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Path Integral Solution for Nonlinear Generalized Langevin Equations

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ABSTRACT

The path-integral solution for nonlinear dynamical systems driven by correlated random forces (colored noise) is developed. We treat three cases: (a) overdamped nonlinear stochastic flows with colored noise in absence of memory-damping, (b) stochastic flows with a nonlinear drift composed of a memory-relaxation of the Mori-Kawasaki-type and arbitrary colored noise, (c) generalized Brownian motion as described in phase-space by a generalized Langevin equation with a nonlinear potential of mean force and a *linear* memory-friction. The results can be cast into the form of a double path integral with a complex-valued non-Markovian-Onsager-Machlup functional. Explicit results can be obtained for colored Gaussian noise and colored Poissonian shot-noise.

1. Introduction

The study of nonlinear systems driven by stochastic, correlated random forces (i.e. colored noise) has attracted a great deal of interest in recent years [1]. Prominent cases for which the colored noise plays a crucial role are the phenomenon of motional narrowing in magnetic resonance [2], or the dynamics for noisy dye lasers [3,4], to name just two examples from nonlinear non-equilibrium statistical mechanics. Within statistical equilibrium mechanics the generalized Langevin equation of the Mori-Kubo-Kawasaki type [5-8] presents a cornerstone result in the description of the relaxation of macroscopic variables. For most applications in chemistry and physics one relies on a generalized Langevin equation with a *linear* memory-function (i.e. a state-independent memory) yielding the generalized Brownian motion for the variable x(t) as [9,10]

$$M\ddot{x} = -\frac{\partial V}{\partial x} - M\int_0^t \gamma(t-s)\,\dot{x}(s)ds + \xi(t) \quad . \tag{1}$$

Here, M denotes the mass of the Brownian particle, V(x) is the generally nonlinear potential of mean force, and $\xi(t)$ is a colored noise source obeying the fluctuation-dissipation theorem

$$<\xi(t)\xi(s)>= D\gamma(|t-s|) \quad , \tag{2}$$

where $D \equiv MkT$ in thermal (canonical) equilibrium at temperature T. It should be pointed out here, that Eq. (1) and Eq. (2) follow rigorously from the statistical mechanics of a Brownian particle embedded into an environment consisting of bilinearly coupled harmonic oscillators. In other words, the Hamiltonian of system plus environment

$$H_{total} = \frac{p^2}{2M} + V(x) + \frac{1}{2}\sum_{i=1}^{N} m_i \left\{ \dot{q}_i + \omega_i^2 \left[q_i + \frac{c_i}{m_i \omega_i^2} x \right]^2 \right\} , \tag{3}$$

wherein $\{m_i\}$ and $\{\omega_i\}$ denote the set of masses and frequencies of the bath-oscillators, respectively, yields upon contraction onto the macrovariable x(t) – from an initially prepared canonical equilibrium for the total system – precisely the set of Eqs. (1) and (2) with the memory-function being

$$\gamma(t) = M^{-1} \sum_{i=1}^{N} \frac{c_i^2}{m_i \omega_i^2} \cos(\omega_i t) , \tag{4}$$

and $\xi(t)$ a Gaussian stochastic force of vanishing mean. The random force is in terms of the microscopic parameters given by

$$\xi(t) = -\sum_{i=1}^{N} c_i \left\{ \left(q_i(0) + \frac{c_i}{m_i \omega_i^2} x(0) \right) \cos(\omega_i t) + \frac{\dot{q}_i(0)}{\omega_i} \sin(\omega_i t) \right\} . \tag{5}$$

Note, that the stochastic force explicitly involves the initial value of the Brownian coordinate x(0)! Moreover, with the coupling given in Eq. (3), the bare potential V(x) undergoes *no coupling-induced frequency shift.* The latter frequency shift would approach infinity for strict Ohmic friction $\gamma(t-s) \to 2\gamma\,\delta(t-s)$ [11].

