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Path integral solutions for non-Markovian processes

Peter Hänggi
Lehrstuhl für Theoretische Physik, Universität Augsburg,
Federal Republic of Germany

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For a nonlinear stochastic flow driven by Markovian or non-Markovian colored noise $\xi(t)$ we present the path integral solution for the single-event probability $p(x, t)$. The solution has the structure of a complex-valued double path integral. Explicit formulas for the action functional, i.e., the non-Markovian Onsager-Machlup functional, are derived for the case that $\xi(t)$ is characterized by a stationary Gaussian process. Moreover, we derive explicit results for (generalized) Poissonian colored shot noise $\xi(t)$. The use of the path integral solution is elucidated by a weak noise analysis of the WKB-type. As a simple application, we consider stochastic bistability driven by colored noise with an extremely long correlation time.

1. Introduction

Some years ago, we have witnessed a considerable activity [1–5] on the path integral solution for the Fokker-Planck equation (see e.g. the collection of references in Chap. VII of [1]). The fundamental role of path integral approaches has its bearing on the possibility for systematic, nonperturbative treatments. The use of a functional integral approach seems particularly important for non-Markovian processes for which the standard (Fokker-Planck) techniques are not readily applicable [6, 7], and good approximate solutions are difficult to obtain [8]. In the following we shall consider nonlinear, non-Markovian Langevin equations for a state variable $x(t)$, i.e. stochastic flows of the form

$$\dot{x} = f(x) + g(x)\,\xi(t). \tag{1.1}$$

Here, the random force $\xi(t)$ is a non-white stochastic process such as for example a non-Markovian or Markovian Gaussian process, or a generalized Poissonian shot noise process. $\xi(t)$ is of vanishing mean and possesses a finite correlation

$$\langle \xi(t)\,\xi(s)\rangle = D\,\sigma(t-s). \tag{1.2}$$

The constant D, i.e.

$$2D \equiv \int_{-\infty}^{\infty} |\langle \xi(t)\,\xi(0)\rangle|\,\mathrm{d}t \tag{1.3}$$

is a measure of the noise intensity. With $\xi(t)$ being a Fokker-Planck process the non-Markovian Langevin equation in (1.1) can be embedded into a multidimensional Fokker-Planck dynamics. An important situation is the case of $\xi(t)$ being an Ornstein-Uhlenbeck process, i.e.

$$\dot{x} = f(x) + g(x)\,\xi(t), \tag{1.4b}$$

$$\dot{\xi} = -\frac{\xi}{\tau} + \frac{\sqrt{D}}{\tau}\,\xi_w(t) \tag{1.4b}$$

with $\xi_w(t)$ being Gaussian white noise $\langle \xi_w(t)\,\xi_w(s)\rangle = 2\,\delta(t-s)$. In other words, the correlation in (1.2) becomes a pure exponential

$$\langle \xi(t)\,\xi(0)\rangle = \frac{D}{\tau}\exp(-|t|/\tau). \tag{1.5}$$

Another important case is the Markovian oscillator dynamics, i.e.

$$\dot{x} = f(x) + g(x)\,\xi(t), \tag{1.6a}$$

$$\dot{\xi} = v, \tag{1.6b}$$

$$\dot{v} = -\omega_0^2\,\xi - \gamma v + \sqrt{D}\,\xi_w(t) \tag{1.6c}$$

yielding

$$\langle \xi(t)\,\xi(0)\rangle = \frac{D}{\gamma\omega_0^2}\exp(-\tfrac{1}{2}\gamma|t|)\cdot\left\{\cos(\omega_1 t) + \frac{\gamma}{2\omega_1}\sin(\omega_1|t|)\right\} \qquad (1.7)$$

