Dye laser with pump and quantum noise

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We consider a single-mode dye laser with colored pump noise and white quantum noise. An analytical approach is applied to study the statistics of the intensity fluctuations for arbitrary correlation times of the pump noise. Full numerical solutions are compared with our approximation and previous approximations schemes. The present approximation scheme compares favorably over wide regimes in parameter space with the numerical data.

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I. INTRODUCTION

The single-mode dye-laser equation [1] has been established as a paradigmatic system for colored-noise theory [2]. It has been shown to describe experimental results very well [3], but is yet simple enough to allow for analytical methods. In contrast to the single-mode laser equation, the dye-laser equation includes a term describing the fluctuations of the pump parameter being imposed on the system by the pump mechanism. To account for the experimental results, the pump noise has to be exponentially correlated [1,3].

Well above threshold, the fluctuations due to spontaneous-emission processes (quantum noise) can be neglected and the dye-laser equation reduces to a Langevin equation with multiplicative colored noise. This reduced equation has been solved analytically for small correlation times in [4] and later on for the entire range of small to large correlation times in Ref. [5]. Full numerical solutions for arbitrary values of the correlation time have been presented in [6].

Closer to threshold, the quantum fluctuations cannot be neglected and one has to deal with a Langevin equation with two noise sources, the colored pump noise and the white quantum noise. Theoretically, this problem is much more challenging since, e.g., the standard small correlation time expansion yields non-Fokker-Planck terms in third order. Full numerical solutions have been presented in [7]. While approximations for small correlation times have been put forth in [8,9], we present in this paper the applications of an alternative approach [10], valid over an *extended* regime of pump noise color.

In Sec. II we present the approximation scheme for the dye-laser equation with colored pump and white quantum noise. In Sec. III we compare the resulting stationary probability density for the laser intensity with full numerical results [7]. In Sec. IV we summarize our results.

II. DYE-LASER EQUATIONS AND APPROXIMATION SCHEMES

An adiabatic elimination of polarization and inversion in semiclassical laser theory yields an equation for the laser-field amplitude. In the presence of white quantum noise the dye-laser equation for the photon intensity $\bar I$ assumes the Stratonovitch stochastic differential equation [7]

$$\dot{\bar I}=2(\bar a-\bar B\bar I)\bar I+\frac{\bar D}{2}+2\bar I\bar\epsilon(\bar t)+\sqrt{2\bar D\bar I}\;\bar\xi_1(\bar t), \qquad (1)$$

where the pump noise $\bar\epsilon(\bar t)$ is exponentially correlated Gaussian noise, i.e.,

$$\dot{\bar\epsilon}=-\frac{\bar\epsilon}{\bar\tau}+\frac{\sqrt{2\bar Q}}{\bar\tau}\bar\xi_2(\bar t). \qquad (2)$$

The fluctuations $\bar\xi_{1,2}(\bar t)$ denote uncorrelated white Gaussian noise forces of vanishing mean, obeying $\langle\bar\xi_i(\bar t)\bar\xi_j(\bar s)\rangle=\delta_{ij}\delta(\bar t-\bar s)$. The pump noise is exponentially correlated as

$$\langle\bar\epsilon(\bar t)\bar\epsilon(\bar s)\rangle=\frac{\bar Q}{\bar\tau}e^{-|\bar t-\bar s|/\bar\tau}, \qquad (3)$$

with variance $\langle\bar\epsilon^2\rangle=\bar Q/\bar\tau$ and noise correlation time $\bar\tau$. Realistic values of the pump parameter $\bar a$, the saturation parameter $\bar B$, and the noise intensities $\bar D$ and $\bar Q$ of the dye-laser model [1] have been obtained in [3] by comparing experimentally obtained, switch-on time distributions with simulations of the model. Operating the laser far above threshold, a typical set of parameters is given by $\bar a=0.7\times10^6\ \mathrm{s}^{-1}$, $\bar B=0.114\times10^6\ \mathrm{s}^{-1}$, $\bar D=8\times10^{-3}\ \mathrm{s}^{-1}$, $\bar Q=4.9\times10^3\mathrm{s}^{-1}$, and $\bar\tau=5\times10^{-7}\ \mathrm{s}$.