2. Path Integral Solution for Overdamped non-Markovian Systems

Before we tackle the path integral solution for Eq. (1) we consider a generalized dynamics in the "overdamped case", formally defined by

$$\dot{x} = f(x) - \int_0^t \gamma(t-s)\, x(s)\, ds + \xi(t) . \tag{6}$$

2.1. The case without memory

First, we consider the Markovian limit $\gamma(t-s) \to 2\gamma_0\, \delta(t-s)$, i.e. we obtain from Eq. (6) the nonlinear colored noise flow

$$\dot{x} = f(x) - \gamma_o x + \xi(t) . \tag{7}$$

Note, that Eq. (1) reduces with $\gamma(t-s) = 2\hat{\gamma}\delta(t-s)$, and $\hat{\gamma}$ very large, to such a structure. Put differently, after an adiabatic elimination of $\ddot{x}$ we have from Eq. (1)

$$\dot{x} = \frac{-1}{M\hat{\gamma}}\frac{\partial V}{\partial x} + \left(\frac{kT}{M\hat{\gamma}}\right)^{1/2} \xi_w(t) \quad , \tag{8}$$

where $\xi_w(t)$ is Gaussian white noise with correlation $< \xi_w(t)\xi_w(s) >= 2\delta(t-s)$. With $\xi(t)$ in Eq. (7) a general colored noise source, the path integral solution for Eq. (7) has been given already previously in the literature [12]: We start with the noise-dependent probability, conditioned at time $t_0 = 0$ at $x(0) = x_0$, i.e.

$$p_\xi(xt|x_0) = \delta\left[x - x_\xi\ (t,x_0)\right] \quad , \tag{9}$$

which by use of a δ-functional reads

$$\begin{aligned} p_\xi(xt|x_0) &= \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}x\ \delta\left(x - x_\xi(s)\right) \\ &= \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}x\ \delta\left(\dot{x} - f(x) - \gamma_0\, x - \xi\right)\mathcal{J} \quad , \end{aligned} \tag{10}$$

where

$$\mathcal{J} \equiv Det\left(\frac{\delta\xi}{\delta x}\right) \equiv \left|\frac{\delta\xi(s_1)}{\delta x(s_2)}\right| = \left(\left|\frac{\delta x}{\delta\xi}\right|\right)^{-1} \tag{11}$$

is the Jacobian of the transformation

$$x_\xi \to \xi = \dot{x} - f(x) + \gamma_0 x \quad \cdot \tag{12}$$

The Jacobian has been evaluated in Ref. [12] to give for this case (within mid-point discretization)

$$\mathcal{J} = \exp\left(-\frac{1}{2}\int_0^t ds\left[\frac{df}{dx} - \gamma_0\right]\right) Det\left(\frac{\partial}{\partial s}\right) \quad , \tag{13}$$

with Det $(\partial/\partial_s) = \lim_{N\to\infty} \varepsilon^{-N}$, where $\varepsilon \equiv t/N$ is the infinitesimal increment. Given the characteristic function

$$\chi(\varepsilon z_1,\ldots,\varepsilon z_N) = \ < \exp\ -i\varepsilon \sum_{n=1}^{N} z_n\xi_n > \quad , \tag{14}$$

it can be inverted to yield

$$p(\xi_1,\dots,\xi_N)=\frac{\varepsilon^N}{(2\pi)^N}\int\dots\int dz_1,\dots,dz_N\,\chi[\dots]\exp\left(i\varepsilon\sum_{n=1}^{N}z_n\xi_n\right)\cdot \tag{15}$$

Thus, by averaging the noise-dependent probability $p_\xi(xt|x_0)$ over $(\xi_1,\dots,\xi_N)$, $N\to\infty$, one finds the double-path integral solution for Eq. (7), i.e. [12]

$$p(xt|x_0)=\int_{x(0)=x_0}^{x(t)=x}\mathcal{D}x\int\mathcal{D}\left(\frac{z}{2\pi}\right)\chi[z]$$

$$\cdot\exp\ i\int_0^t ds\ z(s)\{\dot{x}(s)-f(x(s))-\gamma_0x(s)\}$$

$$\cdot\exp\ -\frac{1}{2}\int_0^t ds[f'(x(s))-\gamma_0]\ \cdot \tag{16}$$

Here, the prime denotes differentiation with respect to x. Note also that the singular part $Det(\partial/\partial s)$ of the Jacobian cancels, in virtue of Eq. (15), with the contributing part ε^N. The result in Eq. (16) holds true for any colored noise source, whose statistics is given by the curtailed characteristic functional $\chi[z]=<\exp-i\int_0^t z(s)\xi(s)ds>$. For colored Gaussian noise, i.e.

