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# Escape over fluctuating barriers driven by colored noise

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Escape over *fluctuating* barriers in the presence of thermal white noise is addressed. Several general results are established, for stochastic barrier fluctuations being controlled by colored Gaussian noise. Our findings are exact in the limit of white noise sources and (partially) in the limit of extreme large noise color, and are approximate for intermediate noise color. As one main result we find that the escape time can generically exhibit a minimum resonant activation, whenever the colored noise intensity is an increasing function of the noise correlation time. The effects induced by correlated *noise sources* are addressed as well.

## 1. Introduction

Ever since the seminal achievements by Svante Arrhenius and Hendrik Antoine Kramers, the problem of escape from metastable states continues to attract ever growing interest, e.g. see refs. [1,2]. In particular, interesting variations of this topic arise when studying transport in complex systems, such as in glasses [3] and in proteins [4]. In this context the problem of escape over stochastic barriers has moved into the limelight within several scientific communities [5-12] [#1]. The interest in this concept of noise-assisted escape over fluctuating barriers has germinated when describing complex non-equilibrium systems such as the migration of ligands in proteins [4], molecular dissociation in strongly coupled chemical systems [5], or electron transport in a quantum double well structure [13], which is subjected to an external fluctuating voltage-bias, to name only a few examples. The problem area is also closely related to noise-assisted escape in systems with *fluctuating* potential parameters [8-10]. A characteristic feature of all these cases is that these are *open* systems, being in contact with one or more fluctuating environments, i.e. we deal with complex nonequilibrium systems, in which the fluctuations are generally not related to a fluctuation-dissipation theorem of the Einstein-Nyquist type. It must further be emphasized that – although related – the fluctuating barrier concept is different from the phenomenon of *stochastic resonance* [14,15], with the latter being characterized by time-dependent, but *deterministic* barrier modulations, i.e. the continuous time-translation symmetry is broken, thereby rendering these latter systems *nonstationary* nonequilibrium systems.

As correctly emphasized already in ref. [6], noise-assisted escape over fluctuating barriers involves several relevant time-scales. In particular the typical fluctuating barrier time-scale can be either very small, comparable to, or even be much larger than the average molecular time-scale characterizing local relaxation within metastable states. Therefore, the escape dynamics for the reaction coordinate $x(t)$ is generally governed by a *non-Markovian process* driven by both, white environmental noise $\xi(t)$, and colored, generally multiplicative barrier fluctuations $\zeta(t)$. The problem of obtaining the average escape time of fluctuating barriers thus becomes a challenging problem, because generally even the stationary probability is not known. Indeed, in all previous studies [6-12], one has been forced to impose severe limitations. These constitute either the restriction to the white or almost white noise limit (i.e. small colored noise limit) for the barrier-fluctuations [8-12], or the discussion had been restricted to both, the use of a very simple colored noise structure, such as exponentially correlated

[#1] In particular note the *News and Views* contribution in ref. [5].

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two-state noise, driving the barrier fluctuations (i.e. dichotomic noise $\zeta(t)$) together with a stylized metastable potential composed of a piecewise linear barrier and piecewise linear wells [6,7]. Even in this case the analytical analysis is already very complex so that Monte Carlo simulations had been invoked [6,7]. Nevertheless, Doering and Gadoua [6] discovered within these latter limitations a most interesting resonance-like phenomenon for the behavior of the average escape time; i.e. the escape time in their study did not grow monotonously with increasing noise color $\tau$, but instead did exhibit a minimum near a "resonant" barrier fluctuation rate $\tau_r^{-1}$. Clearly, the question then arises if this phenomenon is universal, i.e. if it still holds for realistic potential shapes and/or more realistic colored noise sources $\zeta(t)$.

