27 May 1996

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 215 (1996) 26–31

Brownian motors driven by temperature oscillations

P. Reimann [1], R. Bartussek, R. Häußler, P. Hänggi

Department of Physics, University of Augsburg, Memminger Straße 6, D-86135 Augsburg, Germany

Received 26 January 1996; accepted for publication 26 February 1996
Communicated by C.R. Doering

Abstract

We study directed motion of a Brownian particle in a periodic "ratchet"-potential due to a periodically oscillating temperature of the thermal environment. Precise numerical results are compared against analytical approximations for asymptotically slow and fast temperature oscillations. This "diffusion-ratchet" tends to resist carrying a current for slow and fast temperature modulations, while showing a maximal current at moderate frequencies.

PACS: 05.40.+j; 87.10.+e; 82.20.Mj
Keywords: Brownian motor; Ratchet; Non-equilibrium fluctuations

1. Introduction and model

Directed Brownian motion induced by non-equilibrium noise in the absence of macroscopic forces and potential gradients is presently under intense investigation [1]. While in thermal equilibrium such an effect is ruled out by the second law of thermodynamics, in the non-equilibrium case it apparently can be realized always by a suitably tailored "Brownian motor" [2,3]. Besides interesting technological applications like novel mass separation methods, this transport mechanism may also be of relevance for intracellular processes [4].

In this letter we study a one-dimensional overdamped Brownian particle in a periodic potential, $V(x + L) = V(x)$, subject to thermal fluctuations with a periodically modulated temperature,

$$\dot{x}(t) = -V'(x(t)) + \sqrt{2D(t)}\,\xi(t), \qquad (1)$$

where $D(t + \mathcal{T}) = D(t)$ and $D(t) > 0$ for all t. The friction coefficient has been absorbed into the time scale, $D(t)$ and $\mathcal{T} = 2\pi/\omega$ stand for the correspondingly rescaled temperature and period, respectively, and $\xi(t)$ is Gaussian white noise of zero mean and variance $\langle \xi(t)\,\xi(s)\rangle = \delta(t - s)$. This model will be valid under the (very weak) assumption of local (or accompanying) thermal equilibrium, i.e., provided the thermalization of the immediate environment of the Brownian particle is much faster than the time scale $\mathcal{T}$ of the temperature oscillations. In practice, these oscillations may be realized by periodically adding and extracting heat, or alternatively, e.g., by modulating the pressure. From a different point of view, one may also consider $\xi(t)$ in (1) as an externally imposed Gaussian white noise of oscillating intensity $D(t)$.

[1] Present address: Eötvös University, Puskin-u. 5–7, H-1088 Budapest, Hungary.

0375-9601/96/$12.00
PII S0375-9601(96)00222-8

2. General properties

The probability density of the noisy dynamics (1) is governed by the Fokker–Planck equation

$$\frac{\partial}{\partial t}P(x,t) + \frac{\partial}{\partial x}j(x,t) = 0, \tag{2}$$

$$j(x,t) := -\left(V'(x) + D(t)\frac{\partial}{\partial x}\right)P(x,t). \tag{3}$$

Restricting ourselves to the long time limit, the solution $P(x,t)$ will be periodic in time and space, $P(x+L,t) = P(x,t+\mathcal{T}) = P(x,t)$, and is conveniently normalized on the unit interval, $\int_0^L P(x,t)\,dx = 1$. The quantity of central interest is the time-averaged probability current

$$J := \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} j(x,t)\,dt, \tag{4}$$

where the x-independence of J can be readily verified by exploiting (2) and $P(x,t+\mathcal{T}) = P(x,t)$. Note that the average particle velocity $\langle\dot{x}\rangle := \lim_{t\to\infty}(1/t)\int_0^t\langle\dot{x}(s)\rangle\,ds$ is related to the probability current (4) according to [2]

$$\langle\dot{x}\rangle = LJ. \tag{5}$$

Obviously, a non-vanishing current J is only possible for a periodic potential $V(x)$ with broken spatial symmetry ("ratchet"). Even then, in the fast oscillation limit $\omega = 2\pi/\mathcal{T} \to \infty$, the Brownian particle (1) will behave like in the presence of a time-averaged constant temperature. Since this corresponds to thermal equilibrium, we conclude that $J \to 0$ when $\omega \to \infty$. Similarly, in the adiabatic limit $\omega \to 0$ each particle experiences a quenched realization of the temperature, yielding again $J \to 0$. It is therefore not obvious at first glance whether directed motion $J \neq 0$ can be generated *at all* by our "diffusion ratchet" (1). This is in clear contrast to a similar model, but with a *spatial* rather than a *temporal* periodic temperature, where $J \neq 0$ is a trivial consequence of the so-called "blowtorch" effect [5].