$$<\xi(t)>\ =\ 0$$

$$<\xi(t)\xi(s)>=D\sigma(t-s)\ , \tag{17}$$

the result in Eq. (16) simplifies to give [12]

$$p(xt|x_0)=\int_{x_0}^{x}\mathcal{D}x\int\mathcal{D}\left(\frac{z}{2\pi D}\right)\exp\left(-\frac{S[x,z]}{D}\right)\ , \tag{18}$$

with the complex-valued Onsager-Machlup functional given by

$$S[x,z]=-i\int_0^t ds\big(\dot{x}(s)-f(x(s))\ -\gamma_0x(s)\big)\,z(s)$$

$$+\frac{1}{2}\int_0^t ds_1\int_0^t ds_2\ z(s_1)\ \sigma(s_1-s_2)\ z(s_2)$$

$$+\frac{D}{2}\int_0^t ds\ [f'(x(s))-\gamma_0]\ \cdot \tag{19}$$

The double path integral in Eq. (18) can be reduced - in this case - to a single path integral over $\mathcal{D}$x only, if we integrate over the Gaussian functional in z(s) [13]. This however, involves the knowledge of the inverse $\sigma^{-1}(s-s')$, defined by

$$\int_0^t \sigma(s-s')\,\sigma^{-1}(s'-s'')\,ds' \equiv \delta(s-s'') \quad . \tag{20}$$

Cenerally, $\sigma^{-1}(t-s)$ is not known explicitly. For the case of Ornstein-Uhlenbeck noise (i. e. exponentially correlated Gaussian noise) the inverse $\sigma^{-1}(s-s')$ has been determined explicitly in the literature [13]; but it is already a rather complex functional, involving "surface terms" at s = t and $s = t_0 = 0$ as well !

2.2. Overdamped colored noise flow with memory

We now return to Eq. (6) where the nonlinear flow contains explicitly memory. With

$$p(xt|x_0) =< \delta\big(x(t)-x_\xi\big)> \quad , \tag{21}$$

where the average is over the stochastic force $\{\xi(s)\}$, being assumed to be stationary. A master equation for Eq. (21) follows by use of the functional methods developed originally in reference [14]. For the Gaussian noise of vanishing mean in Eq. (17) we find from Eq. (6) the explicit result [15,16]

$$\begin{aligned}\dot{p}(xt|x_0) &= -\frac{\partial}{\partial x}\big[f(x)p(xt,x_0)\big] \\ &+ D\frac{\partial^2}{\partial x^2}\int_0^t ds\,\sigma(t-s) < \delta(x(t)-x)\,\frac{\delta x(t)}{\delta \xi(s)}> \\ &+ \frac{\partial}{\partial x}\int_0^t \gamma(t-s) < x(s)\delta(x(t)-x) > ds \quad .\end{aligned} \tag{22}$$

Thus, this master equation is *not closed*, but depends via the functionals nonlinearly on the noise ξ(s) itself. The correlation $< x(s)\delta(x(t)-x) >$ can be disentangled further by use of equation (25) given in reference [16]. A closed equation for Eq. (22) results if f(x) is linear in x, yielding a non-Markovian, Gaussian process for x(t) [16].

In view of the impossibility to arrive in Eq. (22) at a closed equation for the rate of change $\dot{p}$, we instead seek now a formal solution for Eq. (21), in terms of a path-integral expression.

