

Colored-noise-driven bistable systems

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We consider the escape rate in a bistable potential driven by exponentially correlated noise. Our focus is on the crossover between the small- and large-correlation time behavior. Precise numerical results obtained by using a matrix-continued-fraction technique are compared against recent theoretical predictions.

We consider the escape process in the archetype bistable potential

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 \tag{1}$$

driven by an external Gaussian noise source with a finite correlation time τ , i.e., the Langevin equation with Gaussian noise $\xi(t)$ of vanishing mean

$$\dot{x} = x - x^3 + \xi(t), \tag{2}$$

$$\langle \xi(t)\xi(t') \rangle = \frac{D}{\tau} \exp\left[-\frac{1}{\tau}|t-t'|\right],$$

with x , D , and τ normalized to dimensionless quantities.^{6(b)} The potential $V(x)$ has two wells located at $x = \pm 1$ which are separated by a barrier of height $\Delta V = V(0) - V(\pm 1) = 0.25$ at $x = 0$.

In the white-noise limit $\tau = 0$ the noise becomes δ correlated and for small-noise intensity $D \ll \Delta V$, the escape rate r_K between the two attractors λ is well reproduced by the Kramers formula¹⁻³

$$r_K = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\Delta V}{D}\right]. \tag{3}$$

In the last years several groups⁴⁻¹⁰ studied the problem of determining the corrections to the Kramers rate (3) due to small-correlation times. The difficulty of this problem lies in the fact that there are *two small parameters* D and τ , and the result crucially depends on how the limits $\tau \rightarrow 0$ and $D \rightarrow 0$ are taken. The answer reads⁵

$$r(\tau) = r_K \left(1 - \frac{3}{2}\tau\right) \exp\left[-\frac{\Delta V}{2D}\tau^2\right], \tag{4a}$$

which for $\tau/D \ll 1$ reduces to⁴

$$r(\tau) = r_K \left(1 - \frac{3}{2}\tau\right). \tag{4b}$$

In many real systems, however, the system variables are not much slower than the environmental dynamics represented by the noise source. In these cases the small- τ approximations are of limited use.^{2,4} The most difficult situation arises when the noise correlation time τ is comparable to or larger than the characteristic time

scale of the system. The rate problem associated with the process in Eq. (2) has been studied numerically⁶ by computing the smallest nonvanishing eigenvalue λ_0 of the equivalent two-dimensional Fokker-Planck equation

$$\frac{\partial W(x, \epsilon, t)}{\partial t} = \left[-\frac{\partial}{\partial x}(x - x^3 + \epsilon) + \frac{1}{\tau} \frac{\partial}{\partial \epsilon} \epsilon + \frac{D}{\tau^2} \frac{\partial^2}{\partial \epsilon^2} \right] W(x, \epsilon, t) \tag{5}$$

by means of the matrix continued fraction technique of Ref. 11. In fact, the existence of a separatrix for any finite noise correlation time⁶⁻⁸ which splits the $(x - \epsilon)$ space into two domains of attraction guarantees a clear-cut time scale separation at small-noise intensity between the hopping mechanism (with rate $r = \lambda_0/2$) and the intrawell dynamics [described by the remaining eigenvalues of Eq. (5)].^{6(b)} For $0.2 \lesssim \tau \lesssim 1.5$ and small D the resulting eigenvalue λ_0 is well described by the exponential law^{6(b)}

$$\lambda_0(\tau) \propto \exp\left[-\alpha \frac{\tau}{D}\right], \tag{6}$$

where $\alpha \approx 0.1$ throughout the intermediate range $0.2 \lesssim \tau \lesssim 1.5$.

In this paper we focus on the crossover between such an intermediate- τ regime, where Eq. (6) is valid, and the asymptotically large- τ limit. In the latter regime the authors of Refs. 5(a)-5(c), 8, and 10 all obtain for the exponential leading dependence the result

$$\lambda_\infty(\tau) \propto \exp\left[-\frac{8\Delta V}{27D}\tau\right] \text{ as } \tau \rightarrow \infty, D \rightarrow 0. \tag{7}$$

Note that this law has again the same form as in (6), but with a different value for α , i.e., $\alpha(\infty) = 8\Delta V/27$. This yields a slope for $-\ln\lambda(\tau)$ versus τ of $\alpha(\infty)/D = 8\Delta V/(27D)$, which is by no means amenable to the slope defined through Eq. (6), i.e., $\alpha/D \approx 0.1/D$, as shown in the discussion following by comparison with our numerical data.

