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DISSIPATIVE QUANTUM NOISE IN A PARAMETRIC OSCILLATOR

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ABSTRACT

In this paper we investigate exact solutions for a parametric quantum oscillator. Without dissipation we focus on the propagator and the variances of momentum and position. In the presence of dissipation (Ohmic heat bath) we apply the influence-functional method due to Feynman and Vernon to obtain exact expressions for the time evolution of the reduced density matrix. Knowing this density matrix we calculate and discuss the variances in presence of friction.

1 INTRODUCTION

The potential of the system has the form

$$V(x,t) = \frac{1}{2}m(a - b\cos\Omega t)x^2. \tag{1}$$

This potential has several possible applications. One major application is the study of the Paul trap in the quantum regime[1]. Another suggested application is the generation of squeezed states of light[2]. and particulary interesting is the application to the topic of tunneling through a barrier with a time dependent barrier-width.
Although the system under consideration can be used as an amplifier it should be distinguished from the so-called 'parametric amplifiers' studied by Louisell, Yariv, and Siegman[3] and Mollow and Glauber[4]. Their model consists of two harmonic oscillators coupled bilinearly via a time dependent parameter which oscillates at the combination frequency of the two individual oscillators. The Hamiltonian for this system is time dependent, but elimination of one harmonic oscillator in the Heisenberg equations of motion leads to a differential equation with constant coefficients.

2 THE QUANTUM PARAMETRIC OSCILLATOR WITHOUT DISSIPATION

Introducing the scaled variables $\bar{x} = \sqrt{m\Omega/2\hbar}x$ and $\bar{t} = \Omega t/2$ the dimensionless Schrödinger equation for eq. (1) reads

$$i\dot{\Psi}(\bar{x},\bar{t}) = \left[-\frac{1}{2}\partial_{\bar{x}}^2 + \frac{1}{2}\omega^2(\bar{t})\bar{x}^2\right]\Psi. \tag{2}$$

with $\omega^2(\bar{t}) = \bar{a} - 2\bar{b}\cos 2\bar{t}$, $\bar{a} = 4a/m\Omega^2$, $\bar{b} = 2b/m\Omega^2$. In this section we will use only scaled variables and *henceforth omit the overbars.*

The periodicity of the Hamiltonian leads to Floquet form solutions of the Schrödinger equation. A solution $\Psi_n(x,t)$ of eq.(2) can be factorized as

$$\Psi_n(x,t) = \exp(-i\epsilon_n t)\phi_n(x,t), \qquad \phi_n(x,t) = \phi_n(x,t+\tau). \tag{3}$$

ϕ_n is called Floquet function, ϵ_n a Floquet- or quasienergy. Because of the linearity of the system ϕ_n and ϵ_n are fully determined through the solutions of the corresponding classical problem, i.e.

$$\ddot{x} + \omega^2(t)x = 0. \tag{4}$$

It is *not possible* to obtain the solution in explicit form, but with $\omega^2(t)$ defined like in (2) this is the well-studied Mathieu-equation. Depending on the value of the parameters a and b the solution of (4) can be bounded or increasing with time. Whenever we need explicit solutions of (4) we calculate them numerically.

There are different approaches to the quantum mechanical problem. The Floquet functions and Floquet energies for the three regions were given first by Perelomov and Popov[5]. In the stable region a discrete spectrum of qusienergies exist. In the unstable regions and at the boundaries between these regions the spectrum becomes continous.

The propagator for this system, obtained first by Husimi[6] , can be derived in a variety of ways[7]. One possibility is based on Feynman's path integral method. Given the solutions of eq. (2) it is also possible to construct the propagator directly in terms of a *spectral representation*, i.e.

$$K(x_f,t_f|x_i,t_i) = \sum_{n=0}^{\infty} \phi_n(x_f,t_f)\phi_n^*(x_i,t_i)\exp[-\frac{i}{\hbar}\epsilon_n(t_f-t_i)]. \tag{5}$$

For a continous spectrum the sum becomes an integral and we have to take into consideration possible degeneracies of the quasienergy spectrum. Doing so we obtain for the propagator

$$K(x_f,t_f;x_i,t_i) = \sqrt{\frac{1}{2\pi i X(t_f)}}exp\left[\frac{i}{2X(t_f)}(x_f^2\dot{X}(t_f) - 2x_f x_i + x_i^2 Y(t_f))\right] =$$

$$= e^{-i\frac{\pi}{2}[m(t_f)-m(t_i)]}\sqrt{\frac{1}{2\pi i|X(t_f)|}}exp\left[\frac{i}{2X(t_f)}(x_f^2\dot{X}(t_f) - 2x_f x_i + x_i^2 Y(t_f))\right]. \tag{6}$$

