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Effect of periodic shear on avalanches in granular systems

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Abstract

Surface flow down a granular pile ("avalanches") can be excited by shear forces or by vibration. Based on a generalization of a recently proposed minimal dynamical mean field model, we discuss theoretically how a *periodically modulated* shear rate about a zero mean can change the dynamics of avalanches. In particular, we report the *unexpected* effect of non-periodic avalanching due to periodic driving. The predicted effects should be experimentally observable in two-dimensional drums by modulating the rotation rate.

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1. Introduction

The collective interaction of small macroscopic particles (grains) due to static and dynamic friction and inelastic collisions in the gravitational field forms a state of condensed matter being at the border between fluids and solids. Although this *granular state* (for a recent review see [1]) is a part of daily experience, the understanding of its physical properties, however, is only at its beginning. Its physics has recently triggered considerable interest in the structure and dynamics of granular systems including sophisticated experiments in simple setups [2–7] (cylinders and drums), numerical simulations based on classical many-particle models [8], cellular automata [9] and phenomenological mean field models [10–13].

Any granular dynamics results from driving the system out of equilibrium. Lacking a well-accepted statistical or non-Newtonian-fluid theory for the granular state, the response of granular systems to external (experimentally realizable) excitations such as shear or vibration is the key to obtain deeper insight into their dynamics. Paradigmatic experimental setups are granular piles in horizontal drums [2] or cylinders [3] which can be rotated about their horizontal axes with an externally prescribed rotation rate. An adiabatic rotation of the drum or cylinder shows the following: Whenever the surface of the pile is not horizontal, it undergoes a shear force caused by the along-surface component of the gravitational force. Up to the *maximum angle of repose* (or angle of initial yield) φ_s, the surface can compensate the shear force by intergranular friction. Above φ_s, an avalanche develops in order to decrease the slope of the pile until the *minimal angle of repose* (or angle

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of residual shear) φ_r is reached. Then the pile is stable again. This hysteretic stick–slip–stick behavior is one of the major characteristics of the granular state.

So far, research has been focused on constant shear forces, shear forces that increase linearly in time, and external vibration as driving mechanisms. In this paper, we discuss the effect of a different driving mechanism, time-periodic shear forces, on avalanche dynamics, a driving mechanism that has not been considered in the literature yet. According to the discussion above, this effect can be realized by an external, periodic modulation of the rotation rate of the drum or cylinder about a zero mean. The major questions we will discuss in this paper are: Is the stick–slip–stick dynamics, present without modulation, *significantly* altered by experimentally realizable rotation modulation? How does the dynamics of avalanches depend on the driving frequencies and amplitudes? Do resonance phenomena exist? Is just one avalanche excited or a series? If there is a series of avalanches, is it a periodic process? What is the effect of granular friction? Our theoretical study is based on an extension of a mean field approach [12,13] that we proposed recently. If experimentally verified, our results will lead to a deeper insight into the friction behavior on the surface of granular piles as well as into the validity of mean field approaches.

Our paper is organized as follows. In Section 2, we present the extension of the mean field model to modulated rotation, derive a minimal approximation of the model in the form of a driven nonlinear oscillator in a restricted phase space, and discuss the rigid-body rotation solution. Section 3 discusses the exact solutions of our model for the special case of vanishing nonlinearity. In particular, we study the changes of the duration of the avalanche and the minimum angle of repose. In Section 4 we show, using dominant balance considerations, that the limit of vanishing nonlinearity is sufficient for the qualitative *and* quantitative dynamics of the full model. In Section 5 we demonstrate the effect of *non-periodic* successive avalanching due to *periodic* rotation modulation. We summarize our findings in Section 6.

2. The model

In order to discuss the effect of *periodic* shear forces on avalanches, we extend recent minimal, dynamical mean field equations [12,13] that have been used to model the dynamics of avalanches in granular systems in presence of small constant rotation and vertical vibration. Within the mean field approach, the complexity of the avalanche process is substituted by two *global* variables: the kinetic energy $E_{\text{kin}}(t)$ of the avalanche and the surface angle $\varphi(t)$ of the pile. In order to model the time evolution of $E_{\text{kin}}(t)$, it is more convenient to use the mean velocity $v(t) \sim \sqrt{E_{\text{kin}}(t)}$ (the velocity of a "grain" in the mean field of the others). Then the two coupled mean field equations for the surface velocity $v(t)$ in the corotating frame and the angle $\varphi(t)$ of the surface of the pile can be modeled by generalizing Coulomb's theory of friction of a body on an inclined plane to a surface flow with variable slope of the pile. Note that (i) $v(t)$ and $\varphi(t)$ are ensemble averages over a large number of experimental runs to level out some randomness present for the individual avalanche and (ii) this model applies only to situations when the dynamics of the system takes place in a thin layer at the surface of the pile without any global convection in the bulk. Including the time-dependent external rotation of the drum, $\varphi_{\text{ext}}(t)$, the model reads

$$\dot{v} = g[\sin\varphi - (b_0 + b_2 v^2)\cos\varphi]\chi(\varphi, v), \tag{1a}$$