A nonstationary Fokker-Planck approach has been proposed by Tsironis and Grigolini¹⁰ to bridge the large- τ

and small- τ behavior. In order to assess the validity of this method let us start with the exact equation of motion for the probability of x (Ref. 12),

$$\begin{aligned} \dot{p}_t(x) = & -\frac{\partial}{\partial x}(x-x^3)p_t(x) \\ & + \frac{D}{\tau} \frac{\partial^2}{\partial x^2} \int_0^t \exp\left[-\frac{1}{\tau}(t-s)\right] \\ & \times \left\langle \delta(x(t)-x) \frac{\delta(x(t))}{\delta(\xi(s))} \right\rangle ds. \end{aligned} \quad (8)$$

Here the functional derivative in the integral in (8) is given in terms of the stochastic process $x(t)$ by

$$\frac{\delta x(t)}{\delta x(s)} = \Theta(t-s) \exp\left[\int_s^t [1-3x^2(t')] dt'\right], \quad (9)$$

where $\Theta(t)$ denotes the step function

$$\Theta(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0. \end{cases} \quad (10)$$

Changing the integration variable in (8), i.e., $s \rightarrow u = (t-s)/\tau$, and expanding the argument of the exponential function in (9) up to the first order in τ (note that this expansion is not uniform in x) one finds

$$\begin{aligned} \dot{p}_t(x) = & -\frac{\partial}{\partial x}(x-x^3)p_t(x) \\ & + D \frac{\partial^2}{\partial x^2} \int_0^{t/\tau} \exp(-u) \exp[\tau u(1-3x^2)] \\ & \times du p_t(x). \end{aligned} \quad (11)$$

Performing the integral in (9) without letting the upper limit go to infinity we recover the time-inhomogeneous Fokker-Planck equation (5) of Ref. 10, i.e.,

$$\begin{aligned} \dot{p}_t(x) = & -\frac{\partial}{\partial x}(x-x^3)p_t(x) \\ & + \frac{D}{\tau} \frac{\partial^2}{\partial x^2} \frac{\exp\{[1-3x^2-(1/\tau)]t\} - 1}{1-3x^2-1/\tau} p_t(x). \end{aligned} \quad (12)$$

Note that our derivation of Eq. (12) is only valid under the explicit condition that the correlation time is small. For $\tau < \tau_0 = 1$ the time-dependent diffusion coefficient

$$D(t) = D \frac{1 - \exp\{-[1-\tau(1-3x^2)]t/\tau\}}{1-\tau(1-3x^2)} \quad (13)$$

converges for $t \rightarrow \infty$ to

$$D_{\text{Fox}} = D \frac{1}{1-\tau(1-3x^2)}, \quad (14)$$

thus reproducing Fox's result.⁹ For $\tau \geq \tau_0$, instead, $D(t)$ diverges to $+\infty$ in the domain I , defined by $|x| < \sqrt{(\tau-1)/3}$, thereby leading to a vanishing stationary probability in I . In fact, the correct stationary probability $p_{\text{st}}(x)$ computed numerically (cf. Fig. 1) does not support the existence of this divergence, i.e., $p_{\text{st}}(x)$ does not vanish in I for $\tau > \tau_0$.

The authors of Ref. 10 considered the decay of an initial population confined within one well by solving numerically Eq. (12) for small as well as for large correlation times. They found that the long time tail of this population decays exponentially with decay time T_{TG} , where TG represents Tsironis and Grigolini. For $\tau < \tau_0$, T_{TG} coincides, see (13) and (14), with the reciprocal of the smallest nonvanishing eigenvalue, λ_{Fox} , of the Fokker-Planck equation derived in Ref. 9, i.e.,

$$\begin{aligned} -\lambda_{\text{Fox}} \Psi(x) = & -\frac{\partial}{\partial x}(x-x^3)\Psi(x) \\ & + D \frac{\partial^2}{\partial x^2} \frac{1}{1-\tau(1-3x^2)} \Psi(x), \end{aligned} \quad (15)$$

with the boundary conditions $\Psi(\infty) = \Psi(0) = 0$.