where $\omega_1^2 = \omega_0^2 - \frac{\gamma^2}{4}$. In both cases Eq. (1.4), Eq. (1.6), the Fokker-Planck dynamics for the pair $(x(t), \xi(t))$ does not obey detailed balance and the diffusion matrix does not possess an inverse. The latter property implies that the integration measure for the path-integral representation of the multi-dimensional Fokker-Planck process becomes a singular (δ-function like) quantity [4] which makes it thus difficult to obtain explicit solutions or approximations of the WKB-type. In the following we shall elaborate on the path-integral solution for the non-Markovian process $x(t)$ in (1.1). In doing so, we need not to make reference to an underlying, eventually infinite-dimensional sub-dynamics of the type discussed in (1.4b), or (1.6b), (1.6c). Another advantage with a non-Markovian (NM) path integral representation lies in the fact that with a finite correlation time a lattice discretization involving increments $(dx_{\mathrm{NM}})^2$ are of order ε^2, where $\varepsilon = (t-t_0)/N$ is the infinitesimal time step. Thus, in contrast to the Markovian (M) case there is no need to expand to second order, where $(dx_{\mathrm{M}})^2 \propto \varepsilon$. Moreover, with colored noise one obtains smooth sample paths [6] and therefore the *result* of the functional representation of the probability $p(x, t)$ is independent of the discretization scheme [6, 9]; i.e. no problems of the type of the Stratonovitch-versus-Ito-interpretation do arise.

The paper is organized as follows. In Sect. 2 we consider a one-dimensional, non-Markovian Langevin equation driven by multiplicative colored noise. We derive a general result for the path-integral solution and subsequently present explicit results for Gaussian colored noise and generalized Poissonian colored shot noise. In Sect. 3 we discuss the weak noise analysis, $D \to 0$. Finally, in Sect. 4 we present the result for the multidimensional case and elaborate on limitations and straightforward applications of the non-Markovian path integral formulation.

2. Path integral solution for colored noise

We now consider a solution of (1.1) in terms of the realizations of $x(t)$. Let t_0 denote the initial time of preparation. With ε denoting the infinitesimal time step

$$\varepsilon = \frac{(t-t_0)}{N}, \qquad N \to \infty \qquad (2.1)$$

we set for (1.1) the difference equation

$$\frac{x_n - x_{n-1}}{\varepsilon} = f(\tilde{x}_n) + g(\tilde{x}_n)\,\xi_n. \qquad (2.2)$$

Hereby, we set $x_n = x(t_n)$, $t_n = t_0 + n\varepsilon$, and the notation $\tilde{x}_n$ stands for the symmetrized approximation

$$\tilde{x}_n = \tfrac{1}{2}(x_n + x_{n-1}) \qquad (2.3)$$

while

$$\xi_n = \varepsilon^{-1}\int_{t_{n-1}}^{t_n} \xi(s)\,ds. \qquad (2.4)$$

With this discretization scheme, Eq. (2.2) is accurate to order ε^2. Moreover, the symmetric choice implies the correct transformational properties if we make a change of variables $x \to y(x)$. If we sum Eq. (2.2) from $i=0$ to $i=n$ we find with $x(t_0) \equiv x_0$ for the realization $x(t)$ the approximation

$$x_n = x_0 + \varepsilon \sum_{i=1}^{n} [f(\tilde{x}_i) + g(\tilde{x}_i)\,\xi_i]. \qquad (2.5)$$

For the Jacobian $J_n\left(\frac{\partial x_n}{\partial \xi_n}\right) = |\partial x_n/\partial \xi_n|$ we thus obtain

$$\begin{aligned} J_n\left(\frac{\partial x_n}{\partial \xi_n}\right) &= \varepsilon[f'(\tilde{x}_n) + g'(\tilde{x}_n)\,\xi_n]\,\tfrac{1}{2}J_n + |g(\tilde{x}_n)|\,\varepsilon \\ &= \varepsilon\,|g(\tilde{x}_n)|\,[1 - \tfrac{1}{2}\varepsilon(f'(\tilde{x}_n) + g'(\tilde{x}_n)\,\xi_n)]^{-1}, \end{aligned} \qquad (2.6)$$

where the prime indicates a differentiation after x.