As before, we start from the identity

$$p(x_1,\ldots,x_N|x_0)=p(\xi_1,\ldots,\xi_N)\,Det\left(\frac{\delta\xi}{\delta x}\right)\quad . \tag{23}$$

With

$$p(xt|x_0)=\int\prod_{i=1}^{N-1}dx_i\,p(x_N=x,\,x_{N-1},\ldots x_1|x_0)\quad ,$$

and observing Eq. (15) one finds with $\varepsilon^N=\left[Det(\partial/\partial s)\right]^{-1}$

$$p(xt|x_0)=\int_{x(0)=x_0}^{x(t)=x}\mathcal{D}x\int\mathcal{D}\left(\frac{z}{2\pi}\right)\chi[z]$$

$$\cdot\exp\left(i\int_0^t ds\,z(s)\left\{\dot{x}(s)-f(x(s))+\int_0^s\gamma(s-u)x(u)du\right\}\right)$$

$$\cdot\,\mathcal{J}\left[Det(\partial/\partial s)\right]^{-1}\quad . \tag{24}$$

With a Gaussian colored noise in Eq. (17) we find upon the rescaling $z\to z/D$ the explicit result

$$p(xt|x_0)=\int_{x(0)=x_0}^{x(t)=x}\mathcal{D}x\int\mathcal{D}\left(\frac{z}{2\pi D}\right)\exp(-S[x,z]/D)\quad , \tag{25a}$$

with the complex-valued, *non-Markovian Onsager-Machlup functional* given by

$$S[x,z]=-i\int_0^t\left[\dot{x}(s)-f(x(s))+\int_0^s\gamma(s-u)x(u)du\right]z(s)ds$$

$$+\,\frac{1}{2}\int_0^t ds_1\int_0^t ds_2 z(s_1)\sigma(s_1-s_2)z(s_2)$$

$$-D\ln\left[\mathcal{J}\,/\,Det(\partial/\partial s)\right]\quad . \tag{25b}$$

The results in Eqs. (24) and (25a,b) present our main results. Note, that Eq. (24) holds for arbitrary colored noise $\xi(t)$, while Eq. (25a,b) is restricted to *stationary* Gaussian noise of arbitrary correlation dependence and zero mean value.

We still should evaluate the Jacobian entering Eqs. (24, 25a,b). This Jacobian explicitly depends on the discretization scheme underlying the path-integral measure. The Jacobian is particularly simple to evaluate within a *prepoint-discretization* scheme, i.e.

$$x_i^\alpha = x_i + \alpha(x_{i+1} - x_i) \quad , \tag{26}$$

with α set equal to zero, i.e. α = 0. With $\varepsilon \equiv t/N$, $N \to \infty$, this implies for Eq. (6) the discretization, where x with $x(0) = x_0$ fix

$$(x_{i+1} - x_i)/\varepsilon = f(x_i) + \xi_i - \varepsilon \sum_{n=0}^{i} \gamma(i\varepsilon - n\varepsilon)\, x_n \quad , \quad \xi_i = \xi(i\varepsilon) \quad . \tag{27}$$

The Jacobian, relating the sets of variables $\{x_1, \ldots, x_N\}$ and $\{\xi_0, \ldots, \xi_{N-1}\}$ is then readily evaluated to yield

$$\mathcal{J} \equiv Det\left(\frac{\partial \xi}{\partial x}\right) = \begin{vmatrix} \frac{\partial \xi_0}{\partial x_1} & \frac{\partial \xi_0}{\partial x_2} & \cdots \frac{\partial \xi_0}{\partial x_N} \\ \vdots & & \\ \frac{\partial \xi_{N-1}}{\partial x_1} & \cdots & \frac{\partial \xi_{N-1}}{\partial x_N} \end{vmatrix} = \varepsilon^{-N} \equiv Det[\partial/\partial s] \quad , \tag{28}$$

because the upper triangular matrix in Eq. (28) contains only zeros, – due to the independence of ξ_i from future values x_i –, and $\partial \xi_n / \partial x_{n+1} = \varepsilon^{-1}$. Thus, the last terms in Eqs. (24) and (25b) cancels precisely within the prepoint discretization!