The task to answer these open challenges stimulated the present work. Here, I have succeeded to obtain several results, which describe a variety of general phenomena for noise-assisted escape over fluctuating barriers. Most importantly, one finds that the resonance-phenomenon can occur *generically*, whenever the colored noise intensity,

$$Q \equiv \int_0^\infty |\langle\zeta(t)\zeta(0)\rangle| \,\mathrm{d}t ,$$

*increases with increasing noise-correlation time $\tau$.*

## 2. The approach

The starting point for our considerations is an arbitrary bistable flow for the reaction coordinate $x$. Explicitly, with the static metastable potential denoted by $U(x)$, we have $\dot{x} = -U'(x) = f(x)$, which possesses *two stable* deterministic fixed points $f(x_\pm) = 0$, with $f'(x_\pm) < 0$, and *one unstable* fixed point $f(x^\#) = 0$, with $f'(x^\#) > 0$, see fig. 1. The barrier fluctuations are governed by a fluctuating potential $W(x, \zeta) = -\zeta(t) \int^x g(y)\, \mathrm{d}y$, with $\zeta(t)$ a colored noise source. The function $g(x) = -W'(x, \zeta = 1)$ denotes the corresponding force-profile, which up to the condition $-g(f/g)' \geqslant 0$ within the bistable region $(x_-, x_+)$, can be chosen arbitrarily. Throughout this work, the prime denotes a differentiation with respect to $x$. The escape over a fluctuating barrier is then governed by the nonlinear non-Markovian Langevin equation

$$\dot{x} = f(x) + g(x)\zeta(t) + \sqrt{2T}\,\xi(t) , \tag{1}$$

where $\xi(t)$ is white Gaussian noise of vanishing mean and correlation $\langle \xi(t)\xi(s)\rangle = \delta(t-s)$, reflecting environmental (thermal) noise, while the colored noise $\zeta(t)$ controls the barrier fluctuations. A common example for $f(x)$ is the archetypal Landau flow $f(x) = ax - bx^3$, $a > 0$, $b > 0$; while $g(x)$ could be Gaussian, i.e. $W(x,$

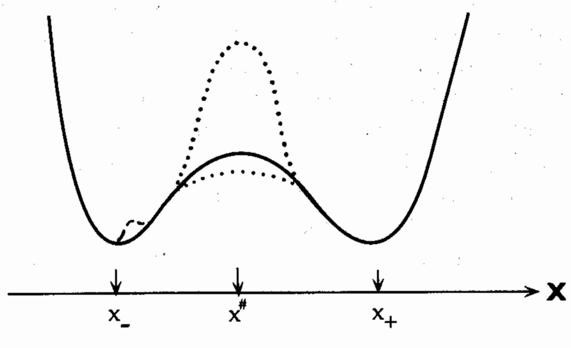


Fig. 1. Escape over a fluctuating barrier. The solid line depicts the static potential with $x_\pm$ denoting the stable states and $x^\#$ the unstable, activated state. The dotted lines present two realizations for the fluctuating barrier. The dashed line shows a slight modification of the static potential away from barrier top which in turn changes the corresponding value for the resonant noise color $\tau_r$, see text below eq. (18).

$\zeta) = -\zeta(2\alpha)^{-1}\exp(-\alpha x^2)$, yielding $g(x) = x\exp(-ax^2)$ [#2]. Note, that for $\alpha=0$ we have $g(x)=x$, which corresponds to a fluctuating barrier curvature [8], i.e. $a\to a+\zeta(t)$. In order to define the stochastic process in eq. (1) completely it is necessary to specify both the individual and the joint statistical properties of $\zeta(t)$ and $\xi(t)$. Bearing in mind the central limit theorem, we use for $\zeta(t)$ a *Gaussian* statistics. For the sake of simplicity only, we choose an exponentially correlated Gaussian noise (Ornstein–Uhlenbeck process) of vanishing mean and correlation, i.e.

$$\langle\zeta(t)\zeta(s)\rangle = \frac{Q}{\tau}\exp\left(-\frac{|t-s|}{\tau}\right). \tag{2}$$

Moreover, to start with we make here the assumption that $\zeta(t)$ and $\xi(t)$ are independent, i.e. $\langle\zeta(t)\xi(s)\rangle=0$ for all $t$, $s$; see however section 4.3 below. The non-Markovian, multiplicative Langevin equation in eq. (1) is then equivalently recast as a two-dimensional (Stratonovitch) Langevin equation, reading

$$\dot{x}=f(x)+g(x)\zeta(t)+\sqrt{2T}\,\xi(t)\,, \tag{3}$$

$$\dot{\zeta}=-\frac{1}{\tau}\zeta+\frac{\sqrt{2Q}}{\tau}\eta(t)\,, \tag{4}$$

where $\eta(t)$ is a again Gaussian white noise, obeying $\langle\eta(t)\eta(s)\rangle=\delta(t-s)$, and $\langle\eta(t)\xi(s)\rangle=0$. The white noise limit then emerges naturally by observing that $\lim_{\tau\to 0}=\langle\zeta(t)\zeta(0)\rangle=2Q\delta(t)$.