X and Y are special solutions of (4) with the initial conditions

$$X(t_i) = 0, \qquad \dot{X}(t_i) = 1, \qquad Y(t_i) = 1, \qquad \dot{Y}(t_i) = 0. \tag{7}$$

$m(t)$ is the number of zeros of X in the interval $[0,t]$, $m(0) = 0$ and we used the definition of the root

$$X^{\frac{1}{2}}(t) = |X|^{\frac{1}{2}}e^{i\frac{\pi}{2}m(t)}.$$

But this propagator (6) is valid only for times $t_f \neq t_n$. With t_n we denote the time when the m-th zero of X occur. To calculate the propagator at these so called *caustics* we use the semigroup property of the propagator

$$K(x_n, t_n; x_0, 0) = \int dx_c \, K(x_n, t_n; x_c, \pi) K(x_c, \pi; x_0, 0). \tag{8}$$

This relation holds for any time order of $0, \pi, t_n$. It is not necessary that $\pi < t_n$. We have chosen a special time $t_c = \pi$ because then we can employ the following relations for the solutions of the Mathieu equation

$$Y(\pi - t) = Y(\pi)Y(t) - \dot{Y}(\pi)X(t), \qquad X(\pi - t) = X(\pi)\dot{Y}(t) - \dot{Y}(\pi)X(t). \tag{9}$$

For the propagator at a caustic we find explicitly the result

$$K(x_n, t_n; x_0, 0) = e^{-i\frac{\pi}{2}m(t_n)} \frac{1}{\sqrt{|Y(t_n)|}} exp\left[\frac{i}{2}\frac{\dot{Y}(t_n)}{Y(t_n)}x_n^2\right] \delta(x_n - Y(t_n)x_0). \tag{10}$$

It was shown before with various methods that a time dependent harmonic oscillator generates squeezed states[2]. To study its squeezing properties explicitly we compute the variances of the operators x and p with the Heisenberg equation of motion. The mean values for this linear system follow the solutions of the classical equation (4).
The variances $U(t) \equiv \langle x^2 \rangle - \langle x \rangle^2$, $V(t) \equiv \frac{1}{2}\langle xp + px \rangle - \langle x \rangle \langle p \rangle$ and $W(t) \equiv \langle p^2 \rangle - \langle p \rangle^2$ satisfy the coupled set of equations

$$\dot{U} = 2V, \qquad \dot{V} = W - \omega^2(t)U, \qquad \dot{W} = -2\omega^2(t)V. \tag{11}$$

By eliminating $V(t)$ and $W(t)$ from eqs. (11) we find an equivalent third-order equation for $U(t)$

$$\dddot{U} + 4\omega^2(t)\dot{U} + 2\left\{\frac{d}{dt}[\omega^2(t)]\right\} U = 0 \tag{12}$$

This equation is solved with $U(t) = W(0)X^2 + U(0)Y^2 + 2V(0)XY$
where X and Y are defined as before in (6). Because of (11)

$$V(t) = W(0)X\dot{X} + U(0)Y\dot{Y} + V(0)(X\dot{Y} + \dot{X}Y) \text{ and}$$

$$W(t) = W(0)\dot{X}^2 + U(0)\dot{Y}^2 + 2V(0)\dot{X}\dot{Y}.$$

We see that the variances are bounded or increasing with time like the solutions of the Mathieu-equation in the corresponding region. Depending on the chosen parameters the form of the results varies strongly. In figs. 1 we plot the time variation of U for a fixed value of a. We start at $t = 0$ with a wavepacket with minimum uncertainty $U(0) = 1/2r\sqrt{a - 2b}$, $V(0) = 0$, $W(0) = r\sqrt{a - 2b}/2$. r is a parameter which characterizes the amount of squeezing of the state: $r = 1$ refers to the unsqueezed state. Fig. 1a shows U for a squeezed state and different