In Fig. 2 we compare $T_{\text{Fox}}(\tau) \equiv 1/\lambda_{\text{Fox}}$, which has been calculated numerically by using a shooting method, with the decay time $T_{\text{TG}}(\tau)$ of Ref. 10 and with the reciprocal

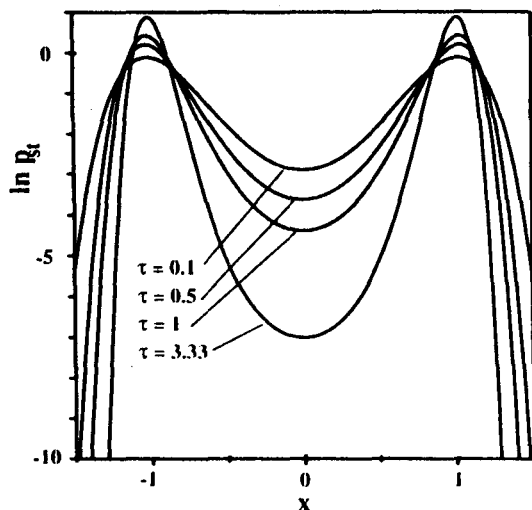


FIG. 1. The stationary probability $p_{\text{st}}(x)$, obtained by numerical integration of (5), is plotted at for several values of τ and $D=0.1$.

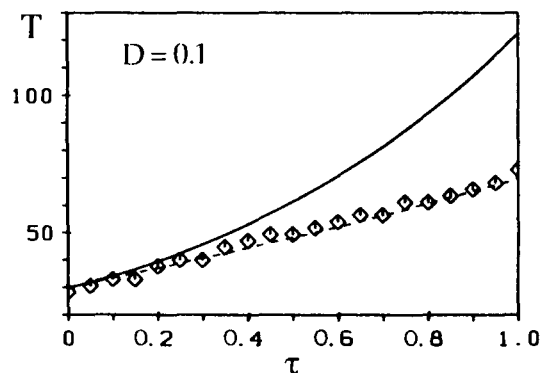


FIG. 2. The numerical values [Eq. (5)] of $T_0(\tau) = 1/\lambda_0(\tau)$ for $D=0.1$ (solid line) are compared against $T_{\text{Fox}}(\tau)$ (dashed line) and $T_{\text{TG}}(\tau)$ (diamonds) for $0 < \tau < \tau_0 = 1$.

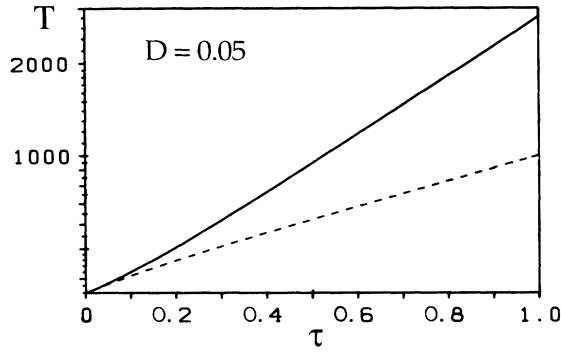


FIG. 3. The relaxation times $T_{\text{Fox}}(\tau)$ (dashed line) are compared with the numerical results (solid line) for $0 < \tau < \tau_0 = 1$ at $D=0.05$.

of the smallest eigenvalue $T_0(\tau) \equiv 1/\lambda_0(\tau)$ (with error less than 0.1%) of the two-dimensional Fokker-Planck equation in (5).^{6(b)} For a detailed description of the numerical matrix continued fraction solution of (5), we refer the reader to Ref. 6(a). The data for $T_{\text{TG}}(\tau)$ lie just on the curve $T_{\text{Fox}}(\tau)$ in the region $0 < \tau < \tau_0$, as predicted by the above argument. The agreement with the exact values $T_0(\tau)$ holds for *small- τ values only*. The discrepancies between the exact result $T_0(\tau)$ and $T_{\text{TG}}(\tau) = T_{\text{Fox}}(\tau)$ do not vanish with decreasing D (Fig. 3), either. In Fig. 3 the exponential behavior of $T_0(\tau)$ is clearly observable. The escape time $T_{\text{Fox}}(\tau)$, or $T_{\text{TG}}(\tau)$ respectively, also exhibits for $D=0.05$ an exponential behavior, but with an *incorrect slope which happens to be very close to the asymptotic value $\alpha(\infty)/D$* . This shows that the exponential behavior (7) of the rate $\lambda(\tau)$ should not be mistaken with the approximate exponential behavior for small to moderate τ values. We conclude that the validity of the nonstationary Fokker-Planck equation (FPE) (12) must be restricted, indeed, to the limit of short noise correlation times, i.e., $\tau \ll 1$.