$$\dot{\varphi} = -av + \dot{\varphi}_{\text{ext}}(t) \tag{1b}$$

with $\chi(\varphi, v) = \Theta(v) + \Theta(\varphi - \varphi_s) - \Theta(v)\Theta(\varphi - \varphi_s)$ and $\Theta(y) = 0$ (1) if $y \leq 0$ $(y > 0)$. Here, g is the gravitational acceleration and a the coupling coefficient between angle and velocity. The cutoff-function χ accounts for the static friction (cf. also [12]) in a way that the pile is at rest if $v = 0$ and $\varphi \leq \varphi_s$. This mimics a Mohr–Coulomb yield criterion (see e.g. [14]) for the viscoplastic behavior of the granular system. The physical origin of the yield criterion is rooted in the fact that the density of a granular system has to fall below a certain threshold in order to flow. Within model (1), an avalanche is represented by a non-zero velocity $v(t)$ with associated changes of the surface angle $\varphi(t)$ due to the cross-coupling of (1a) and (1b). Another consequence of the cutoff-function is that the dynamical evolution of the system is constrained by $v \geq 0$. This reflects the fact that an avalanche at the surface cannot climb up the pile. Therefore, the cutoff-function χ also takes into account that the avalanche is trapped at rest if v reaches zero at an angle $\varphi \leq \varphi_s$. In all other cases, the system evolves dynamically. Dynamic friction is modeled by a nonlinear friction coefficient $k_d(v) = b_0 + b_2 v^2$ with b_0 and b_2 both positive [15]. [1]

The external rotational modulation about a zero mean discussed in this paper varies periodically in time according to

$$\varphi_{ext}(t) = \Delta \sin \nu t \tag{2}$$

with Δ and $\nu > 0$ being the amplitude and the frequency of the external rotation modulation, respectively. Eqs. (1) and (2) are invariant under the combined transformation $\{\Delta \to -\Delta$ and $t \to t + \pi/\nu\}$, i.e. under the inversion of the sign of the driving amplitude and a shift in time by one half of the driving period.

The accessible driving amplitudes Δ in (2) are restricted by the physics of the problem, and are therefore small. This can be seen as follows: The amplitude of the external angle modulation φ_{ext} measured in degrees is related to Δ by $\varphi_{ext} = (180\Delta/\pi)^\circ$. For example, an external modulation of the angle by 1° (5°) corresponds to an amplitude $\Delta = 0.0175$ ($\Delta = 0.087$). Therefore, we restrict the following discussion to modulation amplitudes Δ of the order 0.1 or less. This limitation should also guarantee that generically no global circulation of the system is excited. Finally, we restrict the discussion to driving periods $2\pi/\nu$ that are typically of the order of the duration of an avalanche or larger. This excludes bouncing grain effects on the surface that happen very likely in the high driving frequency limit. To study the effect of the forcing (2) experimentally, one has to create first a stable static granular pile being as close as possible to the maximum angle of repose and then to turn on the sinusoidal external angle variations.

2.1. *Derivation of the minimal model*

If $\Delta = 0$, the dynamics of (1) and (2) is basically centered around the angle $\varphi_d = \arctan b_0$, being smaller than φ_s, by construction of the model [12]. Shifting all angles according to $\Phi(t) = \varphi(t) - \varphi_d$, *non-dimensionalizing time* by $\sqrt{ag}t \to t$ and velocity by $\sqrt{a/g}v \to v$, and combining (1a), (1b) and (2) in order to eliminate v, we arrive at the following driven nonlinear oscillator equation for the variations $\Phi(t)$ about φ_d,

$$\ddot{\Phi} - [\delta(\cos\Phi - \mu \sin\Phi)(\dot{\Phi} - \Delta f \cos ft)^2 - \Omega_0^2 \sin\Phi]\chi(\Phi, \dot{\Phi}) = -\Delta f^2 \sin ft \tag{3a}$$

with the cutoff-function

$$\chi(\Phi, \dot{\Phi}) = \Theta(\Delta f \sin ft - \dot{\Phi}) + \Theta(\Phi - \Phi_s) - \Theta(\Delta f \sin ft - \dot{\Phi})\Theta(\Phi - \Phi_s). \tag{3b}$$

The coefficents in (3) are given by $\delta = (b_2 g/a)\cos\varphi_d$, $\Omega_0^2 = 1/\cos\varphi_d$, and $\mu = \tan\varphi_d$. $\Phi_s = \varphi_s - \varphi_d$ is the shifted maximum angle of repose and $f = \nu/\sqrt{ag}$ denotes the scaled driving frequency. Taking into account that the velocity follows Φ according to $v = -\dot{\Phi} + \Delta f \cos ft$, Eq. (3) is still fully equivalent to (1) and (2). Experiments for $\Delta = 0$ [2,3] show that the variations of Φ are only a few degrees and are therefore small enough to allow a small angle approximation in Φ about φ_d. This will also be the same for not too large external driving amplitude Δ as relevant in this study. Therefore, Eq. (3) can be reduced to

$$\ddot{\Phi} - [\delta(\dot{\Phi} - \Delta f \cos ft)^2 - \Omega_0^2 \Phi]\chi(\Phi, \dot{\Phi}) = -\Delta f^2 \sin ft, \tag{4}$$

[1] The model in [12] is minimal in the sense that its friction coefficient $k_d(v)$ is the simplest to fullfil the physical limits, in particular the quadratic increase with v for large v. In particular it differs from related models [10,11] by assuming that $k_d(v)$ increases monotonically with v. It explains the stick–slip–stick behavior without rotation, shows the periodic slip–stick and constant flow dynamics in presence of constant rotation and explains their transition. In particular, it also shows the correct quadratic dependence of the inclination angle for constant flow on the (constant) rotation rate as found experimentally by Rajchenbach (cf. [3]). Finally, it allows for a simple scenario why external, vertical vibration can lead to a logarithmic decay of the inclination angle of the pile on an intermediate time scale, cf. [13].