For further analysis we adopt a dimensionless form of the laser equation [2]. With use of a dimensionless pump parameter a, we scale as follows: time $t=(\bar a/a)\bar t$; $\tau=(\bar a/a)\bar\tau$; $Q=(a/\bar a)\bar Q$; $D=(a^2/\bar a^2)\bar B\bar D$; $I=(a/\bar a)\bar B\bar I$; $\xi_i(t)=\sqrt{a/\bar a}\,\bar\xi_i(\bar t)$; $i=1,2$, and $\epsilon(t)=(a/\bar a)\bar\epsilon(\bar t)$. This in turn precisely yields the dimensionless equations (1)–(3) without any overbars and with $B\to1$. Most importantly, the scaled correlation time τ *typically*, e.g., at $a=1$, ranges between $\tau=0.1$ and 1. This selects approximation schemes for small and moderate τ, presented below, as most relevant for the photon statistics of the dye laser. For example, the above value of the noise correlation time yields a scaled value of $\tau = 0.35$.

To simplify the notation within our colored-noise approximation to Eq. (1) we set

$$f(I) \equiv 2(a-I)I + D/2, \quad g(I) \equiv 2I, \quad h(I) \equiv \sqrt{I}. \tag{4}$$

In constructing a theoretical approximation to the colored-noise dynamics in Eqs. (1) and (2) we attempt to cover a large regime of correlation times τ. Hence we focus on a nonsystematic approach that globally covers a wide regime of parameter values. In contrast, the previous theories in [8,9] are of an asymptotic nature covering the limiting regime of small noise color $\tau \ll 1$. To construct an approximation that covers a wider regime we follow the reasoning of the theory put forth in Ref. [10].

An effective Markovian approximation to Eq. (1) is based on the principle of adiabatic elimination. At small τ, a simple adiabatic elimination of the ϵ process yields the white-noise Markovian approximation to Eq. (1). Thus, in order to cover noise correlation times τ of small to intermediate order a more judicious choice is needed for the process that is to become adiabatically eliminated. This procedure in turn will yield a one-dimensional Markovian approximation to the joint process in Eqs. (1) and (2) for which the stationary probability can be expressed in terms of quadratures. In doing so, we shall transform *nonlinearly* the pair process (I,ϵ) to a new pair process $(I,\dot{u})$. This comes at a price; the new auxiliary process $\dot{u}$ no longer has a simple physical meaning, but it leads to an effective Markovian approximation that covers also moderate values of noise color; see below in Eq. (11). For our dye-laser system this auxiliary process $\dot{u}$ is given by

$$\dot{u} = \epsilon + \frac{f(I)/g(I)}{A(I,\tau)}, \tag{5}$$

with

$$A(I,\tau) = 1 + \frac{h^2(I)}{Rg^2(I)}\left[1 - \tau g(I)\left(\frac{f(I)}{g(I)}\right)'\right]. \tag{6}$$

Here $R \equiv Q/D$ equals the noise ratio and the prime denotes differentiation with respect to I. In terms of $\dot{u}$, Eq. (1) is recast as

$$\dot{I} = g\dot{u} - \frac{1-A}{A} f + h\sqrt{2D}\xi_1. \tag{7}$$

Because the process $\dot{u}$ does *not contain white-noise forces*, we can perform a derivative with respect to time, yielding

$$\ddot{u} = \dot{\epsilon} + \left(\frac{(f/g)}{A}\right)'\dot{I} \tag{8}$$

$$= -\gamma(I,\tau)\dot{u} + \frac{(f/g)}{A\tau}\left[1 - \tau g(1-A)\left(\frac{(f/g)}{A}\right)'\right] + \frac{\sqrt{2Q}}{\tau}\xi_2 + \left(\frac{(f/g)}{A}\right)' h(I)\sqrt{2D}\xi_1, \tag{9}$$

with the ''effective friction'' $\gamma(I,\tau)$ given by

$$\gamma(I,\tau) = \frac{1}{A}\left\{\left[\frac{1}{\tau} - g(f/g)'\right]\left(1 + \frac{h^2 D}{g^2 Q}\right) + f\frac{A'}{A}\right\}$$
$$= \frac{(1+4RI)(4\tau I^2 + 2I + \tau D)}{\tau[(8R+4\tau)I^2 + 2I + \tau D]} - \frac{8RI[4I(a-I)+D](I+\tau D)}{[(8R+4\tau)I^2 + 2I + \tau D]^2}. \tag{10}$$