The idea underlying our approach is as follows: In realistic situations the (dimensionless) noise intensities $T$ and $Q$ are "small". With $T\ll 1$, $Q\ll 1$, but with the ratio $R\equiv Q/T$ finite, we encounter escape times which are exponentially large. Put differently, the (forward: $x_-\to x_+$) escape time $\mathscr{T}$ exhibits an Arrhenius-like behavior, which is dominated by the ratio of the stationary probability $\bar{p}(x,\tau)$ at the stable state $x_-$ and the unstable state $x^*$. Setting $\bar{p}(x,\tau)=h(x,\tau)\exp[-\Phi(x,\tau,R)/T]$ one has within exponential accuracy

$$\mathscr{T}(R,\tau)\propto\exp\left(\frac{\Delta\Phi(R,\tau)}{T}\right), \tag{5}$$

where in terms of the effective potential $\Phi$ the barrier height equals $\Delta\Phi=\Phi(x^*)-\Phi(x_-)$. Thus, we are not interested in obtaining an accurate approximation of the non-Markovian Langevin dynamics in eq. (1) on all time-scales, but rather are interested in the *long-time* dynamical properties only. Indeed, if we would study the limit of small noise color by expanding eq. (1), via the functional derivative method [16], around the $\tau=0$ limit one finds – in agreement with the general theory [17] – that there exists to leading order in $\tau$ no small-$\tau$ effective Fokker–Planck equation. With $\zeta(t)$ colored, the flow in eq. (1) can thus never be transformed into purely additive noise alone. This fact in turn implies a third order Kramers–Moyal-type contribution for the rate of change $\dot{p}_t(x,\tau)$, being of order $\tau$. Moreover, it is of interest to establish whether novel phenomena such as the resonant-like behavior of Doering and Gadoua [6] persist under realistic conditions; in particular, that they are not the mere result of some "prefactor-effect" appearing only at strong noise intensities $Q$ and/or $T$ within a stylized metastable potential form.

In the presence of a single noise source $\zeta(t)$ only, the unified colored noise approximation (UCNA) [18–20] has proven to accurately model the stationary dynamics of colored noise driven flows [18–21]. Borrowing the reasoning underlying [18–20], one can similarly implement this long-time approximation scheme for the flow in eqs. (3), (4). In terms of the process $u(t)$, i.e. $ug(x)=f(x)+\zeta g(x)$, eqs. (3), (4) can be recast as

$$\dot{x}=ug+\sqrt{2T}\,\xi\,, \tag{6a}$$

$$\dot{u}=-[1/\tau-g(f/g)']u+\tau^{-1}(f/g)+\tau^{-1}\sqrt{2Q}\,\eta+(f/g)'\sqrt{2T}\,\xi\,. \tag{6b}$$

[#2] In this case $-g(f/g)'=2bx^2-2\alpha x^2(a-bx^2)$, being nonnegative within $[x_-=-\sqrt{a/b},x_+=+\sqrt{a/b}]$ for $\alpha<b/a$, see also ref. [24].

By use of the time-scale $\hat{t}=t\tau^{-1/2}$ [18–20], the deterministic equation corresponding to eq. (6b) reads

$$\frac{\mathrm{d}u}{\mathrm{d}\hat{t}}=-\left[\tau^{-1/2}-\tau^{1/2}g(f/g)'\right]u+\tau^{-1/2}(f/g)\,. \tag{7}$$