which will be our *working model* for analytical calculations in the following. Note that the modulation of the rotation rate is reflected in (4) by driving terms that enter *parametrically* and *additively*. From the experiments for $\Delta = 0$ [3,11], one can obtain estimates for the order of magnitude of the parameters entering in Eq. (4); $\delta \simeq 0.1$ and $\Omega_0 \simeq 1.1$ seem to be typical. The dynamics of Eq. (4) for $\Delta \equiv 0$ is a half-oscillation of the harmonic oscillator with quadratic friction [15] previously discussed in [12].

2.2. *Rigid-body rotation*

The simplest possible dynamical evolution of (4) is a modulated rotation of the whole granular pile as a rigid body without avalanches taking place, i.e. $v = 0$ *for all times*. Assuming that the initial condition for the angle at $t = 0$ is given by $\Phi(0) = \Phi_i$, the variation of the surface angle is a pure sinusoidal rotation about the initial angle

$$\Phi(t) = \Phi_i + \Delta \sin ft, \tag{5}$$

which instantaneously follows the external modulation. This type of motion, however, exists as a *permanent solution for all times* $t \geq 0$ only if $\chi = 0$, i.e. if the initial angle Φ_i fulfills the condition $\Delta < \Phi_s - \Phi_i$.

A rigid-body rotation can also appear as a *permanent dynamics after an avalanche came to a halt* at time t_h and angle Φ_h. For $t \geq t_h$, this solution reads

$$\Phi(t) = \Phi_h - \Delta \sin ft_h + \Delta \sin(ft) \tag{6}$$

as long as $\Phi(t) < \Phi_s$. From Eq. (6) we see that the rigid-body rotation (rbr) oscillates about a time average $\Phi_{rbr} = \Phi_h - \Delta \sin ft_h$. The averaged angle Φ_{rbr} is in general different from Φ_h where the avalanche came to a halt; only if $t_h = n\pi/f$ with integer n, $\Phi_{rbr} = \Phi_h$ holds. Eq. (6) also yields a criterion whether rigid-body rotation as long-time dynamics can appear at all. The condition to be satisfied is $\Phi(t) < \Phi_s$ for all times $t > t_h$, or equivalently, $\Phi_h < \Phi_s - \Delta(1 - \sin ft_h)$ for given Φ_h and t_h; otherwise, subsequent avalanches are generated. So far, we have not yet determined the duration of an avalanche t_h; this requires an analysis of the dynamics of (4) given in Sections 3 and 4.

3. Linear dynamics induced by modulated rotation

Previous analysis [12] suggests that δ is very likely a small quantity, $\delta \sim O(0.1)$. Therefore, the limit $\delta = 0$ should already lead to valuable information on the dynamics of (4) for small δ. In Section 4, we will support this statement with a dominant balance argument. If δ were equal to zero, Eq. (3) reduces to an additively driven harmonic oscillator, i.e.

$$\ddot{\Phi} + \Omega_0^2 \Phi\, \chi(\Phi, \dot{\Phi}) = -\Delta f^2 \sin ft, \tag{7}$$

in the restricted phase space $\{\Phi, \dot{\Phi}\}$ with the constraint $\dot{\Phi} < \Delta f \cos ft$. If $\dot{\Phi} = \Delta f \cos ft$, or equivalently $v \equiv 0$, is reached, the system undergoes pure rigid-body rotation as discussed in Section 2.2. Depending on the modulation amplitude Δ, the rigid-body rotation can go on forever or a new avalanche can start if the maximum angle of repose Φ_s is reached again.

3.1. *Exact solution*

Eq. (7) can be solved exactly. The initial conditions at time $t = 0$ we choose are an initial angle given by the maximum angle of repose, $\Phi(0) = \Phi_s$, and zero initial velocity, $v(0) = 0$, or equivalently $\dot{\Phi}(0) = \Delta f$. In the following we suppose that the driving amplitude Δ is positive. If Δ is negative, the system first undergoes a rigid-body rotation for one half-period π/f and then proceeds in the same way as for positive Δ. This reflects the invariance of Eq. (7) under the transformation mentioned in Section 2.

If the driving frequency of the rotation modulation is different from the eigenfrequency, $f \neq \Omega_0$, the solution of (7) reads

$$\Phi(t) = \Phi_s \cos \Omega_0 t + \frac{\Delta f^2}{\Omega_0^2 - f^2}\left[\frac{\Omega_0}{f}\sin \Omega_0 t - \sin ft\right], \tag{8}$$

provided $v > 0$. If the driving frequency equals the eigenfrequency, $f \equiv \Omega_0$, one obtains

$$\Phi(t) = \Phi_s \cos \Omega_0 t + \tfrac{1}{2}\Delta[\sin \Omega_0 t + \Omega_0 t \cos \Omega_0 t], \tag{9}$$

as long as $v > 0$.