For the probability of the discretisized realization we write

$$p(x_1, \ldots, x_n | x_0) = \rho(\xi_1, \ldots, \xi_n)\, J\left[\frac{\partial \xi}{\partial x}\right] \qquad (2.7)$$

where[1]

$$J\left[\frac{\partial \xi}{\partial x}\right] = \prod_{i=n}^{n} J_i^{-1}\left[\frac{\partial x_i}{\partial \xi_i}\right], \qquad (2.8)$$

and $\rho(\zeta_1, \ldots, \xi_n)$ is the multivariate probability of the random noise $\xi(t)$. Following the reasoning of Phyth-

[1] In writing (2.7), we implicitly include multiple contributions that occur for possible multiple roots of the equation $\xi_n = F[x_n]$; i.e. such possible multiple contributions are incorporated into the integration measure

ian [9], we evaluate this Jacobian by use of the matrix-identity for the determinant (Det)

$$\begin{aligned}\mathrm{Det}[\mathbf{I}-\mathbf{M}]&=\exp\{\mathrm{Tr}\ln(\mathbf{I}-\mathbf{M})\}\\&=\exp\{\mathrm{Tr}(-\mathbf{M}-\tfrac{1}{2}\mathbf{M}^2-\ldots)\},\end{aligned} \tag{2.9}$$

where $\mathbf{I}$ denotes the unity matrix. Thus, we have from (2.6)

$$J\left[\frac{\partial\xi}{\partial x}\right]=\varepsilon^{-N}\prod_{i=1}^{N}|g(\tilde{x}_i)|^{-1}$$

$$\cdot\exp-\tfrac{1}{2}\sum_{i=1}^{n}\varepsilon\left[f'(\tilde{x}_i)+g'(\tilde{x}_i)\,g^{-1}(\tilde{x}_i)\left(\frac{x_i-x_{i-1}}{\varepsilon}-f(\tilde{x}_i)\right)\right]. \tag{2.10}$$

Hereby, we do assume that $g(x)$ is not vanishing, i.e. from (2.2) we can solve for ξ_i

$$\xi_i=g^{-1}(\tilde{x}_i)\left[\frac{x_i-x_{i-1}}{\varepsilon}-f(\tilde{x}_i)\right]. \tag{2.11}$$

Note that in (2.10) we have expanded only to first order in ε because contributions of order $\mathbf{M}^2$ are of order ε^2.

The general (Markovian or non-Markovian) noise $\xi(t)$ of vanishing mean is being characterized by its multivariate characteristic function

$$\chi[\varepsilon z_1,\ldots,\varepsilon z_n]\equiv\left\langle\exp-\mathrm{i}\varepsilon\sum_{n=1}^{N}z_n\xi_n\right\rangle, \tag{2.12}$$

which can be inverted to give

$$\rho(\xi_1,\ldots,\xi_n)=\frac{\varepsilon^N}{(2\pi)^N}\int\ldots\int\mathrm{d}z_1\ldots\mathrm{d}z_n$$

$$\chi(\varepsilon z_1,\ldots,\varepsilon z_n)\exp\left(\mathrm{i}\varepsilon\sum_{n=1}^{N}z_n\xi_n\right). \tag{2.13}$$

Apparently, in the limit $N\to\infty$, i.e. $\varepsilon\to0$, Eq. (2.12) approaches the curtailed characteristic functional [10]

$$\chi[z]=\left\langle\exp-\mathrm{i}\int_{t_0}^{t}\xi(s)\,z(s)\,\mathrm{d}s\right\rangle. \tag{2.14}$$

For the initial conditional probability $R(xt|x_0)$ of the non-Markovian process we therefore obtain from

$$R(xt|x_0)=\int\prod_{i=1}^{N-1}\mathrm{d}x_i\,p(x_N=x,\ldots,x_1|x_0), \tag{2.15}$$

and the substitution of (2.13, 2.11, 2.8, 2.7, 2.6) into (2.15) the double path-integral solution

$$R(xt|x_0)=\int_{x(t_0)=x_0}^{x(t)=x}\mathscr{D}x(t)\int\mathscr{D}\left(\frac{z(t)}{2\pi}\right)\chi[z]$$

$$\cdot\exp\left(\mathrm{i}\int_{t_0}^{t}\mathrm{d}s\,z(s)\{\dot{x}(s)-f(x(s))\}/g(x(s))\right)$$

$$\cdot\exp\left(-\tfrac{1}{2}\int_{t_0}^{t}\mathrm{d}s[f'(x(s))+g'(x(s))\{\dot{x}(s)-f(x(s))\}/g(x(s))]\right). \tag{2.16}$$