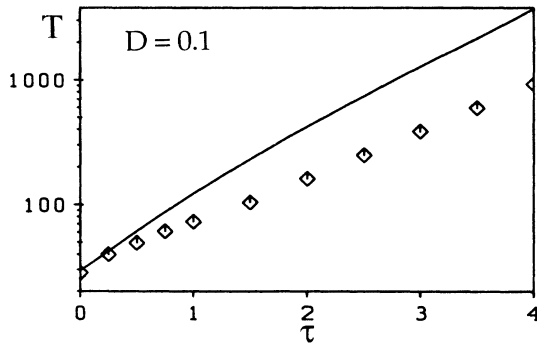


FIG. 4. The relaxation times $T_{\text{TG}}(\tau)$ (diamonds) are compared with the exact results (solid line) for small- to moderate- to large- τ values at $D=0.1$.

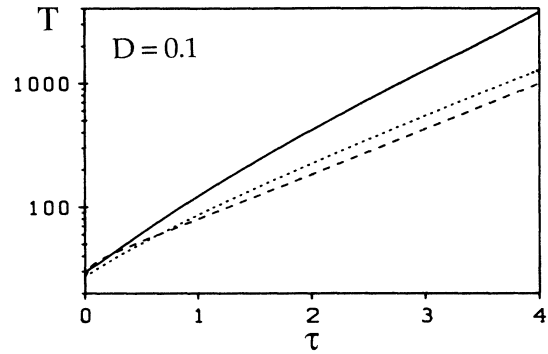


FIG. 5. The bridging formulas (16) (dashed line) and (17) (dotted line) are compared against the exact results (solid line) at small- to moderate- to large- τ values for $D=0.1$.

For $\tau > \tau_0$ (cf. Fig. 4) the disagreement between $T_{\text{TG}}(\tau)$ and $T_0(\tau)$ grows even further; the exact value $T_0(\tau)$ exceeds $T_{\text{TG}}(\tau)$ at $\tau=4$ by a factor of 3.6. Though the absolute values of $T_{\text{TG}}(\tau)$ are off by such an amount, the slope of the logarithmic plot of $T_{\text{TG}}(\tau)$ and $T_0(\tau)$ seemingly converge to the same value $8\Delta V/(27D)$ of Eq. (7) when τ becomes very large. It should be noticed that for $\tau \geq \tau_0$, $T_{\text{Fox}}(\tau)$ diverges. The solid line in Fig. 4 clearly shows that the regime with an exponential τ/D dependence of the rate $\lambda(\tau)$, i.e., $0.2 \lesssim \tau \lesssim 1.5$, is followed by a regime with a nonexponential dependence on τ/D . On further increasing τ , within the domain of reliability of our numerical algorithm [for $0 \leq \tau \leq 4$ and $D=0.1$ $\lambda_0(\tau)$ is determined with an error of less than 1%], the slope of $\ln T_0(\tau)$ seems to converge *slowly* to the asymptotic value α_∞/D .

Finally we discuss the bridging formulas¹⁰

$$T_{\text{TG}}(\tau) = \exp\left[\frac{\Delta V}{D}\right] \times \left[\frac{\pi}{a\sqrt{2}} + \left[\frac{27D\pi\tau}{8\Delta V} \right]^{1/2} \exp\left[\frac{8\Delta V\tau}{27D}\right] \right] \quad (16)$$

and^{5(b),6(b)}

$$T_{LV}(\tau) = \frac{\sqrt{2}}{\pi} (1+3\tau)^{1/2} \times \exp\left[\frac{1}{4D} \left[\frac{1 + \frac{27}{16}\tau + \frac{1}{2}\tau^2}{1 + \frac{27}{16}\tau} \right] \right] \quad (17)$$

proposed to interpolate between the small- τ and large- τ regime. It should be remarked that (16) does not reproduce the correct short- τ behavior (4), while the large- τ limit (7) turns out to be multiplied *ad hoc* by the (large) factor $\exp(\Delta V/D)$. In Fig. 5 the two bridging formulas are compared versus the precise numerical results for

$\lambda_0(\tau)$ obtained by solving the Fokker-Planck equation in (5). Both (16) and (17) have been derived within the steepest descent approach and thus exhibit a difference of $\sim 10\%$ from the exact result at $\tau=0$.^{6(b)} While the bridg-

ing formula due to Luciani and Verga reproduces our numerical data for $\lambda_0(\tau)$ at small τ somewhat more closely, both results in (16) and (17) are off in the region of intermediate-to-large τ values by a considerable amount.

¹For an overview of the present state of the art in rate theory see P. Hänggi, *J. Stat. Phys.* **42**, 105 (1986); **44**, 1003 (1986).

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