The velocity v reaches zero for the first time at time t_h (being the duration of the avalanching process) and at the angle of halt Φ_h, given by the condition $\dot{\Phi}(t_h) \equiv \Delta f \cos f t_h$, or equivalently by

$$\Phi_s(\Omega_0^2 - f^2)\sin(\Omega_0 t_h) + \Delta f \Omega_0(\cos f t_h - \cos \Omega_0 t_h) = 0. \tag{10}$$

The angle of halt $\Phi_h(\Delta)$ is the generalization of the minimal angle of repose, $\Phi_r = \Phi_h(\Delta = 0)$, for non-zero driving. The implicit relation (10) for the duration of an avalanche is one of the central results of the paper. Without driving, $\Delta = 0$, the duration of an avalanche is given by $t_h(\Delta = 0) = \pi/\Omega_0$. For non-zero driving amplitudes Δ, one sees that $t_h = \pi/\Omega_0$ is a solution of (10) provided that f is zero or an odd multiple of Ω_0.

3.2. *Perturbation theory for very small Δ*

With Δ being small, a perturbation analysis for the duration of the avalanche, $t_h(\Delta)$, can be performed. Inserting the expansion

$$\Omega_0 t_h(\Delta) = \pi + \kappa_1 \Delta + \kappa_2 \Delta^2 + O(\Delta^3) \tag{11}$$

in (10) and introducing the *frequency ratio* $r \equiv f/\Omega_0$, one obtains for the expansion coefficients

$$\kappa_1 = \frac{r}{\Phi_s}\frac{\cos(r\pi) + 1}{1 - r^2}, \tag{12a}$$

$$\kappa_2 = -\frac{r^2}{\Phi_s}\frac{\sin(r\pi)}{1 - r^2}\kappa_1. \tag{12b}$$

Since κ_1 is positive (negative) if $f < \Omega_0$ ($f > \Omega_0$), the duration of an avalanche is increased (decreased) for $f < \Omega_0$ ($f > \Omega_0$). The angle of halt, $\Phi_h = \Phi(t_h)$, where the avalanche stops, is given in lowest order correction in Δ by

$$\Phi_h = -\Phi_s - \frac{r^2 \sin(r\pi)}{1 - r^2}\Delta + O(\Delta^2). \tag{13}$$

In Fig. 1 we show the dependence of $\Omega_0 t_h$ and Φ_h on the frequency ratio r in a range $0 < r < 10$ for a modulation amplitude $\Delta = 0.002$. If $r \geq 3$, the duration of the avalanche t_h possesses maxima given by π/Ω_0 at odd values of r and minima close to even values of r; the minima are less pronounced for larger r. For $r < 3$, the behavior is different: $\Omega_0 t_h$ has a very pronounced maximum at about $r = \frac{1}{2}$, reaches $\Omega_0 t_h = \pi$ at $r = 1$, and has a very pronounced minimum close to $r = 2$. If $r \geq \frac{5}{2}$, the angle of halt, $\Phi_h = -\Phi_s$, oscillates about the angle of repose $\Phi_r \equiv \Phi_h(\Delta = 0)$ and possesses alternating minima and maxima at odd multiples of $r = \frac{1}{2}$. For $0 < r < 2$, the modulus of the angle of halt Φ_h is considerably increased in comparison to the non-modulated case, implying the avalanche stops at smaller angles.

Inspection of (11) and (12), however, shows that the radius of convergence of expansion (11) is rather small, In fact, (11) is actually an expansion in terms of Δ/Φ_s. Since Φ_s is half of the difference between the maximum and minimal angle of repose, $\Phi_s = \frac{1}{2}(\varphi_s - \varphi_r)$ when no modulation is present, Φ_s is typically of the order of a few degrees or equivalently, of the order $O(10^{-2})$. Therefore, the validity of (11) is restricted to very small driving amplitudes Δ being of the order 0.001 or less.

3.3. *Numerical results*

For driving amplitudes Δ of the order 10^{-2} or 10^{-1} the avalanche duration t_h has to be calculated numerically, either by finding the roots of (10) or by numerical integration of (7) until $v = 0$ has been reached for the first time. In Fig. 2, we show the dependence of the duration t_h of the first avalanche as function of the forcing frequency f, reduced by Ω_0, for three different forcing amplitudes, $\Delta = 0.01, 0.05, 0.1$. As representative values, we have used $\Omega_0 = 1.1$ and $\Phi_s = 0.014$; they are estimates [12] based on data of the experiment of Jaeger et al. [2]. Several important effects can be observed:

(i) For driving frequencies f smaller than Ω_0, the duration of an avalanche is significantly enhanced. This enhancement is larger for larger Δ, with a maximum that is shifted closer to $f = 0$ for larger Δ.

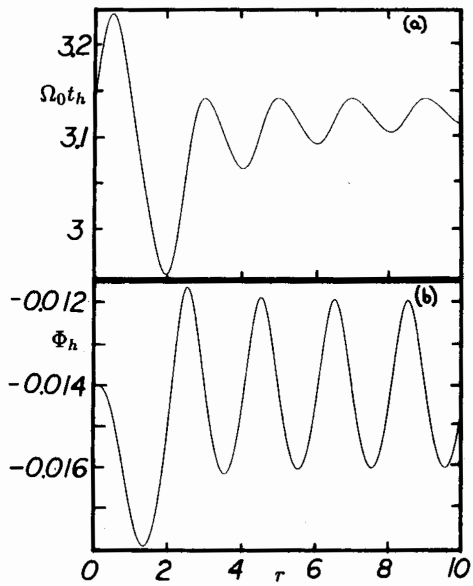

Fig. 1. Dependence of (a) the duration t_h of an avalanche (multiplied by Ω_0) and (b) the angle of halt Φ_h of the surface as function of the frequency ratio $r = f/\Omega_0$, for a *very small* driving amplitude $\Delta = 0.002$. Other parameters are $\Omega_0 = 1.1$ and $\Phi_s = 0.014$. For this value of Δ the perturbation theory (cf. Section 3.2) is still accurate.