[2] Proof: Exploiting (1) one can see that $\langle\dot{x}\rangle = (1/\mathcal{T}) \times \int_0^{\mathcal{T}} dt \int_0^L dx\, P(x,t)\,[-V'(x)]$. With (3) and (4) it then follows that $\langle\dot{x}\rangle - LJ = (1/\mathcal{T})\int_0^{\mathcal{T}} dt \int_0^L dx\, D(t)\,\partial P(x,t)/\partial x = 0$.

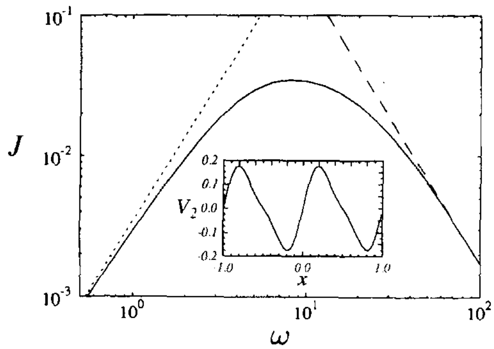

Fig. 1. Probability current J versus oscillation frequency ω for the "diffusion ratchet" (1), (6), (7) at the parameter values $D_0 = 0.1$ and $A = 0.7$. The solid line shows numerical results from a matrix continued fraction calculation and the dashed line is the theoretical large-ω asymptotics (16). The dotted straight line of slope 2 corresponds to the ω^2-asymptotics for small ω, predicted in Section 4.3. The used ratchet potential (6) is plotted in the inset.

If the periodic potential and the oscillating temperature can be expressed in terms of a few Fourier modes, one can evaluate the current J very efficiently and with high accuracy by means of matrix continued fraction techniques, see e.g. in Ref. [6] (note the applications) and further references therein. As an example we choose

$$V(x) = V_2(x) := [\sin(2\pi x) + 0.25\sin(4\pi x)]/2\pi, \tag{6}$$

$$D(t) = D_0[1 + A\sin(\omega t)]^2. \tag{7}$$

In Figs. 1 and 2 the numerically determined current J is plotted for various representative values of the parameters ω, A, and D_0 in (7). In agreement with our prediction, $J(\omega)$ vanishes for asymptotically small and large ω, but is definitely non-zero inbetween, with a pronounced maximum at an A- and D_0-dependent ω-value. For fixed D_0, this maximal current is monotonically increasing with $|A|$. On the other hand, for a fixed parameter A, the maximal current is obviously zero for $D_0 = 0$, reaches an absolute maximum at a D_0-value comparable to the barrier height $\max_x V(x) - \min_x V(x)$, and, not unexpected, vanishes again for $D_0 \to \infty$. For weak noise D_0, the location of the current maximum $J(\omega)$ depends only weakly on A, cf. Fig. 2a. For moderate-to-large noise D_0, this maximum shifts to larger ω-values with de-

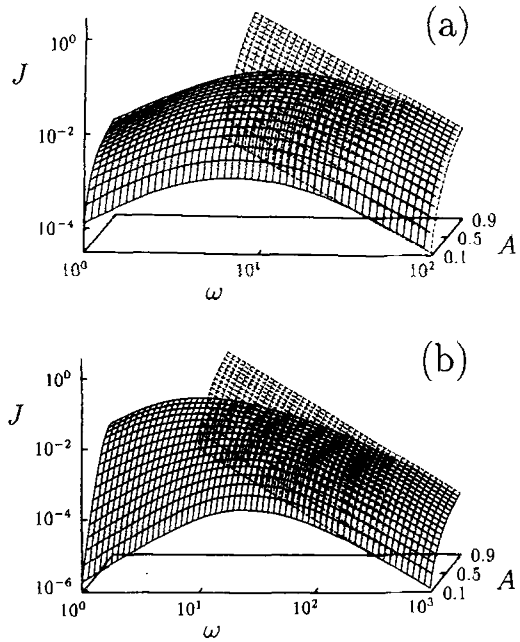

Fig. 2. The probability current J for the "diffusion ratchet" (1), (6), (7) as a function of the temperature modulation frequency ω and amplitude A at different noise intensities $D_0 = 0.1$ (a) and $D_0 = 0.5$ (b). Numerical matrix continued fraction results (solid) are compared with the fast modulation theory in (16) (dashed).

creasing A-values, cf. Fig. 2b. A natural question, which is difficult to answer numerically and thus represents an analytical challenge, is whether the current will have the same sign for all finite ω-values whatever ratchet potential $V(x)$ and diffusion coefficient $D(t)$ is chosen, or not.