With the nonlinear friction obeying $\gamma(x,\tau)\equiv\tau^{-1/2}-\tau^{1/2}g(f/g)'\geqslant 0$ for all $\tau$ when $-g(f/g)'\geqslant 0$ we note that $\gamma(x,\tau)\gg 1$ both for $\tau\to 0$ and $\tau\to\infty$. An adiabatic elimination of $\dot{u}$ thus renders the generalized UCNA for eq. (1), i.e. the (Stratonovitch)–Markovian–Langevin approximation reads in the original time-scale $t$

$$\dot{x}=\left[1-\tau g(f/g)'\right]^{-1}\left(f+g\sqrt{2Q}\,\eta+\sqrt{2T}\,\xi\right), \tag{8}$$

which possesses a corresponding Fokker–Planck equation. In passing, we note that the same (Stratonovitch) Fokker–Planck equation corresponding to eq. (8) results if one performs within the configuration-state path integral representation for the non-Markovian process in eq. (1) a self-consistent Markovian approximation in the non-Markovian–Onsager–Machlup functional, cf. ref. [22]. The stationary probability (up to a normalization constant) thus reads

$$\bar{p}(x,\tau)=\frac{|1-\tau g(f/g)'|}{[1+Rg^2(x)]^{1/2}}\exp\left(\int_0^x\frac{f[1-\tau g(f/g)']}{T(1+Rg^2)}\,\mathrm{d}y\right). \tag{9}$$

The result eq. (9) approximates generally very accurately the stationary non-Markovian probability over the bistable region $(x_-,x_+)$ within its support, i.e. in $x$-regions obeying $[1-\tau g(f/g)']>0$, cf. refs. [18–21].

## 3. Average escape time over fluctuating barriers

With the long-time approximation to eq. (1) in hand, it is smooth sailing towards obtaining the average escape time $\mathscr{T}$. This can be estimated by the mean first passage time (MEPT) expression for the one-dimensional Fokker–Planck process in eq. (8), i.e. $\mathscr{T}_{\mathrm{MFPT}}(x_-\to x_+)\equiv\mathscr{T}(R,\tau)$ is given by the two quadratures

$$\mathscr{T}(R,\tau)=T^{-1}\int_{x_-}^{x_+}\frac{\mathrm{d}x}{D_{\mathrm{eff}}(x,\tau)\bar{p}(x,\tau)}\int_{-\infty}^{x}\bar{p}(y,\tau)\,\mathrm{d}y\,, \tag{10}$$

where $x=x_+$ has been chosen to be an absorbing boundary, and $D_{\mathrm{eff}}(x,\tau)=(1+Rg^2)\,[1-\tau g(f/g)']^{-2}>0$. With weak noise, the steepest descent approximation to eq. (10) explicitly reads

$$\mathscr{T}(R,\tau)=\frac{2\pi}{|(f'(x^{\#})f'(x)|^{1/2}}\left[|1-\tau g(f/g)'|_{x=x^{\#}}\,|1-\tau g(f/g)'|_{x=x_-}\right]^{1/2}\exp\left(\frac{\Delta\Phi(R,\tau)}{T}\right) \tag{11}$$

with the effective barrier height given by

$$\Delta\Phi(R,\tau)=-\int_{x_-}^{x^{\#}}\frac{f(y)\,[1-\tau g(y)\,(f(y)/g(y))']\,\mathrm{d}y}{1+Rg^2(y)}>\Delta\Phi(R,\tau=0)\,. \tag{12}$$

Here, eq. (11) presents a most accurate approximation to the exact non-Markovian escape time for $\tau\to\infty$, and $\tau\to 0$, i.e. $\gamma(x,\tau)\to\infty$. For other values of noise color, the result in eq. (11) implicitly presents a crossover result, bridging smoothly between the limits of small and large noise color.

We remark that the condition $-g(f/g)'\geqslant 0$ in $(x_-,x_+)$ can be relaxed without changing the qualitative features of our results (see also ref. [24]): With $x_-^{\mathrm{s}}(\tau)$, i.e. $1-\tau g(f/g)'=0$ at $x_-^{\mathrm{s}}$, $x_-\leqslant x_-^{\mathrm{s}}(\tau)\leqslant x^{\#}$, the effective diffusion becomes singular at $x_-^{\mathrm{s}}(\tau)$. Escape then predominantly occurs near $x_-^{\mathrm{s}}(\tau)$, i.e. the upper integration limit in eq. (12) is substituted by $x_-^{\mathrm{s}}(\tau)$. The fact that the diffusion is singular at $x_-^{\mathrm{s}}(\tau)$ – just as is also the case for the action–diffusion for the Kramers time at weak friction [1] – nevertheless yields with smooth $\Delta\Phi(R,\tau)$ a well defined escape time. We also like to point out that the use of $x^{s}_{-}(\tau)$, rather than $x^{c}_{-}\equiv x^{s}_{-}(\tau\to\infty)<x^{s}_{-}(\tau)$, also considerably improves the result for colored noise driven escape at finite $\tau$-values, cf. ref. [20].