(ii) If the driving frequency f equals the eigenfrequency Ω_0, the duration of the avalanche is equal to the duration without rotation modulation, $\Delta = 0$.

(iii) For driving frequencies $f > \Omega_0$, the duration of an avalanche cannot exceed $t_h(\Delta = 0) = \pi/\Omega_0$ and is in general significantly decreased.

(iv) For $\Delta = 0.01$, the dependence of the duration of the avalanche is qualitatively similar to the case of very small Δ as discussed in Section 2.2 (cf. also Fig. 1). The major differences are that the wells between the maxima are more asymmetric and the minima are shifted to larger r-values (cf. Fig. 2(a)).

(v) For $\Delta = 0.05, 0.1$, one can see drastic changes of t_h for frequencies $f > \Omega_0$. At odd multiples of the eigenfrequency Ω_0, the avalanche duration does not last until $t_h(\Delta = 0) = \pi/\Omega_0$ any more; t_h is significantly decreased. In addition, the avalanche duration t_h possesses as function of the frequency ratio r step-like discontinuities where t_h jumps to a higher value if r is infinitesimally increased.

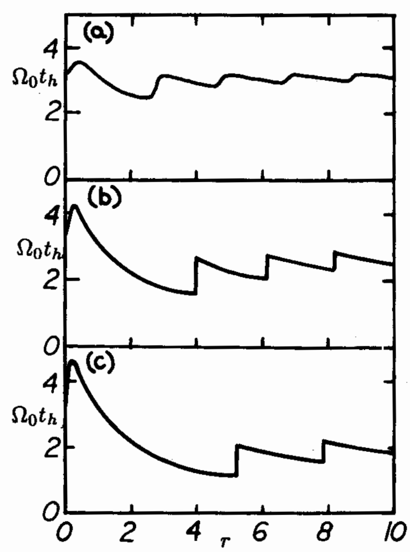

Fig. 2. Dependence of the duration t_h (multiplied by Ω_0) of the avalanche on the frequency ratio $r = f/\Omega_0$ for three different values of the driving amplitude Δ, (a) $\Delta = 0.01$, (b) $\Delta = 0.05$, and (c) $\Delta = 0.1$. Other parameters are $\Phi_s = 0.014$ and $\Omega_0 = 1.1$. Without rotation modulation, $\Delta = 0$, $\Omega_0 t_h$ equals π.

Both phenomena in (v) can be understood by considering the structure of the zeroes of Eq. (10). For $\Delta = 0.01$ and $f > \Omega_0$, Eq. (10) possesses only one real root in the interval $0 < t_h < \pi/\Omega_0$ for each $r = f/\Omega_0$. The dependence of this root of (10) on the frequency ratio r consists of wells with minima between the maxima at integer odd multiples of r (cf. Fig. 2(a)). Increasing Δ changes the well structure of $t_h(r)$; the wells become more and more asymmetric, the minima of the wells bend downwards to smaller values of t_h and to larger values of r. For large enough Δ, the wells "fold" back in a hysteretic way after reaching their minima. The minimum e.g. of the "folded" well between the maxima at $r = 2n+1$ and $r = 2n+3$ (with integer $n > 1$) is then located in the range $2n+3 < r < 2n+5$. Equivalently, for large enough Δ, there exist three real roots of Eq. (10) in the interval $0 < t_h < \pi/\Omega_0$ for each $r = f/\Omega_0 > 1$. Since the avalanche stops at the first positive root of (10), there are discontinuities ("jumps") at values of the frequency ratio where the folded wells cease to exist and therefore, the roots $t_h = \pi/\Omega_0$ at odd integer r-values are covered up.

In Fig. 3, we show the dependence of the angle of halt Φ_h (reduced by its value for $\Delta = 0$, $\Phi_h(\Delta = 0) = -\Phi_s$) versus the frequency ratio $r = f/\Omega_0$ for the same parameter values as in Fig. 2. For $\Delta = 0.01$, the variations of $\Phi_h(r)$ are qualitatively similar to the case $\Delta = 0.002$ (cf. Fig. 1(b)); the minima of $\Phi_h(r)$, however, are wider and the maxima are narrower. The maxima of $\Phi_h(r)$ occur at the minimum values of $t_h(r)$. For larger Δ-values of 0.05 and 0.1, there are again drastic changes in comparison to the behavior for $\Delta = 0.01$. One can see from Fig. 3 that there are discontinuities of Φ_h where an infinitesimal increase of r leads to a strong decrease of Φ_h. These jumps of the angle of halt occur at the same values of r as the jumps of the duration of the avalanche, t_h. From Fig. 3(b) and (c), one can also see that the angle of halt Φ_h is strongly decreased in comparison to the non-modulated case $\Phi_h(\Delta = 0)/\Phi_s = -1$ for most of the frequency ratios r. In particular, the first minimum of Φ_h is the most pronounced for $\Delta = 0.05$ ($\Delta = 0.1$) about 7.5 (14)