With the *midpoint-discretization*, i. e. $\bar{x}_i = \frac{1}{2}(x_i + x_{i-1})$, the Jacobian must be evaluated with greater care. From Eq. (6) we find

$$\frac{\delta \xi(s)}{\delta x(s')} = \frac{\partial}{\partial s}\delta(s-s') - f'\delta(s-s') + \gamma(s-s') \quad . \tag{29}$$

Thus

$$Det\left(\frac{\delta \xi}{\delta x}\right) = Det\left(\left[\frac{\partial}{\partial s} - f'\right]\delta(s-s') + \gamma(s-s')\right).$$

With Tr denoting the "trace" we can write further

$$Det\left(\frac{\delta \xi}{\delta x}\right) = \exp\left[Tr \ln\left\{\left(\frac{\partial}{\partial s} - f'\right)\delta(s-s') + \gamma(s-s'\right\}\right]$$

$$= \exp\left[Tr \ln\left\{\frac{\partial}{\partial s}\left[\delta(s-s') - \theta(s-s')f' + \int_0^t \theta(s-v)\gamma(v-s')dv\right]\right\}\right]$$

$$= \exp\left[Tr\left\{ \ln(\partial/\partial s) + \ln\left[\delta(s-s') - \theta(s-s')f' + \int_0^t \theta(s-v)\gamma(v-s')dv \right]\right\}\right]$$

$$= \exp[Tr \ln(\partial/\partial s)] \, \exp\left[Tr \ln\left\{ \delta(s-s') - \theta(s-s')f' + \int_0^t \theta(s-v)\gamma(v-s')dv \right\}\right] \cdot$$

Doing the usual expansion for $\ln(1+x) = x - (x)(x')/2 + \ldots$, we obtain with $\theta(0) \equiv 1/2$

$$Det\left(\frac{\delta\xi}{\delta x}\right) = Det(\partial/\partial s) \, \exp\left\{ -\frac{1}{2}\int_0^t f'(x(s))ds + \frac{1}{2}\int_0^t ds \int_0^t \gamma(v-s)dv \right\} \cdot \tag{30}$$

For a Markovian memory $\gamma(t-s) = 2\gamma_0\delta(t-s)$, this result reduces to the previous expression in Eq. (13). Quadratic, and all subsequent higher order terms, do not contribute to Eq. (30), because of the identity $\theta(s-s')\theta(s'-s) = 0$. Again we emphasize that the determinant $\mathcal{J}$ is proportional to $Det(\partial/\partial s)$. Hence, this divergent term always cancels out in Eq. (24) and in Eq. (25b)!

3. Path Integral for Generalized Brownian Motion

We next consider the generalized Langevin equation in Eq. (1). First of all, it can be recast as a two-dimensional non-Markovian flow, i.e.

$$\dot{x} = u$$

$$\dot{u} = -\frac{1}{M}\frac{\partial V}{\partial x} - \int_0^t \gamma(t-s)\, u(s)\, ds + \xi(t) \tag{31}$$

where

$$< \xi(t)\xi(s) > \; = \frac{kT}{M}\, \gamma(t-s) \equiv D\gamma(t-s) \quad \cdot \tag{32}$$

Again we point out that in presence of a anharmonic potential V(x) no exact, closed master equation, nor any other exact time-dependent solution of Eq. (32) is known. With $\xi(t)$ being Gaussian, and V(x) a quadratic potential the pair (x,u) constitutes a non-Markovian, Gaussian process whose multivariate (conditional) probabilities can all be constructed in terms of the correlations of the fluctuations [17]. A formal path-integral solution for Eq. (31) follows if we first recast Eq. (31) as

$$\dot{x} = u + \zeta(t)$$

$$\dot{u} = -M^{-1}\frac{\partial V}{\partial x} - \int_0^t \gamma(t-s)u(s)ds + \xi(t) \quad , \tag{33}$$

and where $\zeta(t)$ is *white* Gaussian noise of vanishing mean with correlation