## 4. Results

From eqs. (9)–(12) we can now establish a variety of general findings. First we shall consider the limit of white noise for both the barrier fluctuations $\zeta(t)$ and the internal thermal noise $\xi(t)$.

### 4.1. The limit of white noise

For zero noise color $\tau=0$, the above results in eqs. (7)–(12) become exact. The MFPT in eq. (10) can be evaluated at weak noise up to order $O(T^2)$ to give

$$\begin{aligned}
&\mathcal{T}(R,\tau=0)\equiv\mathcal{T}(R)\\
&=\exp[\Delta\Phi(R)/T]\,\frac{2\pi}{[\,|\Phi''(x^*)|\,\Phi''(x_-)]^{1/2}}\Bigg\{h(x^*)h(x_-)+T\Bigg[\frac{1}{2}\left(\frac{h''(x^*)h(x_-)}{|\Phi''(x^*)|}+\frac{h''(x_-)h(x^*)}{\Phi''(x_-)}\right)\\
&+\frac{1}{2}\left(\frac{h'(x^*)h(x_-)\Phi'''(x^*)}{|\Phi''(x^*)|^2}-\frac{h(x^*)h'(x_-)\Phi'''(x_-)}{[\Phi''(x_-)]^2}\right)\\
&+\frac{1}{8}\left(\frac{h(x^*)h(x_-)\Phi''''(x^*)}{|\Phi''(x^*)|^2}-\frac{h(x^*)h(x_-)\Phi''''(x_-)}{[\Phi''(x_-)]^2}\right)\\
&+\frac{5}{24}\left(\frac{h(x^*)h(x_-)[\Phi'''(x^*)]^2}{|\Phi''(x^*)|^3}+\frac{h(x^*)h(x_-)[\Phi'''(x_-)]^2}{[\Phi''(x_-)]^3}\right)\Bigg]\Bigg\},
\end{aligned}\tag{13}$$

where $\Phi(x)$ is the effective potential

$$\Phi(x)=\int^{x}-f(1+Rg^2)^{-1}\,\mathrm{d}y\,,\tag{14a}$$

and $h(x)$ is a state-dependent form function given by

$$h(x)=[1+Rg^2(x)]^{-1/2}\,,\tag{14b}$$

which is assumed to be smoothly varying.

The third (1/8...) and fourth term (5/24...) within the squared brackets in eq. (13) describe the well-known [23] steepest-descent correlation to the Smoluchowski escape time, while the additional first and second contribution emerge due to the multiplicative character of the white noise sources, cf. eq. (1).

From eq. (12) we further find

$$\Delta\Phi(R)\leqslant\Delta\Phi(R=0)\,.\tag{15}$$

Moreover, from eqs. (11), (15) the escape time $\mathcal{T}(R)$ is monotonously *decreasing* with increasing $R=Q/T$, i.e.

$$\mathcal{T}(R)\leqslant\mathcal{T}(R=0)\,.\tag{16}$$

The result in eq. (16) is in agreement with prior studies investigating white noise driven escape over fluctuating barriers [8,11,12].

### 4.2. Case with colored noise

We now turn to the main focus of our work, namely the escape over fluctuating barriers which are modulated by colored noise of *weak-to-moderate-to-strong noise correlation time* $\tau$:

(i) Fixed colored noise intensity. With $T \ll 1$, $Q \ll 1$ let us keep fixed the ratio $R = Q/T$. The colored noise assisted escape time over a fluctuating barrier is then always *enhanced*, i.e.

$$\mathcal{T}(R,\tau) \geqslant \mathcal{T}(R,\tau=0)\,. \tag{17}$$

We note from eq. (11) that this increase is Arrhenius-like, and it occurs monotonously. The characteristic behavior in eqs. (16), (17) can be made plausible if we observe that in the white noise limit the escape is driven by an *enhanced* state-dependent temperature $T(x,R) = T[1+Rg^2(x)] > T$, while from eqs. (9), (11) colored noise driven escape (at $R$ fixed) is governed by a *lower*, effective temperature $T(x,\tau) = T/[1-\tau g(f/g)'] < T$.