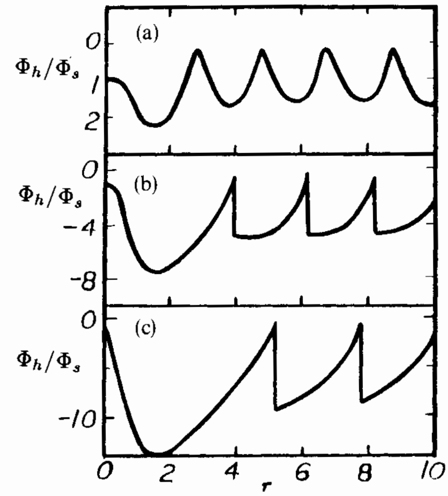

Fig. 3. Dependence of the angle of halt $\Phi_h(r, \Delta)$ of the avalanche (reduced by its angle of halt for $\Delta = 0$,$\Phi_r = -\Phi_s$) on the frequency ratio $r = f/\Omega_0$ for three different values of the driving amplitude Δ, (a) $\Delta = 0.01$, (b) $\Delta = 0.05$, and (c) $\Delta = 0.1$. Other parameters are $\Phi_s = 0.014$ and $\Omega_0 = 1.1$. Without rotation modulation , $\Delta = 0$, Φ_h/Φ_r equals -1.

times $-\Phi_s$. Only for values of r in the neighborhood of the discontinuities, is the angle of halt Φ_h larger than $-\Phi_s$.

4. Nonlinear dynamics induced by modulated rotation

In this section, we discuss the avalanche dynamics of the fully nonlinear model [2] (see also footnote 1), Eq. (4), and, in particular, the impact of non-zero δ. We distinguish between two ranges of driving frequencies f.

4.1. *Very small frequencies – adiabatic limit*

In the adiabatic limit, $f \to 0$, the system evolves like the unmodulated system until it reaches zero velocity at Φ_h, basically given by the minimal angle of repose $\Phi_r = -\Phi_s(1 - \frac{4}{3}\delta\Phi_s) + O(\delta^2)$ [12].

[2] The distinction between linear and nonlinear dynamics is based on the behavior of the dynamical friction coefficent $k_d(v)$. All terms proportional to δ result from the nonlinear (velocity-dependent) contributions to $k_d(v)$, since $\delta \sim b_2$.

The duration of the avalanche is determined by $t_r = (\pi/\Omega_0)(1+\frac{1}{6}\Delta^2\Phi_s^2)+O(\delta^4)$. Since $\Delta \simeq O(0.1)$ and $\Phi_s \simeq O(0.01)$, the corrections to these results for zero non-linearity are small and probably hard to observe experimentally. After having reached $v = 0$, the system will follow instantaneously the rigid-body rotation, Eq. (5), with $\Phi_i = \Phi_r$ provided $\Delta < \Phi_s - \Phi_r$.

4.2. *Larger frequencies – dominant balance*

In order to see the effect of the nonlinear terms in Eq. (4), we have performed numerical integrations of Eq. (4) to determine the duration of an avalanche and the angle of halt for the parameter sets used in Figs. 2 and 3. The results of the nonlinear model agree within line width with the results in section 3C for $\delta = 0$. The reason why there are no significant changes of the duration of an avalanche due to the incorporation of the nonlinearity δ, can be understood on the basis of the following dominant balance argument: If there is no rotation modulation present, $\Delta = 0$, Φ and its time derivatives are of the order Φ_s [12]. For small driving amplitudes Δ (as relevant in our study), this scaling is unaltered. Therefore, we can rescale Eq. (4) by Φ_s. Introducing $x(t) = \Phi(t)/\Phi_s$ being a quantity of unit order, Eq. (4) reads

$$\ddot{x} + \Omega_0^2 x = -\frac{\Delta f^2}{\Phi_s}\sin ft + \delta\Phi_s\dot{x}^2 - 2\delta\Delta f\,\dot{x}\cos ft + \frac{\delta\Delta^2 f^2}{\Phi_s}\cos^2 ft. \tag{14}$$

After multiplication with $\Phi_s/\Delta f^2$, the four terms on the right-hand side of (14) scale like

$$1 \;:\; \frac{\Phi_s^2}{\Delta f^2}\delta \;:\; \frac{\Phi_s}{f}\delta \;:\; \Delta\delta, \tag{15}$$

where we have used that $\dot{x}(t)$ and the trigonometric functions are quantities of unit order. Since $\delta \sim O(10^{-1})$ and since the physically accessible ranges of Δ reach from $O(10^{-3})$ to $O(10^{-1})$, relation (15) implies that the three δ-dependent terms are small in comparison to the first term in (15). This holds provided that $\Delta f^2 \gg \delta\Phi_s^2 \simeq O(10^{-5})$ and $f \gg \delta\Phi_s \simeq O(10^{-3})$. Therefore, the impact of the terms proportional to δ in Eq. (4) do not change the dynamics qualitatively or even quantitatively in a significant way in comparison to the case $\delta = 0$ discussed in Section 3. As a consequence, the avalanche duration t_h determined by Eq. (10) and the angle of halt, determined by $\Phi_h = \Phi(t_h)$ using Eq. (8), approximate the exact values very accurately. Only for very small driving frequencies $f \simeq O(10^{-3})$ or less, the second term in (15) dominates. In this case, the dynamics of Eq. (4) is basically unaffected by the modulation and the adiabatic limit discussed above applies.