The prerequisites for an effective adiabatic elimination of the auxiliary process $\dot{u}$ are as follows [5]. (i) The effective friction must assume large values, implying $\ddot{u} \approx 0$. (ii) The reduction to a single process requires a smooth force $K(I,\tau)$, given by the second term on the right-hand side in Eq. (9). This means that $[K(I,\tau)/\gamma(I,\tau)]'$ shall not assume large values. (iii) The stationary probability $p(I,\dot{u})$ should approximately separate, i.e., $p(I,\dot{u}) \approx p(I)p(\dot{u})$. In applying the resulting approximation, conditions (i) and (ii) must be checked self-consistently as a function of the parameters; see below Eq. (22). With our choice for $\dot{u}$, the latter property (iii) holds true exactly for a linear dynamics [10].

With $\tau \ll 1$, the effective friction $\gamma(I,\tau)$ takes on large values. Setting $\ddot{u} = 0$, one finds the Stratonovitch stochastic differential equation

$$\dot{I} = \frac{1}{\tau\gamma(I,\tau)}\left\{2(a-I)I + \frac{D}{2} + \sqrt{2D(4RI^2+I)}\xi(t)\right\}, \tag{11}$$

with $\xi(t)$ being a Gaussian white noise, obeying $\langle \xi(t)\xi(s)\rangle = \delta(t-s)$. Its stationary solution can readily be obtained in terms of quadratures, i.e.,

$$p(I,\tau) = \frac{Z^{-1}}{[D_{\mathrm{eff}}(I,\tau)]^{1/2}} \exp[-\Phi(I,\tau)/D], \tag{12}$$

where Z^{-1} is a constant of normalization. D_{eff} is the effective diffusion coefficient, reading

$$D_{\mathrm{eff}} = \frac{D(I+4RI^2)}{[\tau\gamma(I,\tau)]^2}, \tag{13}$$

and $\Phi(I,\tau)$ is given by

$$\Phi(I,\tau) = -\int^I \frac{f(y)\left[1 - \tau g(y)\left(\frac{f(y)}{g(I)}\right)'\right]dy}{Rg^2(y)A(y,\tau)} - \tau\int^I \frac{f^2(y)A'(y,\tau)dy}{A^2(y,\tau)[h^2(y)+Rg^2(y)]}. \tag{14}$$

In the opposite limit $\tau \gg 1$, the auxiliary variable $\dot{u}$ goes to zero as $\tau \to \infty$. Then the first term on the right-hand side of Eq. (7) goes to zero while the second term approaches $f(I)$. An approximative Markovian stochastic differential equation for large noise color hence reads

$$\dot{I} = \frac{f(I)\left[1 - \tau g(I)\left(\frac{f(I)}{g(I)}\right)'\right]h^2(I)}{A(I,\tau)Rg^2(I)} + h(I)\sqrt{2D}\xi_1. \tag{15}$$