Hereby, we introduced the integration measures

$$\mathscr{D}x(t)=\lim_{N\to\infty}\left(\prod_{i=1}^{N-1}dx_i\right)\exp\left(-\varepsilon^{-1}\int_{t_0}^{t}\ln|g(x(s))|\,\mathrm{d}s\right) \tag{2.17a}$$

and

$$\mathscr{D}\left(\frac{z(t)}{2\pi}\right)=\lim_{N\to\infty}\prod_{i=1}^{N}\frac{\mathrm{d}z_i}{2\pi}. \tag{2.17b}$$

With $g(x)=1$, (additive noise) our general result in (2.16) simplifies[2] to give

$$R(xt|x_0)=\int_{x(t_0)=x_0}^{x(t)=x}\mathscr{D}x(t)\int\mathscr{D}\left(\frac{z(t)}{2\pi}\right)\chi[z]$$

$$\cdot\exp\left(\mathrm{i}\int_{t_0}^{t}\mathrm{d}s\,z(s)(\dot{x}-f(x(s)))\right)\exp\left(-\tfrac{1}{2}\int_{t_0}^{t}f'(x(s))\,\mathrm{d}s\right). \tag{2.18}$$

Next we consider two important special cases for the colored noise $\xi(t)$.

2.A. $\xi(t)$: *General Gaussian noise*

The characteristic functional of a general (Markovian or non-Markovian) Gaussian process of vanishing mean and correlation Eq. (1.2) reads [10]

$$\chi[z]=\exp\left(-\frac{D}{2}\int_{t_0}^{t}\mathrm{d}u\int_{t_0}^{t}\mathrm{d}s\,z(u)\,\sigma(u-s)\,z(s)\right). \tag{2.19}$$

[2] Note that with $g(x)\neq0$ the multiplicative flow in (1.1) can be transformed in additive noise by setting $x\to y=\int g^{-1}(u)\,\mathrm{d}u$

Thus, upon a rescaling of $z \to z/D$ we have for (2.16) the central result

$$R(x\,t\,|\,x_0) = \int_{x(t_0)=x_0}^{x(t)=x} \mathscr{D}x(t) \int \mathscr{D}\left(\frac{z(t)}{2\pi D}\right) \exp\left(-\frac{S[x,z]}{D}\right) \tag{2.20a}$$

where the complex-valued action functional (*non-Markovian Onsager-Machlup functional*) reads

$$S[x,z] = -\mathrm{i}\int_{t_0}^{t} \mathrm{d}s \frac{(\dot{x}(s) - f(x(s)))\,z(s)}{g(x(s))}$$

$$+\tfrac{1}{2}\int_{t_0}^{t} \mathrm{d}u \int_{t_0}^{t} \mathrm{d}s\, z(u)\,\sigma(u-s)\,z(s)$$

$$+\frac{D}{2}\int_{t_0}^{t} \mathrm{d}s \left[f'(x(s)) + g'(x(s))\left(\frac{\dot{x}(s) - f(x(s))}{g(x(s))}\right)\right]. \tag{2.20b}$$

For additive noise, i.e. $g(x)=1$, and exponentially correlated Gaussian noise, $\sigma(t)=\tau^{-1}\exp-|t|/\tau$, i.e. $\xi(t)$ an Ornstein-Uhlenbeck process, see Eq. (1.4), the result in (2.20) has been presented previously in [11] by use of a rather complex method involving δ-functionals for functionals $F[Q]$ of the time s on the interval in (t_0, t) [12], i.e. $F[Q] = \int \mathscr{D}x\, F[x]\,\delta[x-Q]$.

2.B. $\xi(t)$: Generalized colored shot noise

Let us consider stationary, generalized shot noise composed of pulse functions $h(t)$, with $h(t)=0$ for $t<0$ and pulse area

$$\int_0^\infty h(t)\,\mathrm{d}t = H. \tag{2.21}$$

We shall assume that the pulses occur at Poissonian arrival times $\{t_i\}$ which are distributed with a density $q(t)=\lambda\exp-\lambda t$. The generalized shot noise is then given by

$$\xi(t) = \sum_i a_i\, h(t-t_i). \tag{2.22}$$

The set $\{a_i\}$ are random variables which are independent of $\{t_i\}$, and independent among each other. The random variables are distributed with a common probability $\pi(a)$ obeying $\langle a\rangle = 0$, i.e. $\xi(t)$ is of vanishing mean. The cumulant averages, $\langle\xi(t_n)\ldots\xi(t_1)\rangle_c$, $t_n \geqq t_{n-1} \geqq \ldots \geqq t_1$ read explicitly [10]

$$\langle\xi(t_n)\ldots\xi(t_1)\rangle_c = \lambda\langle a^n\rangle \int_{-t_1}^{\infty} h(t_n+s)\ldots h(t_1+s)\,\mathrm{d}s. \tag{2.23}$$