5. Successive avalanching

So far, we have discussed the changes of the duration and the angle of halt of the initial avalanche due to additional rotation modulation. As already mentioned in Section 2.2, the dynamics can be even richer: Depending on the modulation frequency and modulation amplitude, a series of successive avalanches can be excited. This successive avalanching differs significantly from the *periodic* avalanching due to rotation with a *constant*, small rotation rate [3,11,12]. Successive avalanching due to periodic rotation modulation about a zero mean is generically a *non-periodic* process.

5.1. *Numerical example*

As a typical example we show in Fig. 4(a) the time evolution of velocities of the successive avalanches and in Fig. 4(b) the trajectories in the $\{v, \Phi\}$-phase space obtained from a numerical integration of Eq. (4) for $\Delta = 0.1$ and $f = \frac{5}{2}\Omega_0$ in a time range $0 < t < 15$. The system was started at time $t = 0$ infinitesimally above the maximum angle of repose. From Fig. 4(a) one sees that a first avalanching event with a large maximum velocity and duration is followed by subsequent avalanches with much smaller maximum velocity and duration. The spacings between successive avalanches, i.e. the duration of the rigid-body rotation with $v = 0$, are not constant. In addition, one can see from the phase space plot in Fig. 4(b) that the angle of halt of the subsequent avalanches slightly increases.

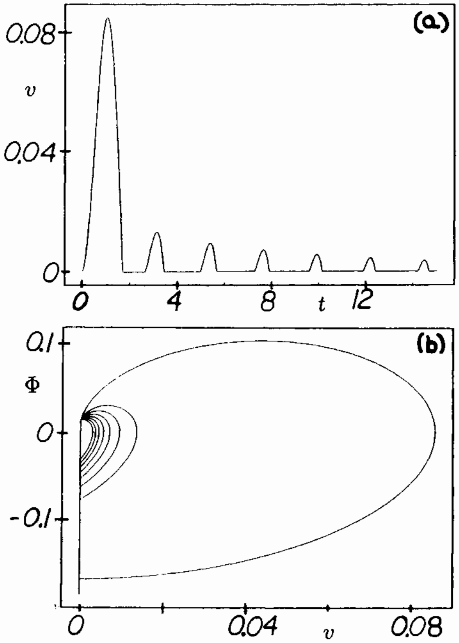

Fig. 4. (a) Velocity $v(t)$ of successive avalanches and (b) trajectories in the phase space spanned by $\{v, \Phi\}$ obtained by numerical integration of Eq. (4). All avalanches start at the maximum angle of repose determined by $\Phi_s = 0.014$; other parameters in (4) were fixed to $\Omega_0 = 1.1$, $\delta = 0.1$, $\Delta = 0.1$ and $f = (5/2)\Omega_0$.

5.2. *Theoretical considerations*

To understand the effect of successive avalanching analytically, we have to combine several results obtained previously in this study. Since we have numerically checked that setting $\delta = 0$ does not show any significant differences of the velocity evolution in comparison to realistic values of $\delta \simeq O(0.1)$, it is sufficient to discuss the approximation $\delta = 0$. The first avalanche starts at time $t = t_{s1} = 0$ and at the maximum angle of repose $\Phi(0) = \Phi_s$ with $\dot{\Phi}(0) = \Delta f$ and stops at time $t_{h1} \equiv t_h$ and at the angle of halt $\Phi_{h1} \equiv \Phi_h$; the values can be calculated from Eqs. (9) and (10). Then, the system undergoes rigid-body rotation according to $\Phi(t) = \Phi_{h1} - \Delta \sin f t_{h1} + \Delta \sin f t$. If $\Phi_s - \Phi_{h1} < \Delta(1 - \sin f t_{h1})$, the system will eventually reach the maximum angle of repose, Φ_s, again at time t_{s2}. In contrast to the first avalanche, the time derivative $\dot{\Phi}(t_{s2}) = \Delta f \cos f t_{s2}$ is in general different from and smaller than the time derivative of the angle of the first avalanche at its start. This explains why the first avalanche possesses the largest maximum velocity and the largest duration and also why subsequent avalanches differ in their maximum velocity and duration. With the new initial condition for the time derivative $\dot{\Phi}(t_{s2})$, the next avalanche starts. In Appendix A, we present a general analytical method to derive the process of successive avalanching. Depending on the entering parameters, this process can go on forever or end up in a pure rigid-body rotation. The iterative procedure in Appendix A also shows that (i) the successive avalanching process depends strongly on the history of the process, particularly on the duration of *all* previous avalanches and on *all* previous rigid-body rotations and (ii) the duration of the avalanches as well as the duration of the rigid-body rotations between them are not constant and do not seem to vary in a periodic way.

6. Summary

We have studied the dynamics of avalanches along granular piles induced by periodic rotation modulation, $\varphi_{\text{ext}} = \Delta \sin ft$, about a zero mean rotation rate. Although this effect has not been considered in the literature yet, it should be realizable in standard rotating drum experiments (as used e.g. in [2,11]). In our theoretical study which is based on a generalization of a recent mean field model [12,13], we have found a variety of new interesting dynamical phenomena. The most important results are:

(i) Due to rotation modulation, the duration of an avalanche and the angle of halt of the surface of the granular pile are significantly altered; in particular, the duration of the first avalanche is prolonged for small driving frequencies and shortened for larger driving frequencies.