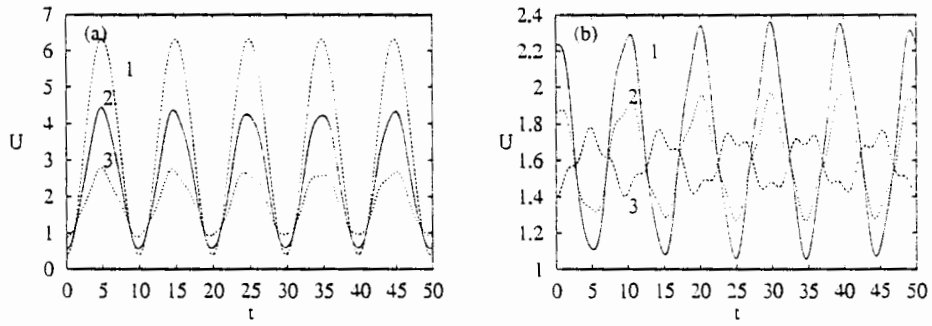

Figure 1: Variance U for various amplitudes of the parametric modulation and squeezing parameters for $a = 0.1$. (a): $b = 0$ (1), 0.025 (2), 0.04 (3) and $r = 4$ (b): $r = 1$ (1), 1.2 (2), 1.6 (3) and $b = 0.025$

amplitudes of the parametric modulation b. It can be seen how the variation of b changes the form and also the amplitude of the oscillations. In fig. 1b variances are plotted for different squeezing parameters r.

3 THE DAMPED QUANTUM PARAMETRIC OSCILLATOR

To describe damping we couple our system linearly to an environment[8]. This environment is modeled as a linear system consisting of a set of noninteracting harmonic oscillators . The Hamiltonian of the coupled system assumes then the following form

$$H = H_A + H_I + H_B$$

$$\text{with } H_A = \frac{p^2}{2m} + \frac{1}{2}(a - b\cos\Omega t)mx^2, \qquad H_B = \sum_{n=1}^{N} \frac{p_n^2}{2m_n} + \frac{1}{2}\omega_n^2 m_n x_n^2,$$

$$H_I = x\sum_{n=1}^{N} c_n x_n + \sum_{n=1}^{N} \frac{c_n^2}{2m_n\omega_n^2}x^2. \tag{13}$$

H_A and H_B are the Hamiltonians of the parametric quantum oscillator and the bath oscillators, respectively. The first term in H_I couples the system to the bath. This coupling leads to a frequency shift of our system, that is removed with the second term in H_I. To gain explicit results we consider from now on an Ohmic heat bath. As we are not interested in the dynamics of the environment we elimimate the bath and calculate the *exact reduced density operator* of the system at time t, i.e.

$$\rho_R(x_f, y_f, t) = \int dx_i dy_i J(x_f, y_f, t|x_i, y_i, 0)\rho_R(x_i, y_i, 0). \tag{14}$$

where J is calculated by the influence-functional method. Introducing the sum and difference variables $q = x - y$, $r = \frac{1}{2}(x + y)$ yields for J

$$J(q_f, r_f, t|q_i, r_i, 0) = \frac{\dot{u}_2(t,0)}{2\pi\hbar} \exp\left[\frac{-1}{\hbar}\{a_{11}(t)r_i^2 + [a_{12}(t) + a_{21}(t)]r_i r_f + a_{22}(t)r_f^2\}\right]$$

$$\times \exp\left[\frac{i}{\hbar}\{[\dot{u}_1(t,0)r_i + \dot{u}_2(t,0)r_f]q_i - [\dot{u}_1(t,t)r_i + \dot{u}_2(t,t)r_f]q_f\}\right]. \qquad (15)$$

a_{ij} is given through

$$a_{ij}(s) = \frac{1}{2}\int_0^s \int_0^s ds_1 ds_2 v_i(t,s_1)v_j(t,s_2)K(s_1 - s_2)$$

with the noise kernel

$$K(s) = \int_0^\infty \frac{d\nu}{\pi} m\gamma\nu \coth(\frac{\nu\hbar}{2k_BT})\cos(\nu s). \qquad (16)$$