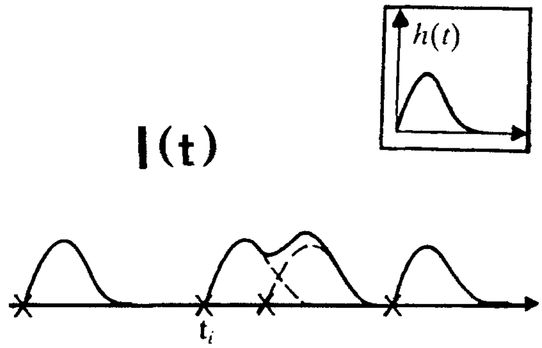

Fig. 1. Realization of colored Poissonian shot noise, Eq. (2.26), with its pulse function $h(t)$ depicted in the inset

In particular, the correlation function reads

$$\langle\xi(t+\tau)\,\xi(t)\rangle = \lambda\langle a^2\rangle \int_0^\infty h(\tau+u)\,h(u)\,\mathrm{d}u \tag{2.24}$$

which with $\langle a^2\rangle \propto D$ is of the form in (1.2).

The characteristic functional is also explicitly known [10, 13], and reads

$$\chi[z] = \exp\left(\lambda\int_{t_0}^{t}\mathrm{d}s\right.$$

$$\left.\cdot\left[-1 + \int_{-\infty}^{\infty}\mathrm{d}a\,\pi(a)\exp\left\{\mathrm{i}a\int_{t_0}^{t}\mathrm{d}u\,z(u)\,h(u-s)\right\}\right]\right). \tag{2.25}$$

Considering the usual shot noise [14]

$$I(t) = \sum_i h(t-t_i) \tag{2.26}$$

whose realization is depicted in Fig. 1, we find with $\pi(a)=\delta(a-1)$ for shot noise $\xi_I(t)$ of vanishing mean

$$\xi_I(t) = I(t) - \lambda H, \tag{2.27}$$

with the correlation function

$$\langle\xi_I(t)\,\xi_I(0)\rangle = \lambda\int_0^\infty h(t+u)\,h(u)\,\mathrm{d}u. \tag{2.28}$$

Thus its characteristic functional simplifies to give

$$\chi[z] = \left(\exp - \mathrm{i}\lambda H\int_{t_0}^{t} z(s)\,\mathrm{d}s\right)$$

$$\cdot\left(\exp\lambda\int_{t_0}^{t}\mathrm{d}s\left[-1 + \exp\mathrm{i}\int_0^\infty h(u)\,z(u+s)\,\mathrm{d}u\right]\right). \tag{2.29}$$

The substitution of (2.25) or (2.29) into (2.16) then yields the explicit path integral solution for generalized colored shot noise.

With the scaling $z \to z/D$, a WKB analogue of the form (2.20) with $S[x, z]$ being of order $O(D)$ does generally not exist. It is readily seen from (2.29) that the scaling $z \to z/D$ introduces singular terms D^{-n} with $n \geqq 1$. Formally, such a form exists for generalized shot noise of the type in (2.22), however, when $\langle a^n \rangle \propto D^{n-1}$ for all $n > 1$. For the remainder of the paper, we shall now restrict ourselves to Gaussian noise forces only.

