B}{2}\right)\left[\frac{1}{2} - \frac{(2n+2)!}{2^{2(n+1)}(n+1)!^2}c^n\right]\right.$$

$$+\frac{\alpha}{4}\left[1-\lambda(4\ln 2+\frac{7}{24})c^2\right]\Big] \qquad (24).$$

From eqn(23.a) it is clear that τ_B determines the amplitude of the bounce. The pair of simultaneous equations, (23), can be solved analytically for the case of zero non-linear damping, yielding

$$(\frac{\omega_0\tau_B}{2}) = \frac{1}{2n}\left[(\alpha^2+\frac{n(n+4)}{2})^{\frac{1}{2}}+\alpha\right] \qquad (25.a),$$

for the bounce time and on re-writing (23.a) as

$$\frac{(2n+1)!}{2^{2n}n!(n+1)!}c^n = \left[1-\frac{1}{2^3}(\frac{\omega_0\tau_B}{2})^{-2}\right] \qquad (25.b),$$

the amplitude follows imeadiately. One sees that the bounce time increases with increasing dissipation, and hence the termination point of the bounce moves to larger q values. This corresponds to the extremal trajectory "slowing down" and experiencing an "energy loss" ΔE during the traversal, due to the effect of the dissipation[49]. The maximal energy loss occurs in the limit, $\alpha \rightarrow \infty$, and has the value

$$\Delta E = M\omega_0^2\Delta q^2 2^3\left[\frac{2(n+1)!^2}{(2n+2)!}\right]^{\frac{2}{n}}\left[\frac{2^{(2n+1)}(n+1)!^2}{(2n+2)!}-1\right] \qquad (26).$$

The effect of the non-linear dissipation can be inferred from eqn(25.b), which shows the bounce time is decreased on increasing λ. Since eqn(25.b) is valid for all values of λ, one can see that the exit point is decreased and the energy loss is reduced by increasing λ from zero to finite values. A similar, but more tedious, analysis shows that the bounce action is decreased on increasing λ. This is consistent with the effect of λ on the action at the cross-over temperature, as represented by eqns.(21) and (22). Hence, we can safely conclude that the non-linear dissipation, consistently, effectively reduces the strength of the linear dissipations.

6 Conclusions

The above analysis was performed in the regime where the WKB approximation is valid. This roughly corresponds to the regime where the bounce action S_B exceeds $\hbar\omega_0$ and presents no serious limitation. An estimate of the error involved in the use of the variational method can be found by comparing exact analytic[20,24–26] and numerical[28,29] results for the case of linear dissipation.

The largest discrepancy occurs, for zero temperature, in the limit $\alpha \rightarrow 0$, where the exact bounce solution is found to be

$$q(\tau) = \Delta q \ \mathrm{sech}^{\frac{2}{n}}\left(\frac{n\omega_0\tau}{2}\right) \qquad (27),$$

and the corresponding value of the action is

$$S_B = M\omega_0 \Delta q^2 \frac{2^{\frac{4}{n}}\Gamma^2(\frac{2}{n})}{(n+4)\Gamma(\frac{4}{n})} \qquad (28),$$

where $\Gamma(z)$ is the gamma function. Comparison with the results of the variational method show that the error is limited to be less than 5.96 %, which occurs when $n = 1$. For finite values of α the error is rapidly diminished, and therefore since λ only plays a role for finite α our conclusions are quite reliable.

The effect of the non-linear part of the dissipation is seen to reduce the effect of the linear dissipation, in that it causes both the cross-over temperature and the low temperature tunneling rate to increase back towards their undamped values. The effect may be attributed to the fact that the non-linear coupling increases the frequency of the thermal reservoir's normal modes, as seen in eqn(2). The resulting shift of the oscillators from smaller frequencies to higher frequencies can be expected to yield a reduced effective dissipation strength. The conclusion that the non-linear dissipation causes an increase inthe tunneling rate can be seen more directly from eqn(3), which clearly shows that the effect of the non-linear coupling is to reduce the distortion of the heat bath caused by the linear coupling. The reduction in the tunneling rate due to the linear dissipation is due to a polaronic effect, whereby the moving particle has its' effective mass increased to include the distortion of the heat bath[11-12]. Hence, by eqn(3), the non-linear dissipation reduces the distortions of the heat bath and thereby increases the tunneling rate towards its' undamped value.

Our results also indicate that it is the bounce time τ_B that is the relevant time scale which determines the interaction between the interaction between the particle's tunneling motion and the heat bath's normal modes and it is this time scale which determines the difference between the bounce amplitude and the dissipationless exit point Δq. Hence, it is the bounce time which determines the energy of the tunneling particle that is lost to the heat bath.

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