New Analogies between the Noisy Feigenbaum Scenario and Critical Phenomena

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The concepts of universality and scaling play an important role in various fields of theoretical physics like e.g. critical phenomena, nonlinear dynamics, hydrodynamics etc. [1]. In this note we work out further striking analogies of the Feigenbaum route to chaos [2] in presence of weak noise [3,4] to the theory of critical phenomena.

The noisy Feigenbaum scenario is described by the one-dimensional Langevin equation for the dynamical variable \( z \) in discrete time \( n [3,4] \)

\[
x_{n+1} = f_\mu(z_n) + g_\mu(z_n)\xi_n
\]  

where \( f_\mu(z) \) is a map with an invariant interval \( I_\mu \) with negative Schwarzian derivative [2] smoothly depending on the control parameter \( \mu \) and \( \alpha = 0 \) with the possible exception \( z = 0 \in I_\mu \) where the map has its global maximum of order \( z > 1 \). The noise coupling function \( g_\mu(z) \) is required to be smooth, positive, and bounded. The distribution of the uncorrelated random numbers \( \xi_n \) is given by

\[
P_\sigma(\xi_n) = u_\sigma(\xi_n)\exp\{-h_\sigma(\xi_n)/\sigma^2\}
\]  

where the small parameter \( \sigma \) determines the noise strength, the prefactor \( u_\sigma(\xi_n) \) is smooth, positive and bounded, and the exponentially leading part \( h_\sigma(\xi_n) \) has a global minimum of order \( \alpha > 0 \) at \( \xi_n = 0 \) and is smooth everywhere else.

Iteration of (1) leads to the renormalization group transformation that can be formulated in terms of the deterministic renormalization operator [3,4] \( \mathcal{T}f_\mu(x(z) = -\alpha f_\mu(f_\mu(-x/\alpha)) \).

Using an appropriate parametrization of \( \mu \) and \( z \) one obtains [2]

\[
(\mathcal{T}^k f_\mu(x(z \to f_\mu(x(z
\]  

for \( k \to \infty, z \in I_\mu \). Here \( \alpha \) and \( \delta \) are the universal Feigenbaum numbers and the maps \( f_\mu(z) \) constitute the unstable invariant manifold of the operator \( \mathcal{T} \). For the full renormalization group this manifold yields one relevant direction in function space at the critical point \( f_\mu(0) \). Similarly the scaling limit \( g_\mu^*(x) \) of the noise coupling functions gives another relevant direction.

The renormalization group yields an approximate scaling relation for the invariant density \( W_{\sigma\mu}(x) \), \( x \in I_\mu \), of the stochastic process (1) [5-7]

\[
\alpha W_{\sigma\mu}(x) \simeq 2 W_{\sigma|\mu|/\alpha}(-x/\alpha)
\]  

where \( \kappa \) is the scaling factor of the noise strength \( \sigma \). Strict scaling holds only in the trivial case with vanishing noise \( \sigma = 0 \). However, for the invariant functions \( f_\mu(x) \), \( g_\mu^*(x) \) the deviations from exact scaling become extremely small.

As a further approximative result a Frobenius-Perron-like relation [6]

\[
W_{\sigma\mu}(x) \simeq \sum_{f_\mu(y) = x} W_{\sigma\mu}(y) |f_\mu'(y)|^{-1}
\]
is found, valid for such \( x \in I_e \) for which there exists at least one \( y \in I_e \) with \( f_{\mu}(y) = x \).

For \( a \leq 1/2 \) (5) is exact in exponentially leading order in \( \sigma \) but not for \( a > 1/2 \), although it may still yield good approximations.

Using (3)-(5) one recovers the scaling behavior of the envelope \( \lambda \) of the Lyapunov exponent \( \lambda(\sigma, \mu) = \int W_{\sigma, \mu}(x) \ln|f_{\mu}'(x)| \, dx \) [3,4]

\[
\lambda(\sigma, \mu) = |\mu|^\frac{1}{1-\alpha} L(\sigma; |\mu|^{-\frac{1}{1-\alpha}})
\]

where \( \tilde{\psi} \) is a scaling function.

In terms of critical phenomena [8] the noise strength \( \sigma \) can be identified with the relevant scaling field, \( \lambda \) is then the order parameter, and \( \ln \tilde{\psi}, \ln \alpha \) are the critical exponents [3,4]. Any further critical exponent can be expressed in terms of \( \ln \delta \) and \( \ln \kappa \), i.e. we are dealing with a two exponent theory. The Feigenbaum number \( \alpha \) must not be confused with a critical exponent. It rather corresponds to the spatial scaling factor in a real space renormalization theory [4]. Eq.(4) corresponds to scaling of the coarse grained partition function and (6) to the equation of state.

The scaling factor \( \delta \) as well as the Feigenbaum number \( \alpha \) depend on \( x \) [2] whereas \( \kappa \) depends on both \( x \) [9] and \( a \) [7] but are independent of further details of (1), (2). Thus the numbers \( x \) and \( \alpha \) are related to the dimensionality of space and number of components of the order parameter which completely determine the universality class, fixed point and critical exponents [8].

As in (5) fluctuations are no longer taken fully into account this may be considered as the counterpart of mean field approximation for critical phenomena. Within this approximation one finds \( \tilde{\psi}_{\alpha''} = \alpha'' \) independent of \( a \). This result follows from the full renormalization theory only if \( a \leq 1 \). Hence, we identify \( \alpha'' \) with the dimensionality \( d \) of space and find as its critical value \( \alpha_c = 1 \). For \( d < \alpha_c \) the noise scaling factor \( \kappa \) is given by

\[
\kappa = \alpha''(1 + |c(\lambda, d)/\alpha''|^{\frac{1}{2}})^{\frac{1}{2-d}}
\]

with \( 1 > c(\lambda, d) > 0 \) having well defined limits \( d \to 0 \) [7] and \( d \to \infty \). Eq.(7) may be read either as \( \alpha' \) expansion or as hyperscaling relation. Note that the second scaling factor \( \kappa \) is independent of \( d \).

Finally we note that the whole analysis can be generalized to a large class of correlated noise including Ornstein-Uhlenbeck noise. In particular these systems belong to the same two exponent universality class independent of the correlation of the noise.

References