3. Weak-noise analysis

With $g(x) \neq 0$, we restrict without loss of generality the discussion to additive noise only. The result for the conditional probability in (2.20) is then in a suitable form for a weak noise analysis, i.e. $D \to 0$. With $g(x) \equiv 1$, we now invoke the multi-dimensional saddle point method. To leading order in the noise strength, the action functional $S^0[x, z]$ in (2.20) reads

$$S^0[x, z] = \tfrac{1}{2} \int_{t_0}^{t} \mathrm{d}u \int_{t_0}^{t} \mathrm{d}s\, z(u)\, \sigma(u-s)\, z(s) - \mathrm{i} \int_{t_0}^{t} \mathrm{d}s [\dot{x}(s) - f(x(s))]\, z(s). \tag{3.1}$$

The equation for the extremal path $Q(s) \equiv (x_e(s), z_e(s))$ satisfies

$$\delta S^0 = 0 \tag{3.2}$$

i.e.

$$\left(\int_{t_0}^{t} z(u)\, \sigma(s-u)\, \mathrm{d}u - \mathrm{i}(\dot{x}(s) - f(x(s))) \right) \delta z + \left(\mathrm{i}\, z(s) \frac{\partial f(x(s))}{\partial x(s)} + \mathrm{i}\, \dot{z}(s) \right) \delta x = 0. \tag{3.3}$$

The extremal path $(x_e(s), z_e(s))$ thus satisfies the equation of motion

$$\dot{z}_e = -\frac{\partial f}{\partial x_e} z_e, \tag{3.4a}$$

$$\dot{x}_e = f(x_e) - \mathrm{i} \int_{t_0}^{t} \sigma(s-u)\, z_e(u)\, \mathrm{d}u, \tag{3.4b}$$

and the boundary conditions are given by

$$x_e(t_0) = x_0, \quad x_e(t) = x. \tag{3.4c}$$

With this equation of motion, the extremal action becomes from (3.1) solely a function of $z_e(t) \equiv z_e(t; x, x_0)$; i.e.

$$S^0[x_e, z_e] = -\tfrac{1}{2} \int_{t_0}^{t} \mathrm{d}s \int_{t_0}^{t} \mathrm{d}u\, z_e(s)\, \sigma(s-u)\, z_e(u). \tag{3.5}$$

For the Gaussian fluctuation analysis we follow standard methods [15, 16]. By setting $x = x_e + u_1$, $z = z_e + u_2$, and expanding $S^0[x, z]$ up to second order we find after Gaussian integrations, and the definition

$$W \equiv |\mathrm{Det}\, \delta^2 S^0[x_e, z_e]| = \left| \mathrm{Det}\left(\frac{\partial^2 S^0[Q]}{\partial Q_{in} \partial Q_{jm}} \right) \right|,$$

$$Q_1 \equiv x_e, \quad Q_2 = z_e; \quad i, j = 1, 2 \tag{3.6}$$

the WKB-type approximation for $R(xt|x_0)$, i.e. with (2.20b), (3.5)

$$R(xt|x_0) = (2\pi D W)^{-1/2} \left(\frac{z_e(t)}{z_e(t_0)} \right)^{1/2} \exp\left(-\frac{S^0[Q]}{D} \right). \tag{3.7}$$

An alternative approach consists in transforming the complex-valued functional in (2.20) into a real-valued functional. In doing so, one must integrate out the z-path integration in (2.20a). This suggestion has been made by Phythian [9], who introduces the (left) inverse $\sigma^{-1}(v-s)$, obeying (see also Ref. 17)

$$\int_{t_0}^{t} \mathrm{d}s\, \sigma^{-1}(v-s)\, \sigma(s-u) = \delta(v-u). \tag{3.8}$$

Thus, the result in (2.20) can (*formally*) be recast as a single, real-valued path integral

$$R(xt|x_0) = \int_{x(t_0)=x_0}^{x(t)=x} \mathscr{D}x(t) \exp\left(-\tfrac{1}{2} \int_{t_0}^{t} \mathrm{d}s f'(x(s)) \right) \cdot \exp -\frac{1}{2D} \int_{t_0}^{t} \mathrm{d}u \int_{t_0}^{t} \mathrm{d}s\, \sigma^{-1}(u-s) \cdot (\dot{x}(u) - f(x(u))(\dot{x}(s) - f(x(s))). \tag{3.9}$$

The memory in (3.9) clearly exhibits the non-Markovian character induced by colored noise, Eq. (1.2). Equation (3.9) yields from the corresponding action functional for the extremal path $x_e(s)$ the equation of motion

$$f'(x_e(s)) \int_{t_0}^{t} \sigma^{-1}(s-u)[\dot{x}_e(u) - f(x_e(u))]\, \mathrm{d}u + \frac{\mathrm{d}}{\mathrm{d}s} \int_{t_0}^{t} \sigma^{-1}(s-u)[\dot{x}_e(u) - f(x_e(u))]\, \mathrm{d}u = 0 \tag{3.10}$$

which with $x_e(t_0)=x_0$, $x_e(t)=x$, is not at all simple to solve. Moreover, generally, even the operator $\sigma^{-1}(u-s)$ is not readily constructed. Therefore, a formulation of the form Eqs. (3.8)–(3.10) does not provide any simplification in practice.