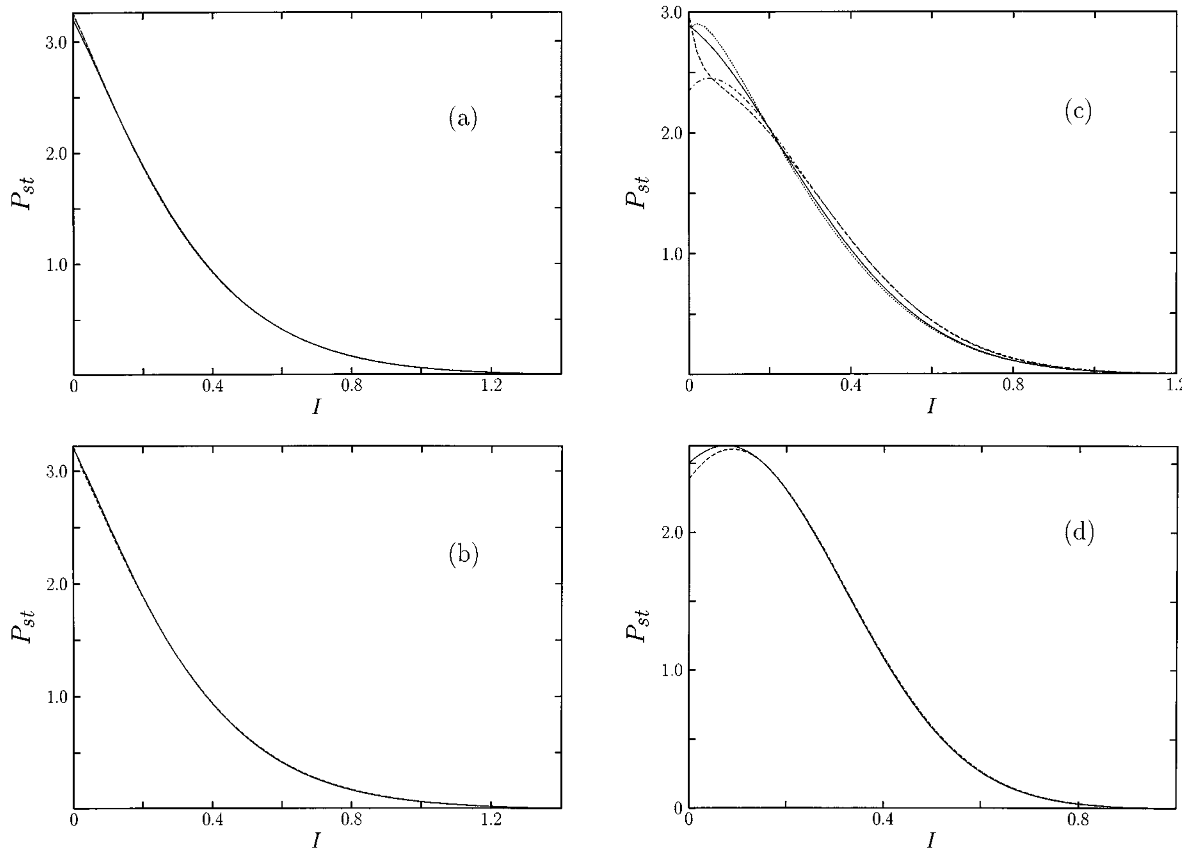

FIG. 1. Stationary probability $p(I,\tau)\equiv P_{st}$ shown at $a=0$, $R=1$, and $D=0.1$ for different values of the correlation time τ. In (a) we compare the results of Peacock-López *et al.* [9] (dashed) and Aguado and San Miguel [8] (dotted) at $\tau=0.1$ with numerical results (full line). (b) shows the comparison of our approach (small-τ approximation) (dashed) with numerical results (full line) for $\tau=0.1$. In (c) we compare the small-τ (dashed), large-τ (dash-dotted), and crossover approximations (dotted) with numerical results (full) for $\tau=1$. For $\tau=10$ the large-τ approximation (dashed) is compared with numerical results (full) in (d).

The stationary solution is given by Eq. (11) with $D_{\text{eff}}(I,\tau)$ substituted by

$$D_{\text{eff}}(I,\tau)=Dh^2(I)=DI. \tag{16}$$

The corresponding $\Phi(I,\tau)$ coincides precisely with the first contribution in Eq. (14). On observing that this $\Phi(I,\tau)$ approaches the correct limit both at $\tau=0$ and as $\tau\to\infty$, one can concoct a crossover approximation. With the drift part taken from Eq. (4) one then finds

$$\dot{I}=f(I)+\sqrt{2D_C(I,\tau)}h(I)\xi(t), \tag{17}$$

where

$$D_C(I,\tau)=\frac{DRA(I,\tau)g^2(I)}{\left[1-\tau g(I)\left(\dfrac{f(I)}{g(I)}\right)'\right]h^2(I)}. \tag{18}$$

Its effective diffusion coefficient equals $D_C(I,\tau)h^2(I)$ and $\Phi(x,\tau)$ coincides again with the first term in Eq. (14).

III. STATIONARY PROBABILITY

In this section we shall compare our approximation schemes put forth in the preceding section versus precise numerics of the exact stationary probability as given by the corresponding two-dimensional Fokker-Planck equation; cf. Eqs. (1) and (2). This exact stationary probability is evaluated by the method of matrix continued fractions detailed in [7]. In the following we compare the quality of the colored-noise approximation in Eq. (11) (for small τ), in Eq. (15) (for large τ), and for the crossover approach in Eqs. (17) versus numerically precise results. In addition, for small values of τ, we compare the results with the previously obtained small-τ theories by Aguado and San Miguel (A) [8] and by Peacock-López *et al.* (P) [9]. The corresponding approximations for the stationary probability read

$$p(I,\tau)=\frac{Z^{-1}\sqrt{I}}{\sqrt{D_{A,P}(I,\tau)}}\exp\left[\int_0^I\frac{2y(a-y)dy}{D_{A,P}(y,\tau)}\right], \tag{19}$$

where the diffusion coefficient $D_A(I,\tau)$ refers to the small-τ theory in Ref. [8], i.e.,

$$D_A(I,\tau)=DI(-8\tau RI^2+4RI+1-2\tau RD), \tag{20}$$

and $D_P(I,\tau)$ in [9]