3. Transformation into a flashing ratchet

One can get rid of the unusual time-dependent diffusion coefficient in the Langevin equation (1) by means of the common transformation $y(t) := x(t)\,[D(t)]^{-1/2}$, yielding

$$\dot{y}(t) = -\frac{\partial}{\partial y}\hat{V}(y(t),t) + \sqrt{2}\,\xi(t), \tag{8}$$

$$\hat{V}(y,t) := \frac{V(yD(t)^{1/2}) - \dot{D}(t)\,y^2/4}{D(t)}. \tag{9}$$

Unfortunately, the transformed potential (9) now depends periodically on time and, worse, is not even periodic in space anymore. Hence, we are not able to draw any useful conclusion from (8). However, prominent insight is gained by the time transformation [7]

$$\hat{t}(t) := \int_0^t D(s)\,\mathrm{d}s, \tag{10}$$

which has a well-defined inverse $t(\hat{t})$ due to the positivity of $D(t)$. One readily sees that this transformation is equivalent to considering the noisy dynamics

$$\dot{x}(\hat{t}) = -\frac{V'(x(\hat{t}))}{D(t(\hat{t}))} + \sqrt{2}\,\xi(\hat{t}), \tag{11}$$

instead of (1). Moreover, one finds that the averaged currents J are equal for the processes (1) and (11). In particular, one recovers $J \to 0$ for $\omega \to 0$ and $\omega \to \infty$ from (11). The dynamics (11) describes a Brownian particle in a fixed thermal environment but with a spatially homogeneous time-periodic modulation of the ratchet potential, i.e., a variant of the so-called "flashing ratchet" or "fluctuating potential ratchet" [3]. While this equivalence of (1) and (11) considerably extends the scope of our present work, it does not lead to significant simplifications and therefore is not used in the subsequent analytical approximations.

4. Asymptotic analysis

After a couple of unsuccessful attempts it becomes apparent that a closed analytical solution of (2)–(4), like for most non-equilibrium problems, is probably impossible. In the following we focus on an asymptotic analysis for fast and slow oscillations. For convenience we will work throughout this section with the rescaled time $\tilde{t} := t\omega$ and the correspondingly rescaled $\tilde{D}(\tilde{t}) := D(t(\tilde{t}))$ and $W_\omega(x,\tilde{t}) := P(x,t(\tilde{t}))$, where $t(\tilde{t}) := \tilde{t}/\omega$. Finally, we use for this new time $\tilde{t}$ again the previous symbol t. Hence, $\tilde{D}(t)$ is now a 2π-periodic, ω-independent function given by

$$\tilde{D}(t) = D(t/\omega), \tag{12}$$

while $W_\omega(x,t) = P(x,t/\omega)$ is also 2π-periodic in time but still parametrically ω-dependent.

4.1. Fast oscillations

For fast oscillations $\omega \gg 1$ we expand the rescaled density $W_\omega(x,t)$ with respect to the parameter ω as $W_0(x,t)+\omega^{-1}W_1(x,t)+\ldots$. For physical reasons it is clear that in the limit $\omega=\infty$ the density $W_\omega(x,t)$ and therefore $W_0(x,t)$ is time-independent, $W_0(x,t)=W_0(x)$. Further, the normalization and periodic boundary conditions on $W_\omega(x,t)$ imply $W_i(x+L,t)=W_i(x,t+2\pi)=W_i(x,t)$ and $\int_0^L W_i(x,t)\,\mathrm{d}x=\delta_{i,0}$, where $i\geqslant 0$ and $\delta_{i,j}$ is the Kronecker delta. From the Fokker–Planck equations (2), (3) the functions $W_i(x,t)$ can now be readily determined by comparing the coefficients of equal order ω^{-i}. We briefly exemplify the typical line of reasoning for the order ω^0-terms. The corresponding coefficients in the Fokker–Planck equation (2) lead to the identity