Finally, with $\xi(t)$ being a Markovian (Fokker-Planck) process, such as in (1.4) and in (1.6), we like to point out another pitfall: With the Onsager-Machlup functional for the Ornstein-Uhlenbeck process in (1.4, 1.5) given by [1–5]

$$S[\xi]=\frac{1}{4D}(\tau\dot{\xi}+\xi)^2, \tag{3.11}$$

or for the Markovian oscillator process $\dot{\xi}(t)$ in (1.6, 1.7) (see [4])

$$S[\dot{\xi}]=\frac{1}{4D}(\ddot{\xi}+\gamma\dot{\xi}+\omega_0^2\,\xi)^2, \tag{3.12}$$

respectively, one is tempted to construct the corresponding non-Markovian action-functional in (2.20) for the non-Markovian process in (1.1) by simply substituting, via a subsequent *differentiation* of (1.1), the ξ-dynamics in (3.11, 3.12), by the related dynamics of $x(t)$, i.e. $\xi(t)=(\dot{x}-f(x))/g(x)$, $\dot{\xi}(t)=[(\ddot{x}-f'\dot{x})\,g-(\dot{x}-f)\,g'\dot{x}]/g^2(x)$, etc. By doing so, however, one formally introduces unknown, additional constants of integration. In particular, the corresponding equation of motion for the extremal path, $x_e(t)$, is readily seen to be a fourth-order nonlinear differential equation for $x_e(t)$ in the case of (1.4), and a sixth-order nonlinear differential equation for $x_e(t)$ in the case of (1.6). With the only known two boundary condition given in (3.4c), we are thus left with no complete set of boundary conditions for the extremal path. This fact clearly reflects the loss of information introduced upon the formal differentiation of (1.1).

4. Generalizations, applications, and conclusions

All of the foregoing discussions can naturally also be generalized to multidimensional non-Markovian flows of the type

$$\dot{x}_\alpha=f_\alpha(\mathbf{x})+\sum_{i=1}^{n} g_{\alpha i}(\mathbf{x})\,\xi_i(t). \tag{4.1}$$

Moreover, if the condition [5c, 18, 19]

$$\frac{\partial g^{-1}_{\alpha\beta}}{\partial x_\gamma}=\frac{\partial g^{-1}_{\alpha\gamma}}{\partial x_\beta} \tag{4.2}$$

holds, the transformation, $\dot{y}_i=g^{-1}_{i\alpha}\,\dot{x}_\alpha$ yields a non-Markovian flow with additive noise forces only. Thus, for the non-Markovian Langevin equations

$$\dot{x}_i=f_i(\mathbf{x})+\xi_i(t) \tag{4.3}$$

we obtain with colored Gaussian noise

$$\langle\xi_i(t)\,\xi_j(s)\rangle=D\,\sigma_{ij}(t-s) \tag{4.4}$$

for the generalization of (2.20) the multidimensional non-Markovian Onsager-Machlup functional

$$S[\mathbf{x},\mathbf{z}]=-i\int_{t_0}^{t}\mathrm{d}s\,z_i(s)[\dot{x}_i(s)-f_i(\mathbf{x}(s))]$$
$$+\tfrac{1}{2}\int_{t_0}^{t}\mathrm{d}u\int_{t_0}^{t}\mathrm{d}s\,z_i(u)\,\sigma_{ij}(u-s)\,z_j(s)+\frac{D}{2}\int_{t_0}^{t}\mathrm{d}s\,\frac{\partial f_i(\mathbf{x}(s))}{\partial x_i} \tag{4.5}$$

wherein the standard summation convention over equal indices is implied.