(ii) For the physically relevant parameter ranges, a discussion of the *linearized* dynamics seems to be sufficient. The velocity dependence of the dynamical friction coefficient has no significant effect on the dynamics in presence of modulated rotation about a zero mean. This implies that the core of our study, Sections 3 and 5, also applies to mean field models with small negative differential friction [10,11]. Moreover, the dynamical behavior under periodic rotation is quite different from the one under constant rotation and/or vertical vibration [12,13] where the nonlinearity in the friction coefficient plays the key role. Therefore, the relevance of the velocity dependence in the friction strongly depends on the type of driving.

(iii) Rotation modulation can excite successive avalanching if the modulation amplitude Δ is large enough. In the example that we have studied, a large first avalanche is followed by many smaller ones with smaller and smaller maximum velocities. This successive *aperiodic* avalanching due to *periodic* rotation moduation seems to be generic.

An implicit assumption of our model is that the inclined surface remains basically flat during the modulation. This assumption might not be true if the rotation frequency is too large, and might restrict the applicability of our model. Therefore, we hope that our theory will stimulate further experimental studies on this subject in the near future.

Appendix A

Here, we provide – in form of an iterative procedure – the general formulas to analyze the successive avalanching process discussed in Section 5. The successive avalanching process alternates between avalanche ("slip") and rigid-body rotation ("stick") contributions. The main point in the following is that avalanches can only start at the maximum angle of repose Φ_s. The slip and stick contributions can be calculated as follows.

(1) *Avalanches:* The avalanche i started at time t_{si} and at the maximum angle of repose Φ_s obeys – in $\delta = 0$-approximation – Eq. (7), i.e. $\ddot{\Phi}_{\text{av}} + \Omega_0^2 \Phi_{\text{av}} = -\Delta f^2 \sin ft$, with initial conditions $\Phi_{\text{av}}(t_{si}) = \Phi_s$ and $\dot{\Phi}_{\text{av}}(t_{si}) = \Delta f \cos(f t_{si})$. Its solution for $t \geq t_{si}$ is given by

$$\begin{aligned}\Phi_{\text{av}}(t) &= \Phi_s \cos[\Omega_0(t - t_{si})] + \frac{\Delta}{\Omega_0^2/f^2 - 1} \\ &\times \left[\sin(f t_{si}) \cos[\Omega_0(t - t_{si})] + \frac{\Omega_0}{f} \cos(f t_{si}) \right. \\ &\left. \times \sin[\Omega_0(t - t_{si})] - \sin(ft)\right]. \qquad \text{(A.1)}\end{aligned}$$

The avalanche i stops at time t_{hi} and at the angle $\Phi_{hi} = \Phi_{\text{av}}(t_{hi})$ given by the first zero which fulfills the zero-velocity condition $\dot{\Phi}(t_{hi}) \equiv \Delta f \cos f t_{hi}$ and is larger than t_{si}. Explicitly, the condition for t_{hi} reads

$$\begin{aligned}0 \equiv\ & \Phi_s(\Omega_0^2 - f^2) \sin[\Omega_0(t_{hi} - t_{si})] + \Delta f \Omega_0 \\ & \times [\cos(f t_{hi}) - \cos(f t_{si}) \cos[\Omega_0(t_{hi} - t_{si})]] \\ & + \Delta f^2 \sin(f t_{si}) \sin[\Omega_0(t_{hi} - t_{si})]. \qquad \text{(A.2)}\end{aligned}$$

Setting $t_{si} = 0$ yields the solutions for the initial avalanche given in Eqs. (9) and (10).

(2) *Rigid-body rotation:* The solution of the rigid-body rotation after the avalanche i came to a halt at time t_{hi} and at the angle of halt Φ_{hi} can be obtained from $\dot{\Phi}_{\text{rbr}}(t) = \Delta f \cos ft$ with initial conditions $\Phi_{\text{rbr}}(t_{hi}) = \Phi_{hi}$. For $t > t_{hi}$ one obtains

$$\Phi_{\text{rbr}}(t) = \Phi_{hi} - \Delta \sin f t_{hi} + \Delta \sin ft \qquad \text{(A.3)}$$

and holds as long as $\Phi_{\text{rbr}}(t)$ is smaller than the maximum angle of repose Φ_s. If $\Phi_{\text{rbr}}(t) = \Phi_s$ is fulfilled for the first time again, the rigid-body rotation ceases to exist. The corresponding time t_{si+1} is the initial time of next avalanche, given by the first zero of

$$\sin f t_{si+1} = \frac{1}{\Delta}(\Phi_s - \Phi_{hi}) + \sin f t_{hi}, \tag{A.4}$$

which is larger than t_{hi}. Eq. (4) shows that a finite t_{si+1} can only exist if $\Delta \sim O(\Phi_s - \Phi_{hi})$; for too small driving amplitudes Δ, there does not exist any solution of (4) implying that no next avalanche develops.

From the preceding discussion follows: For each avalanche, one has to calculate three quantities: (i) the initial time t_{si} (ii) the time when the avalanche stops again, t_{hi}, and (iii) the angle of halt Φ_{hi}. In general, these three quantities differ in general from avalanche to avalanche and lead to aperiodic avalanching. An expansion in small Δ to obtain explicit results for t_{hi} and Φ_{hi} is not helpful; for the range of validity of this expansion, one can generically observe only one avalanche.

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