(ii) Resonant activation. If one notes the two inequalities in eqs. (16), (17), which are obeyed monotonously, one finds that the overall effective temperature in eqs. (9)–(11), i.e.

$$\hat{T}(x,\tau) = T[1+Rg^2(x)][1-\tau g(x)(f(x)/g(x))']^{-1} \equiv T^{-1}T(x,R)\,T(x,\tau)\,,$$

which in turn controls the escape process, can be either smaller or larger than $T$. Therefore, with $R$ not held fixed, but being a function of noise color $\tau$, such that $R(\tau)$ increases with increasing noise color, a *competition* between the monotonous decrease in eq. (16) and the monotonous increase in eq. (17) becomes possible. With $R = R(\tau)$ being increasing with $\tau$, the escape time $\mathcal{T}(R(\tau),\tau)$ can thus attain a minimum at a "resonant" noise correlation time $\tau_r$ for which the effective barrier height $\Delta\Phi(R(\tau),\tau)$ assumes a *minimal value*! Setting from eq. (11) $\mathcal{T}(R,\tau) = A(\tau)\exp[\Delta\Phi(R,\tau)/T]$, with $\Delta\Phi(R(\tau),\tau)$ given by eq. (12), the resonant value (or values) $\tau_r$ obeying $\mathrm{d}\mathcal{T}(R(\tau),\tau)/\mathrm{d}\tau = 0$ can, with $A(\tau) \approx$ const, be estimated from the minimum of $\Delta\Phi(R(\tau),\tau)$, i.e.

$$\int_{x_-}^{x^\#} \frac{f\{[1+R(\tau)g^2]g(f/g)' + (\mathrm{d}R/\mathrm{d}\tau)g^2[1-\tau g(f/g)']\}}{[1+R(\tau)g^2]^2}\,\mathrm{d}y = 0\,, \tag{18}$$

which with $f \leqslant 0$, $g(f/g)' \leqslant 0$ within $(x_-, x^\#)$ and $\mathrm{d}R/\mathrm{d}\tau > 0$ always possesses, with sufficiently strongly increasing $R(\tau)$, a solution for $\tau_r$. The width of the "resonance" can further be estimated from the inverse of $\mathrm{d}^2\Delta\Phi(R(\tau),\tau)/\mathrm{d}\tau^2$. Most importantly, we note that the value of the "resonant"-color time $\tau_r$ is not attained at the adiabatic minimum of the fluctuating barrier, but depends *globally* on both the static metastable potential shape (or its force $f(x)$) and the barrier-modulation function $g(x)$. Put differently, modifying slightly the potential *away from the barrier top* dictates already a different "resonant" noise color value $\tau_r$, cf. fig. 1. This resembles very much quantum tunneling where the Gamow-factor for barrier transmission depends globally on the potential shape and not just on the barrier height, as is the case for thermally activated escape [1]. Within the piecewise linear barrier model driven by two-state noise in ref. [6] the authors implicitly used $Q/\tau = \text{const} = C$; i.e. $R(\tau) = Q/T = C\tau/T$ indeed increases with increasing noise color $\tau$.

Thus, upon inspecting eq. (11) with $R(\tau)$ *being a function of noise color* $\tau$, *the effective barrier* $\Delta\Phi(R,\tau)$ in eq. (12) exhibits one amongst the following three characteristic behaviors: (I) with a solution of eq. (18) at finite $\tau$, the effective barrier depicts a minimum as a function of increasing $\tau$, (II) with eq. (18) obeyed only for $\tau \to \infty$ the effective barrier increases towards an asymptotically flat value as $\tau \to \infty$, (III) with $R(\tau)$ not sufficiently increasing with $\tau$ the behavior is as in eq. (17), [redacted]

(iii) Symmetries. We next consider *symmetric* bistable potentials $U(x)$ such that $-U'(x) = f(x) = -f(-x)$

is an odd function. The flow in eqs. (3), (4) then exhibits a different symmetry depending whether the barrier modulation function $g(x)$ is even or odd, respectively; i.e.

$$\text{inversion symmetry:} \quad x \to -x, \zeta \to -\zeta, \quad \text{if } g(x) = g(-x), \tag{19}$$

$$\text{reflection symmetry:} \quad x \to -x, \zeta \to +\zeta, \quad \text{if } g(x) = -g(-x). \tag{20}$$

These symmetries drastically impact the behavior of the separatrix, which divides the deterministic domain of attraction of the bistable flow in eqs. (3), (4). With an odd modulation, the separatrix is described by the line $x=0$, whereas for even $g(x)$ (e.g. $g(x)=\text{const}$) the separatrix is moving into the $x$–$\zeta$ plane, crossing at ($x=\zeta=0$) from left to right [20,25].