$$\frac{\partial W_1(x,t)}{\partial t}-\Delta(t)\frac{\partial^2 W_0(x)}{\partial x^2} = \frac{\partial}{\partial x}\left(V'(x)+\bar{D}\frac{\partial}{\partial x}\right)W_0(x), \qquad (13)$$

where $\bar{D}$ and $\Delta(t)$ denote the averaged and the oscillating parts of $\tilde{D}(t)$,

$$\bar{D}:=\frac{1}{2\pi}\int_0^{2\pi}\tilde{D}(t)\,\mathrm{d}t, \quad \Delta(t):=\tilde{D}(t)-\bar{D}. \qquad (14)$$

Clearly, both sides of (13) are equal to a time-independent, spatially periodic function. Hence, $[V'(x)+\bar{D}\partial/\partial x]W_0(x)$ must be equal to an unknown function $f(x)$ satisfying $f(x+L)=f(x)$. Upon integration this yields

$$W_0(x)=\mathrm{e}^{-V(x)/\bar{D}}\left(C+\int_0^x \mathrm{d}y\,\mathrm{e}^{V(y)/\bar{D}}f(y)/\bar{D}\right), \qquad (15)$$

where C is an integration constant. Exploiting the x-independence of the time-averaged current (4) one can conclude that $f(x)$ must be a constant. Further, $W_0(x+L)=W_0(x)$ implies $f(x)\equiv 0$ and C is fixed by the normalization condition $\int_0^L W_0(x)\,\mathrm{d}x=1$. For the current (4) this finally yields $J=0$ in leading order ω. Proceeding in the same way up to the next order ω^1 still gives a zero result for the current! The first non-vanishing contribution is obtained in order ω^2 and reads

$$J=\frac{2\int_0^{2\pi}\mathrm{d}t\,[\int_0^t \mathrm{d}s\,\Delta(s)]^2\int_0^L \mathrm{d}x\,V'(x)\,V''(x)^2}{\omega^2\,\bar{D}^2\,\pi\int_0^L \mathrm{d}x\,\mathrm{e}^{V(x)/\bar{D}}\int_0^L \mathrm{d}x\,\mathrm{e}^{-V(x)/\bar{D}}} + \mathrm{O}(\omega^{-3}). \qquad (16)$$

As expected, the current vanishes for $\bar{D}\to 0$ and $\bar{D}\to\infty$ as well as for potentials $V(x)$ with spatial inversion symmetry. For small $\bar{D}$ the integral $\int_0^L \mathrm{d}x\,\mathrm{e}^{V(x)/\bar{D}}\int_0^L \mathrm{d}x\,\mathrm{e}^{-V(x)/\bar{D}}$ becomes comparable to the inverse transition rates between adjacent potential minima of $V(x)$ and, together with ω^{-2}, dominates the magnitude of the current J. The invariance under $\omega\mapsto-\omega$ is not obvious and indeed is broken in the higher-order terms $\mathrm{O}(\omega^{-3})$ unless this symmetry is already present in the underlying dynamics (1). The integral $\int_0^{2\pi}\mathrm{d}t\,[\int_0^t \mathrm{d}s\,\Delta(s)]^2$ alludes to an effective time correlation of $\tilde{D}(t)$ and, surprisingly, the factor $\int_0^L \mathrm{d}x\,V'(x)\,V''(x)^2$ also arises in the completely different context of a "correlation ratchet" [8]. Yet, a reasonably convincing intuitive explanation of these terms seems impossible to us. The comparison of (16) with the numerical results in Figs. 1–3 is excellent.

4.2. Current inversion

The *sign* of $\int_0^L \mathrm{d}x\,V'(x)\,V''(x)^2$ and thus of J in (16) is positive for the specific ratchet potential $V_2(x)$ from (6), but may be negative for other examples like

$$V(x)=V_3(x):=\{\sin(2\pi x)+0.2\sin[4\pi(x-0.45)] + 0.1\sin[6\pi(x-0.45)]\}/2\pi, \qquad (17)$$

see the inset of Fig. 3. By continuously deforming the potential $V_2(x)$ into $V_3(x)$ it follows that *there must exist certain $V(x)=\hat{V}(x)$ for which J changes sign as a function of ω*, where we tacitly excluded the highly ungeneric possibility that J identically vanishes for all ω. An example for such a potential $\hat{V}(x)$ is $V_3(x)$ itself, see Fig. 3. Recalling that in (1) we already absorbed the friction coefficient of the Brownian particle into the time scale, it follows that in the original (unscaled) system, particles with different friction coefficients will move in opposite directions for the same properly chosen ratchet potential and the same periodically varying thermal environment. Similar conclusions apply for a periodically oscillating ratchet and a