The set $\{u_1, u_2\}$ determines the solution of the equation of motion

$$\ddot{u} - \gamma\dot{u} + (a - b\cos\Omega s)u = 0 \qquad (17)$$

with the conditions $u_1(t,0) = 1$, $u_1(t,t) = 0$, $u_2(t,0) = 0$, $u_2(t,t) = 1$ and $v_{1,2}(t,s) = u_{1,2}(t,s)\exp(\gamma s)$. To arrive at this form we assumed that the system and environment were initially ($t_0 = 0$) *uncoupled* and the bath was in equilibrium at temperature T. Knowing the density matrix we are able to calculate expectation values of the variables. The mean values of space and coordinate follow the trajectories of a damped classical parametric oscillator. Next we study the time development of the variances $U(t)$, $V(t)$ and $W(t)$. The Ohmic damping leads to a divergence in W, just as with a damped quantum oscillator[9]. We introduce an abrupt high frequency cutoff ν_c of the bath frequencies ν in the frequency integral in (16) to remove this divergence. This is correct as long as we consider only times that are large compared to ν_c^{-1}.The results are plotted in figs 2. Fig.2a shows $m\Omega U/2\hbar$ for increasing modulation amplitude $|b|$. The initial values of the variances are like in section 2. For curves labeled (1), (2) and (3) b has values that lead to decaying solutions of the damped Mathieu equation in eq.(17), i.e. $\langle x(t)\rangle \to 0$ as $t \to \infty$. After a short time, U becomes a constant for curve labeled (1), whereas for curves (2) and (3) U becomes a *periodic function* which oscillates with the frequency Ω. This frequency is *not affected* by the strength of the friction γ. The amplitude of the oscillations is increasing with increasing modulation strength $|b|$. For b in the unstable region the variances become unbounded also, as can be seen in curve labeled (4). In fig.2b we start with a squeezed state and compare it with the unsqueezed state. As an interesting result we find that the effect of initial squeezing relaxes on a fast time scale. This relaxation time depends only weakly on temperature, but depends on the strength of Ohmic friction γ.

We also like to point out here that the system dynamics $x(t)$ of the coupled system in eq. (13) obeys an exact equation of motion. The Heisenberg operator

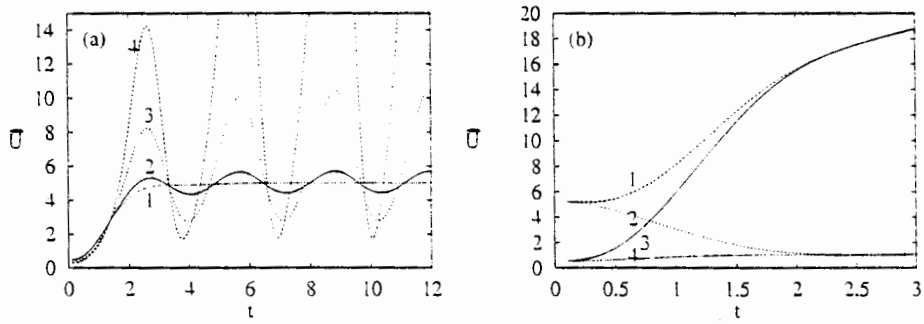

Figure 2: Variance $\bar{U} = m\Omega U/2\hbar$ for various amplitudes of the parametric modulation and different squeezing parameters, $\nu_c = 50$, $a = 1$, $\Omega = 2$, $\gamma = 1$, $\omega_0^2 = a$,
(a): $|b| = 0$ (1), 0.1 (2), 1 (3), 2 (4) and $r = 1$, $k_BT/\hbar\omega_0 = 5$,
(b): $r = 1$ (3,4), 0.1 (1,2), $k_BT/\hbar\omega_0 = 1$ (2,4), 20 (1,3) and $|b| = 0.1$

$x(t)$ obeys - after elimination of the bath degrees of motion - an *exact Quantum Langevin equation*, which in terms of the initial time of preparation reads explicitly

$$m\ddot{x} + m\int_{t_0}^{t} \gamma(t-s)\dot{x}(s)ds + m(a - b\cos\omega t)x + m\gamma(t-t_0)x(t_0) = \xi(t) \qquad (18)$$

where $m\gamma(t) = \sum_n c_n^2 \cos\omega_n t/m_n\omega_n^2$; $\frac{1}{2}\langle\xi(t)\xi(0) + \xi(0)\xi(t)\rangle = \hbar K(t)$, with $K(t)$ given in eq. (16). The limit of Ohmic friction - without cutoff - is obtained from (18) by setting $\gamma(t-t_0) = 2\gamma\delta(t-t_0)$. We note that (18) presents a suitable starting point to investigate initial and long-time (aged) correlation function properties of the damped parametric quantum oscillator[9].

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