(iv) Behavior at extreme noise ratios $R$. It turns out that the behavior for $\mathscr{T}(R, \tau)$ exhibits a different asymptotic behavior depending on whether $Q/T=R \ll 1$, or $R \gg 1$. In the latter case the escape is dominated by the noise intensity $Q$, rather than $T$. Putting a particle initially at $x=x_-$, the escape dynamics within the $(x, \zeta)$-phase space of eqs. (3), (4) closely follows for $R \gg 1$ the line $\epsilon(x) = -f(x)/g(x)$, where the deterministic flow lines (i.e. $T=Q=0$ in eqs. (3), (4)) exhibit turning points, i.e. $d\zeta/dx=\infty$, see fig. 2. If we denote by $\zeta_M$ the maximum of $|f(x)/g(x)|$ within $(x_-, x^*)$, the asymptotic behavior for $\mathscr{T}(R \gg 1, \tau)$ reads

$$\mathscr{T}(R \gg 1, \tau) = \mathscr{T}(R)[1+O(\tau)] \exp\{\tau[\zeta_M^2 + O(R^{-1})]/2Q\}. \tag{21}$$

Note that for $\tau \to \infty$, the exponential increase given by the last term dominates over all the remaining contributions.

For the bistable symmetric Landau potential with a fluctuating curvature [24], i.e. $f(x)=ax-bx^3$, $g(x)=x$, one finds $\zeta_M=a$, i.e. eq. (11) yields

$$\mathscr{T}(R \gg 1, \tau) = \mathscr{T}(R)(1+2a\tau)^{1/2} \exp\{\tau[a^2 + O(R^{-1}, R^{-1}\ln R)]/2Q\}, \tag{22}$$

while in the opposite limit, and not too large noise color, the escape time behaves as

$$\mathscr{T}(R \ll 1, \tau) = \mathscr{T}(R)(1+2a\tau) \exp(a^3\tau/6bT), \quad a\tau < 1. \tag{23}$$

This makes explicit that in the latter case with $Q \to 0$, $T \to 0$, $T \gg Q$, the escape is dominated by the additive thermal noise $\xi(t)$.

### *4.3. Correlated noise sources*

Throughout the above analysis we assumed that the colored noise $\zeta(t)$ driving the barrier fluctuations and the internal white noise $\xi(t)$ were *not* correlated. This assumption, however, might not always hold a priori. In particular, when the barrier fluctuations are not imposed externally by the experimenter, but rather are the result

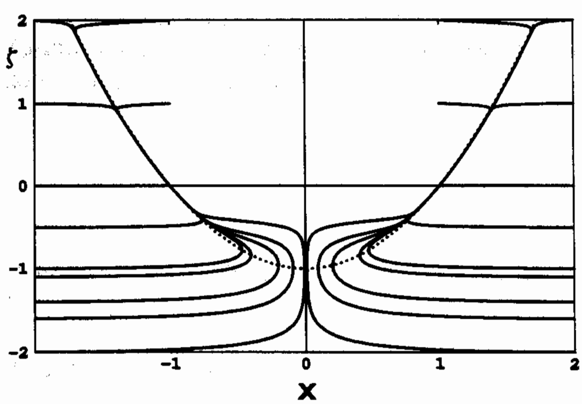


Fig. 2. Deterministic trajectories for the archetypal Landau flow: $\dot{x}=x-x^3+x\zeta$; $\dot{\zeta}=-\zeta/\tau$, for the noise correlation time $\tau=15$. The dotted line depicts the line of turning points $\epsilon(x)=x^2-1$. The separatrix is given by the line $x=0$.