$$< \zeta(t)\zeta(s) >= 2\hat{D}\delta(t-s) \quad , \tag{34}$$

where $\hat{D} \to 0$ at the end of the derivation. Then, setting in terms of the functional joint-probability $p[x,u]$

$$p(x,u,t|x_0,u_0) = \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}x \int_{u(0)=u_0}^{u(t)=u} \mathcal{D}u \, p[x,u] \tag{35}$$

$$= \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}x \int_{u(0)=u_0}^{u(t)=u} \mathcal{D}u \;\; p[\zeta,\xi] \, \mathcal{J} \tag{36}$$

where $\mathcal{J} = \text{Det}\,(\delta\zeta\delta\xi \,/\, \delta x\delta u)$ is the Jacobian of the transformation $(x,u) \to (\zeta,\xi)$. With the white noise $\zeta(t)$ not being correlated with the colored noise $\xi(t)$, the functional probability $p[\zeta,\xi]$ *separates*, with the white Gaussian noise functional explicitly given by

$$p[\zeta] = Z^{-1}\exp\left\{-\frac{1}{4\hat{D}}\int_0^t [\zeta(s)]^2 ds\right\} \quad . \tag{37}$$

Proceeding just as before in section 2, a few manipulations in terms of the curtailed characteristic functional for the colored noise $\xi(t)$ in Eq. (33), i.e.

$$\chi[z] = < \exp - i\int_0^t \xi(s)z(s)ds > \quad , \tag{38}$$

yield for the path-integral representation the central result

$$p(x,u,t|x_0,u_0) = \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}x \int_{u(0)=u_0}^{u(t)=u} \mathcal{D}u \;\; \exp\left\{-\frac{1}{4\hat{D}}\int_0^t ds\,[\dot{x}-u]^2\right\}$$

$$\int \mathcal{D}\left(\frac{z}{2\pi}\right)\chi[z]\,\exp\left\{i\int_0^t ds\, z(s)\left[\dot{u}(s) + M^{-1}\frac{\partial V}{\partial x} + \int_0^s \gamma(s-s')u(s')ds'\right]\right\}\,\exp\{\ln[\mathcal{J}\,/\,Det(\partial/\partial s)]\} \cdot \tag{39}$$

Within *prepoint-discretization* we find $\mathcal{J} = Det(\partial/\partial s)$, i.e.the last contribution in Eq. (39) cancels out, yielding unity. The functional $\chi[z]$ is known explicitly both for Gaussian noise and generalized colored Poisson noise [14]. With a Gaussian noise of vanishing mean and general correlation $<\xi(t)\xi(s)>= D\sigma(t-s)$, we have

$$\chi[z] = \exp - \frac{D}{2}\int_0^t ds_1 \int_0^t ds_2\, z(s_1)\, \sigma(s_1 - s_2)\, z(s_2) \quad . \tag{40}$$

With $\hat{D} \to 0$, the explicit result for generalized Brownian motion driven by Gaussian colored noise thus reads

$$p(x,u,t|x_0,u_0) = \int_{x_0,u_0}^{x,u} \mathcal{D}x \int \mathcal{D}\left(\frac{z}{2\pi D}\right) \delta(\dot{x}(s) - u(s))$$

$$\cdot \exp\left(-S[x,z]/D\right) \tag{41a}$$

where

$$S[x,z] = -i\int_0^t ds\, z(s)\left[\ddot{x}(s) + M^{-1}\frac{\partial V}{\partial x(s)} + \int_0^s \gamma(s-s')\dot{x}(s')ds'\right]$$

$$+\frac{1}{2}\int_0^t ds_1 \int_0^t ds_2\, z(s_1)\, \sigma(s_1 - s_2)\, z(s_2) \quad . \tag{41b}$$

For a *thermal equilibrium system* we have $D = kT/M$, and $\sigma(t-s) = \gamma(t-s)$, thereby obeying the fluctuation-dissipation theorem. Let us comment on this main result further:

(i) The result is expressed in terms of a (constrained) double-path integral with the complex-valued non-Markovian Onsager-Machlup functional given in Eq. (41b).