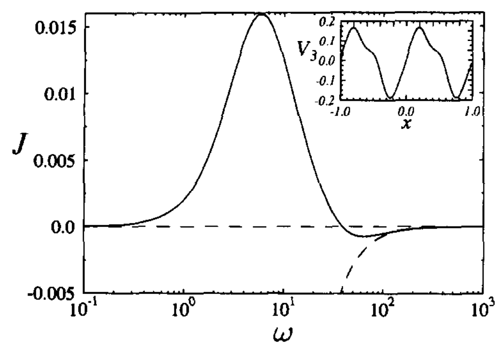

Fig. 3. Same as Fig. 1, but for the ratchet potential $V_3(x)$ from (17). Note that the current inversion is not a consequence of the slight extra "shoulder" of $V_3(x)$ as compared to $V_2(x)$ from Fig. 1, but rather of the proper interplay of $V'(x)$ and $V''(x)$ in (16).

fixed thermal environment, cf. (10) and (11). Interesting applications of this effect like friction-sensitive separation methods for Brownian objects seem likely.

4.3. Slow oscillations

Finally, we briefly turn to the analytic treatment of the slow oscillation limit $\omega \to 0$. By a similar line of reasoning as for $\omega \to \infty$ one can rewrite $W_\omega(x,t)$ under the form $W_0(x,t) + \omega W_1(x,t) + \ldots$ and by comparing powers of ω in the Fokker–Planck equation (2) one finds for the current (4) that

$$J = \frac{\omega}{2\pi} \int_0^{2\pi} dt \frac{\dot{\tilde{D}}(t)}{\tilde{D}(t)^2} \times \left[\langle \Theta(x-y)\, V(y) \rangle_t - \langle \Theta(x-y) \rangle_t \langle V(y) \rangle_t \right] + O(\omega^2), \tag{18}$$

where $\Theta(x)$ denotes the Heavyside step function and the time-dependent average $\langle f(x,y) \rangle_t$ of a function $f(x,y)$ is defined as

$$\langle f(x,y) \rangle_t := \frac{\int_0^L dx \int_0^L dy\, f(x,y) \exp\{[V(x) - V(y)]/\tilde{D}(t)\}}{\int_0^L dx \int_0^L dy\, \exp\{[V(x) - V(y)]/\tilde{D}(t)\}}. \tag{19}$$

The leading-order contribution in (18) vanishes if the dynamics is invariant under spatial inversion $V(x) \mapsto V(-x)$ but also if time-inversion symmetry $\omega \mapsto -\omega$ is respected, as for instance for the example (7). In the latter case one has to proceed to second-order perturbation theory. Though the calculations are straightforward, the resulting expression is quite lengthy but not very illuminating and is therefore not given here. We, however, observe that the numerical results from Fig. 1 indeed show the expected ω^2-asymptotics for small ω-values.

5. Conclusions

We studied the overdamped Brownian motion in a one-dimensional ratchet potential in the presence of an oscillating temperature. By means of matrix continued fraction methods we obtained accurate numerical results for the current in the long time limit. For asymptotically fast and slow oscillations we derived analytical approximations which compare very well with the numerics. In particular, we demonstrated that for properly chosen ratchet potentials $V(x)$ a current inversion as a function of the oscillation frequency ω arises.

Apparently, our "diffusion ratchet" tends to resist carrying a finite current: It starts only proportional to ω^{-2} for fast oscillations and, in many cases, vanishes again like ω^2 for slow oscillations. A further remarkable observation is that such a diffusion ratchet is equivalent to a "flashing ratchet" as given in Eq. (11). We finally note that a modulation of the diffusion typically yields a current in the opposite direction as compared to that induced by applying a modulated force ("rocking ratchet") (see Fig. 1a in Ref. [9], where the ratchet potential $V(x) = -V_2(x)$ is used.).

Acknowledgement

We gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (R. B. and P. H., Az. Ha1517/13-1) and the Holderbank foundation, Switzerland (P. R.).

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