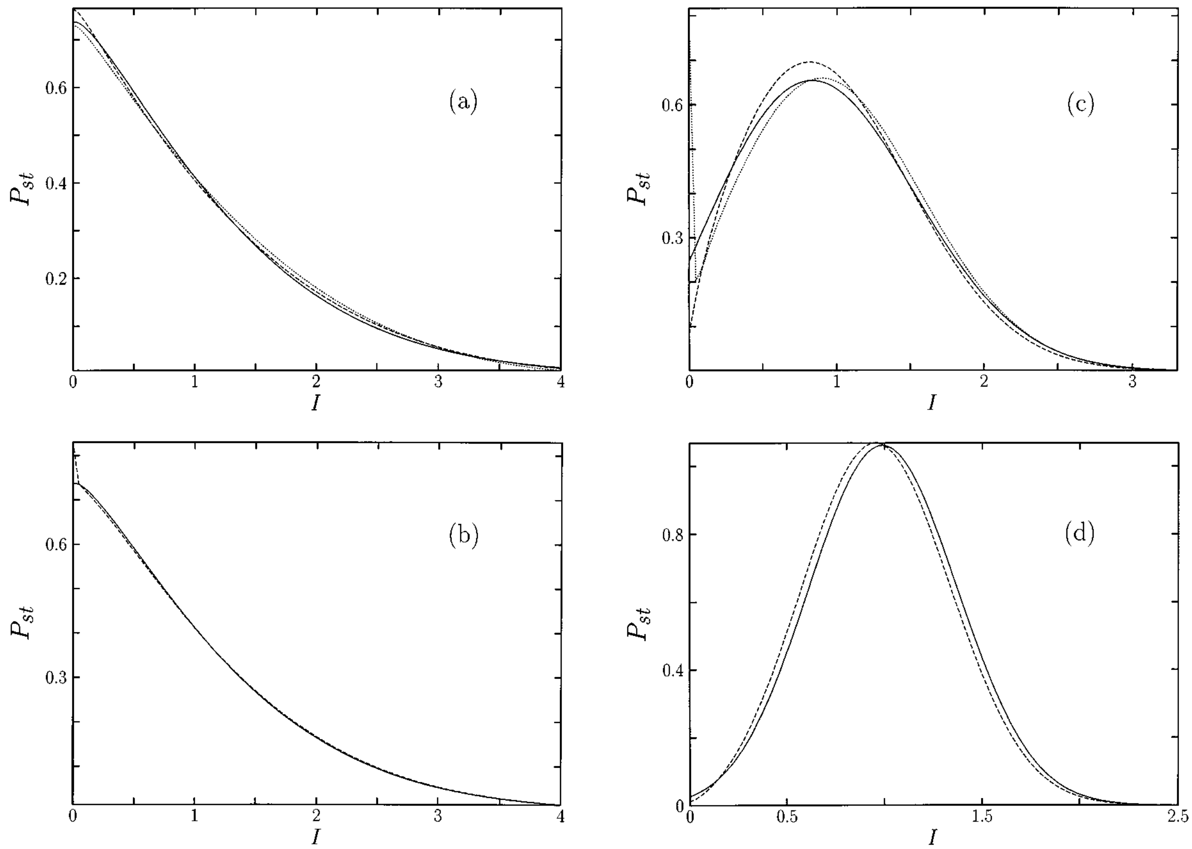

FIG. 2. Stationary probability $p(I,\tau)\equiv P_{st}$ shown at $a=1$, $R=5$, and $D=0.1$ for different values of the correlation time τ. In (a) we compare the result of Peacock-López *et al.* [9] (dashed), Aguado and San Miguel [8] (dotted) at $\tau=0.1$ with numerical results (full line). (b) shows the comparison of our approach (small-τ approximation) (dashed) with numerical results (full line) for $\tau=0.1$. In (c) we compare the large-τ (dashed) and small-τ approximations (dotted) with numerical results (full) for $\tau=1$. For $\tau=5$ the large-τ approximation (dashed) is compared with numerical results (full) in (d).

$$D_P(I,\tau)=DI\left[1+2R\left(\frac{-4\tau(1-2\tau a)I^2+2(1-4\tau^2a^2)I-\tau D(1+2\tau a)}{(1-4\tau^2a^2-4\tau^2 D)}\right)\right], \tag{21}$$

respectively. These effective diffusion coefficients are not positive as I assumes very large values. Nevertheless, the effective diffusion is positive valued within the regime of validity of the small-τ theory, $\tau\leqslant 0.1$ within the physically relevant range of laser intensities.