(ii) The limit $\hat{D} \to 0$ eliminates the functional integral over $\mathcal{D}u$, but introduces instead the path-dependent constraint $\dot{x}(s) = u(s)$, with the noisy realization $u(s)$ determined from Eq. (31).

(iii) The path integral is over all those noisy trajectories x(t), which are differentiable, $\dot{x}(s) = u(s)$, with initial values at $x(0) = x_0$, and $\dot{x}(0) = u_0$, and final values at x(t)=x, with $\dot{x}(t) = u$, fix.

(iv) For a *Markovian* damping $\gamma(t-s) \to 2\hat{\gamma}\,\delta(t-s)$, and corresponding white Gaussian noise $\xi(t)$, the path integral over z(s) is readily evaluated, yielding for S[x] a real-valued functional

$$S[x] = \exp\left(\frac{1}{4}\int_0^t ds\left[\ddot{x} + \hat{\gamma}\,\dot{x} + M^{-1}\frac{\partial V}{\partial x}\right]^2\right) \tag{42}$$

with $D \equiv kT/M$. This result then precisely coincides with the constrained path integral solution for Markovian Brownian motion, derived first by Grabert et al. [18].

(v) In terms of the inverse $\sigma^{-1}(t-s)$ defined in Eq. (20) the functional probability for $p[\xi]$ reads

$$p[\xi] = \exp\ -\frac{1}{2D}\int_0^t ds_1 \int_0^t ds_2\ \xi(s_1)\ \sigma^{-1}(s_1 - s_2)\ \xi(s_2)\ . \tag{43}$$

Then, the $\mathcal{D}z(s)$-path integral can readily be evaluated to yield the constrained, single-path integral result

$$p(x,u,t|x_0,u_0) = \int_{x_0,u_0}^{x,u} \mathcal{D}x\ \delta(\dot{x}(s) - u(s))\ \exp\left(-\frac{S[x]}{D}\right)\ , \tag{44a}$$

where

$$S[x] = \frac{1}{2}\int_0^t ds_1 \int_0^t ds_2\ \xi(s_1)\ \sigma^{-1}(s_1 - s_2)\ \xi(s_2)\ , \tag{44b}$$

with $\xi(s) = \left(\ddot{x}(s) + M^{-1}\ \frac{\partial V}{\partial x(s)} + \int_0^s \gamma(s-s')\dot{x}(s')ds'\right)$.

4. Conclusions

In the preceiding sections we derived the formal path integral solution for non-Markovian stochastic flows driven by colored noise and containing a memory-relaxation. The results possess the structure of a double-path integral, and in the case of generalized Brownian motion in phase-space a *constrained* double-path integral over differentiable random trajectories. The double path integrals can formally be reduced to a single-path integral only at the expense of introducing the generally unknown inverse functional $\sigma^{-1}(t-s)$ of the noise correlation $\sigma(t-s)$, obeying $\int^t \sigma^{-1}(s-s'')\,\sigma(s''-s')ds'' = \delta(s-s')$. This inverse $\sigma^{-1}(s-s'')$ is rather complex; even for exponentially correlated Gaussian noise, it involves surface terms, as well as higher order derivatives [19]. This then implies for the trajectory extremalizing the action integral S[x] an Euler-Lagrange equation which is already of 6-th order in the Brownian motion coordinate x(t)!

The results for Gaussian colored noise in Eqs. (19), (25b) and (41b) are in a form suitable to study the weak-noise asymptotics as the correlation strength $D \to 0$, (see also in Ref. 12). An unsolved difficulty in this context surely presents the study of the "prefactor", because it seems not plausible that the determinant for the 2-nd variation of the action can be cast into a form similar to the "Van-Vleck-Morette-Pauli-expression" known in quantum mechanics. This open problem certainly limits the wide-spread use of these non-Markovian path integral results. In contrast, however, the corresponding non-Markovian-Onsager-

Machlup functional provides essential information about the exponential leading behavior, being otherwise not readily available (for applications within this spirit see in the references [12,13,20,21]).

5. References

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