of strong couplings to random nonequilibrium environments, additive and multiplicative noise contributions likely become correlated. Within our approach, such a correlation can be described by setting in eqs. (3), (4) $\langle\eta(t)\xi(s)\rangle = \rho\delta(t-s)$ with $|\rho|\leqslant 1$, which guarantees a positive definite diffusion tensor. The corresponding UCNA Fokker–Planck equation becomes rather complex, reading explicitly

$$\dot{p}_t = -\frac{\partial}{\partial x}\Big(\{C^{-1}(x,\tau)f + Q[gC^{-1}(x,\tau)](gC^{-1}(x,\tau))' + TC^{-1}(x,\tau)(C^{-1}(x,\tau))' + \rho\sqrt{QT}\,C^{-1}(x,\tau)[(gC^{-1}(x,\tau))' + g(C^{-1}(x,\tau))']\}p_t\Big) + T\frac{\partial^2}{\partial x^2}[C^{-2}(x,\tau)(1+Rg^2+2\rho\sqrt{R}\,g)p_t], \quad (24)$$

where $C(x,\tau) = 1-\tau g(f/g)'$ has been used. For the corresponding Arrhenius factor one obtains from eq. (24) the result

$$\Delta\Phi(R,\tau,\rho) = -\int_{x_-}^{x^*} \frac{f[1-\tau g(f/g)']}{1+Rg^2+2\rho\sqrt{R}\,g}\,\mathrm{d}y\,. \quad (25)$$

For the important case of a symmetric static barrier, $f(x) = -f(-x)$ and symmetric barrier modulations $W(x) = W(-x)$, i.e. $g(x) = -g(-x)$, the forward and backward escape times $\mathcal{T}(R,\tau)$ are no longer equal. From eq. (25) we find with $\rho>0$: $\mathcal{T}_+(R,\tau=0,\rho) < \mathcal{T}_+(R,\tau,\rho)$, and $\mathcal{T}_+(R,\tau,\rho) > \mathcal{T}_-(R,\tau,\rho) > \mathcal{T}_-(R,\tau=0,\rho)$, where we assumed $g(x-x^*)<0$, for $x-x^*<0$. Here $\mathcal{T}_\pm$ denotes the forward $(x_-\to x_+)$ and backward $(x_+\to x_-)$ escape time, respectively. For $\tau=0$, the above effective barrier becomes exact, yielding $\mathcal{T}_+(R,\rho) > \mathcal{T}_-(R,\rho)$, $\rho>0$. However, we find that $\mathcal{T}_+(R,\rho) \leqslant \mathcal{T}(R=0)$ generally is *no longer* obeyed. This is so, because the two contributions $Rg^2$ and $2\rho\sqrt{R}\,g$ making up the diffusion coefficient are with $g<0$ in $(x_-, x^*)$ of different sign. In particular, with $R(\tau)$ increasing with $\tau$ the possibility of a "resonance-behavior" is not necessarily guaranteed.

## 5. Conclusions and outlook

There are a number of further investigations suggested by our general study of noise-assisted escape over fluctuating barriers. The role of *correlated* noise sources certainly deserves future research efforts. Another area which remained untouched is the study of inertia effects, and more generally, the influence of additional relevant degrees of freedom, i.e. the role of multidimensional (fluctuating) barrier crossing [26]. In presence of colored noise sources this latter task obviously becomes very difficult [1,20].

We could demonstrate that the phenomenon of "resonance-like" escape in ref. [6] is generic if the noise temperature $Q(\tau)$ is sufficiently strongly increasing with increasing noise color $\tau$. This resonance essentially occurs when the color-induced effective barrier in eq. (12) assumes a minimal value: Its minimum depends globally on both the static potential shape $U(x)$ and the barrier modulation $W(x)$. In the context of surmounting fluctuating barriers in metastable nanostructures, the influence of *non-Gaussian statistics* (e.g. shot-noise) for both the colored noise source $\zeta(t)$ and the white noise $\xi(t)$ is of interest as well. The author hopes to return to this latter area in a future study [#3].

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[#3] For two-state noise, the *exact* result for the escape time $\mathcal{T}(R,\tau,\rho=\pm 1)$ follows from eq. (11) in ref. [27], or eq. (3.5) in ref. [28], which both give the inverse escape time (i.e. the flux-over-population escape rate) in terms of two quadratures.

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