The results at threshold $a=0$ are depicted in Fig. 1. For $\tau=0.1$, both small-τ theories essentially agree within line thickness with the precise numerics; cf. Fig. 1(a). The present small-τ approximation is shown in Fig. 1(b). The agreement with the exact result is of the same quality as for the theories in [8,9]; cf. Fig. 1(a). The value $\tau=1$ lies beyond the regime of validity of the small-τ theories in [8,9]. With γ $(I,\tau=1)$ being positive valued, the small-τ theory in Eq. (10) still yields reasonable agreement with the precise result; cf. Fig. 1(c). The crossover approximation yields even better agreement. For $\tau=10$ [see Fig. 1(d)], the large-τ and crossover approximations agree within line thickness, yielding good agreement with the precise numerics.

The behavior near threshold at $a=1$ is depicted for typical laser operation values in Figs. 2(a)–2(d). At $\tau=0.1$, the theory of Aguado and San Miguel [8] is compared with the approach by Peacock-López *et al.* [9] in Fig. 2(a). Small deviations occur near $I\approx 0$. The theory of Peacock-López *et al.* approaches the exact result for large intensities I somewhat faster than the theory in [8]. The small-τ theory based on the adiabatic elimination of the auxiliary variable $\dot{u}$ in the Eq. (10) is depicted in Fig. 2(b). The agreement at finite intensities is superior if compared with Fig. 2(a). At very small intensities around $I\approx 0$, however, a systematic deviation occurs. This difference is not of numerical origin (see below). It becomes even more pronounced at $\tau=1$; see Fig. 2(c). The behavior at large correlation times can very well be characterized within the large-τ approximation Eq. (14). For a typical set of parameters the behavior is exhibited in Fig. 2(d). Characteristic for the intensity statistics is the observation that the maximum becomes shifted monotonically to larger intensities $I\leqslant a$ with increasing noise color τ.

Overall, we find good agreement between precise numerics for the exact stationary probability and the global colored-noise approximation schemes in the Eqs. (10), (14), and (17). The effective friction $\gamma(I,\tau)$ assumes positive values over an extended regime of the correlation time $\tau \leqslant 1$. Its asymptotic behavior at small τ behaves as

$$\gamma(I,\tau)=\begin{cases} \dfrac{4RI}{(\tau+2R)}, & I \gg 1 \\ \dfrac{1}{\tau}-\dfrac{4RDI}{(\tau D+4I)}, & I \ll 1. \end{cases} \tag{22}$$

The above-mentioned systematic deviations between theory and exact results at very small intensities I is rooted in the breakdown of the condition (ii) stated below Eq. (9), i.e., $[K(I,\tau)/\gamma(I,\tau)]'$ assumes *large* values near $I \approx 0$. Thus the regime of validity of the adiabatic approximation begins to break down as $I \rightarrow 0$; cf. Figs. 2(a) and 2(c). In terms of laser operation, the approximation thus fails to describe accurately the photon statistics as $I \rightarrow 0$.

IV. CONCLUSION

In summary, we have studied the probability distribution of a dye-laser system driven by both colored pump noise of small to moderate to large correlation times and white quantum noise fluctuations. A global approximation scheme aimed at covering a wide regime of correlation times has been tested against precise numerical results, obtained by the matrix continued fractions method applied to the exact two-dimensional Fokker-Planck equation. The present approximation schemes compare favorably over wide regimes in the parameter space of realistic laser operation values for the pump parameter a, noise ratio $R=Q/D$, and the pump noise correlation time τ. At intermediate noise color $\tau \sim O(1)$ the approximations still provide qualitative correct predictions where no other theoretical estimates are presently available. At very small correlation time, the theory yields essentially the same qualitative predictions as previously developed small-τ theories [8,9].

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