From a practical point of view, the path-integral formulation put forward in Sect. 2 has its use in non-perturbative, e.g. "instanton-like" treatments, such as the weak noise analysis in Sect. 3. Clearly, with Gaussian colored noise forces $\xi(t)$ of the type in (1.4), or (1.6), a corresponding weak noise analysis in the corresponding Markovian multi-dimensional phase space is certainly more complex, due to the presence of a singular diffusion matrix [4] which in turn implies the highly singular path integration measure. As a simple illustration let us consider a bistable flow,

$$\dot{x}=f(x)+g(x)\,\xi(t) \tag{4.6}$$

where with $x_1<x_2<x_3$, $f(x_1)=f(x_2)=f(x_3)=0$, and x_1, x_3 are locally stable states, i.e. $f'(x_1)<0$, $f'(x_3)<0$, while x_2 is a locally unstable state with $f'(x_2)>0$. Moreover, we assume that $\xi(t)$ is Gaussian with a large correlation time τ

$$\tau=\frac{\int_0^\infty |\langle\xi(t)\,\xi(0)\rangle|\,\mathrm{d}t}{\langle\xi^2\rangle}\to\infty, \tag{4.7}$$

and extremely slowly varying correlation, i.e. with $\beta>0$

$$\langle\xi(t)\,\xi(0)\rangle\cong\langle\xi^2(0)\rangle=\frac{D}{\tau^\beta},\quad \text{as } \tau\to\infty. \tag{4.8}$$

Then, one readily finds from (3.4, 3.5), for the asymptotic leading part of the extremal action S_e^0, as $\tau \to \infty$, the result

$$S_e^0(\tau \to \infty) = \tfrac{1}{2} \tau^\beta \max{}^\pm \left|\frac{f(x)}{g(x)}\right|^2. \quad (4.9)$$

The notation $(+)$ indicates that the particle has started out at $x_1 < x_2$ with the maximum of the absolute value $|f/g|$ taken over the interval (x_1, x_2), while $(-)$ indicates the reverse situation with the particle starting out at $x_3 > x_2$, and the maximum of $|f/g|$ taken over (x_2, x_3). Therefore, the forward rate Γ^+ and the backward rate Γ^-, respectively, assume for $\tau \to \infty$ the asymptotic behavior

$$\Gamma^\pm \propto \exp\left\{\frac{-\tau^\beta}{2D} \max{}^\pm \left|\frac{f(x)}{g(x)}\right|^2\right\}. \quad (4.10)$$

The result in (4.9) presents a generalization of the asymptotic study for stochastic bistability being driven by the additive Ornstein-Uhlenbeck process Eq. (1.5) (i.e. $\beta = 1$) in [11] to a general (Markovian, or non-Markovian) process $\xi(t)$ obeying Eqs. (4.7), (4.8), and general multiplicative noise $g(x)$, with $g(x) \neq 0$ in (x_1, x_3). The result in (4.9) is also intuitively clear by noting that with $\xi(t)$ extremely slow, we can set in (1.1) $\dot{x} \cong 0$, i.e. $\xi(t)$ follows a Gaussian process with variance, $\sigma = D\tau^{-\beta}$, and with maximal mean value given by $|\langle \xi(t) \rangle| \cong \max^\pm |f/g|$, which must be overcome in order to reach the unstable state x_2.

In conclusion, we have presented the path integral solution for non-Markovian stochastic flows of the form in (1.1). Explicit results have been obtained for general colored Gaussian noise (Sect. 2.A), and generalized colored Poissonian shot noise (Sect. 2.B). From a practical point of view however, we suspect that – just as with the case of white noise (Fokker-Planck and master equations) discussed in [1–5] –, use of non-Markovian path integral methods will not necessarily undergo a flurry. This is so because often alternative, problem-specific methods [6–8] provide the same information more simply, as illustrated for example below Eq. (4.10). Also, the solution of equations such as the extremal path solution in (3.4) usually requires already extensive numerical methods. *Nevertheless, the path integral formulation does in many cases provide additional insight.* Moreover, it can be utilized in non-perturbative approximation methods which are a priori not available otherwise (for example, see (3.7) in Sect. 3).

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P. Hänggi
Lehrstuhl für Theoretische Physik
Universität Augsburg
Memminger Strasse 6
D-8900 Augsburg
Federal Republic of Germany